KYUNGPOOK Math. J. 51(2011), 385-393
http://dx.doi.org/10.5666/KMJ.2011.51.4.385

On Some Lacunary Generalized Difference Sequence Spaces of Invariant Means Defined by a Sequence of Modulus Functions

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Abstract. The aim of this paper is to introduce and study the sequence spaces $[w,\theta,F,p,q]_\infty(\Delta_v^m)$, $[w,\theta,F,p,q]_1(\Delta_v^m)$ and $[w,\theta,F,p,q]_0(\Delta_v^m)$, which arise from the notions of generalized difference sequence space, lacunary convergence, invariant mean and a sequence of Moduli $F=(f_k)$. We establish some inclusion relations between these spaces under some conditions.

1. Introduction

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, with $x_k\in\mathbb{R}$ or $\mathbb{C}$, normed by $\|x\|=\sup_k|x_k|$, respectively.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, is said to be an invariant mean or $\sigma-$mean if and only if

(i) $\phi(x)\geq 0$ when the sequence $x=(x_n)$ has $x_n\geq 0$ for all n,

(ii) $\phi(e)=1$,

(iii) $\phi(x_{\sigma(n)})=\phi(x)$ for all $x\in\ell_\infty$.

In case σ is the translation mapping $n\to n+1$, a $\sigma-$mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences. If $x=(x_k)$, write $Tx=(Tx_k)=(x_{\sigma(k)})$. It can be shown that

$$V_\sigma=\left\{x\in\ell_\infty:\lim_k t_{kn}(x)=l,\text{ uniformly in } n\right\},$$

* Corresponding Author.
Received August 31, 2010; revised October 8, 2010; accepted December 9, 2010.
2010 Mathematics Subject Classification: 46A45, 40A05.
Key words and phrases: Invariant mean, Difference sequence spaces, lacunary sequence, modulus function.

$l = \sigma - \lim x$ where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + x_{\sigma^2(n)} + ... + x_{\sigma^k(n)}}{k+1} \text{ [16].}$$

A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all of its Banach limits coincide (see Banach [1]). Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [11] proved that

$$\hat{c} = \left\{ x \in \ell_\infty : \lim_{m\to\infty} t_{mn}(x) \text{ exists, uniformly in } n \right\},$$

where $t_{mn}(x) = (x_n + x_{n+1} + ... + x_{n+m})/(m+1)$.

The notion of difference sequence space was introduced by Kızmaz [10]. It was generalized by Et and Çolak [6] as follows:

$$X(\Delta^m) = \{x = (x_k) : \Delta^m x \in X\},$$

for $X = c_0, c$ and ℓ_∞, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $(\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Et and Esi [7] generalized the above sequence spaces to the following sequence spaces:

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $\Delta_v^0 x = (v_k x_k)$, $(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and $(\Delta_v^m x_k) = \left(\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}\right)$ and so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}.$$

Let $\theta = (k_r)$ be a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by ρ_r.

The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [9] as

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

A function $f : [0,\infty) \to [0,\infty)$ is called a modulus function if
(i) $f(t) = 0$ iff $t = 0$,
(ii) $f(t+u) \le f(t) + f(u)$, $\forall t, u \ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

The following inequality will be used throughout the article. Let $p=(p_k)$ be a positive sequence of real numbers with $0<p_k\le \sup p_k=G$, $D=\max(1,2^{G-1})$. Then for all $a_k,b_k\in\mathbb{C}$ for all $k\in N$, we have

$$|a_k+b_k|^{p_k}\le D\{|a_k|^{p_k}+|b_k|^{p_k}\}[12]. \tag{1}$$

Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if there exists a constant K such that $q_2(x)\le Kq_1(x)$ for all $x\in X$. If each is stronger than the other, q_1 and q_2 are said to be equivalent [29].

Ruckle [15] used the idea of a modulus function to construct some spaces of complex sequences. Maddox [13] investigated and discussed some properties of some sequence spaces defined using a modulus function f. After Bektaş and Çolak [2] used a sequence of moduli $F=(f_k)$ to define some sequence spaces. Later on Et [4] defined some sequence spaces by using a modulus function. Recently Bektaş and Çolak [3] introduced some new sequence spaces by using a sequence of moduli $F=(f_k)$.

Et and Gökhan [8] have defined the following sequence spaces:

$$\begin{aligned}(w,\theta,f,p,q) &= \{x\in w(X):\lim_r \frac{1}{h_r}\sum_{k\in I_r}[f(q(t_{km}(x-L)))]^{p_k}=0,\\ &\qquad \text{uniformly in } m, \text{ for some } L\}\\ (w,\theta,f,p,q)_0 &= \{x\in w(X):\lim_r \frac{1}{h_r}\sum_{k\in I_r}[f(q(t_{km}(x)))]^{p_k}=0 \text{ uniformly in } m\},\\ (w,\theta,f,p,q)_\infty &= \{x\in w(X):\sup_{r,n} \frac{1}{h_r}\sum_{k\in I_r}[f(q(t_{km}(x)))]^{p_k}<\infty\}.\end{aligned}$$

2. Main results

Definition 2.1. Let $F=(f_k)$ be a sequence of moduli, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q and $p=(p_k)$ be a sequence of strictly positive real numbers. By $w(X)$, we shall denote the space of all sequences defined over X. Now we define the following sequence spaces:

$$[w,\theta,F,p,q]_\infty(\Delta_v^m)=\left\{x\in w(X):\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r}[f_k(q(t_{kn}(\Delta_v^m x_k)))]^{p_k}<\infty\right\},$$

$$[w,\theta,F,p,q]_1(\Delta_v^m)=\left\{x\in w(X):\lim_r\frac{1}{h_r}\sum_{k\in I_r}[f_k(q(t_{kn}(\Delta_v^m x_k-L)))]^{p_k}=0 \text{ for some } L>0\right\},$$

$$[w,\theta,F,p,q]_0(\Delta_v^m)=\left\{x\in w(X):\lim_r\frac{1}{h_r}\sum_{k\in I_r}[f_k(q(t_{kn}(\Delta_v^m x_k)))]^{p_k}=0\right\},$$

uniformly in n. Throughout the paper Z denotes 0, 1 or ∞.

For $p_k = 1$ for all $k \in \mathbb{N}$, we write these spaces as $[w, \theta, F, q]_Z(\Delta_v^m)$.

For $f_k(x) = x$ for every k and $p_k = 1$ for all $k \in \mathbb{N}$, we write these spaces as $[w, \theta, q]_Z(\Delta_v^m)$.

Theorem 2.2. *$p = (p_k)$ be a bounded sequence, then $[w, \theta, F, p, q]_0(\Delta_v^m)$, $[w, \theta, F, p, q]_1(\Delta_v^m)$ and $[w, \theta, F, p, q]_\infty(\Delta_v^m)$ are linear spaces over the set of complex numbers.*

Proof. We give the proof for $[w, \theta, F, p, q]_0(\Delta_v^m)$ only. The others can be treated similarly. Let $x, y \in [w, \theta, F, p, q]_0(\Delta_v^m)$. For $\alpha, \beta \in \mathbb{C}$ there exist positive integers N_α and M_β such that $|\alpha| \le N_\alpha$ and $|\beta| \le M_\beta$. Since each f_k is subadditive, q is a seminorm, Δ_v^m linear and by inequality (1)

$$\frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m (\alpha x_k + \beta y_k)))))]^{p_k}$$

$$\le \frac{1}{h_r} \sum_{k \in I_r} [|\alpha| f_k (q (t_{kn}(\Delta_v^m x_k))) + |\beta| f_k (q (t_{kn}(\Delta_v^m y_k)))]^{p_k}$$

$$\le D (N_\alpha)^G \frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m x_k)))]^{p_k} + D (M_\beta)^G \frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m y_k)))]^{p_k}$$

$$\to 0$$

as $r \to \infty$ and uniformly in n. Thus $[w, \theta, F, p, q]_0(\Delta_v^m)$ is a linear space. □

Theorem 2.3. *Let $F = (f_k)$ be a sequence of moduli, then $[w, \theta, F, p, q]_0(\Delta_v^m) \subset [w, \theta, F, p, q]_1(\Delta_v^m) \subset [w, \theta, F, p, q]_\infty(\Delta_v^m)$.*

Proof. We prove the second inclusion, since the first inclusion is obvious. Let $x \in [w, \theta, F, p, q]_1(\Delta_v^m)$. We have

$$\frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m x_k)))]^{p_k}$$

$$\le D \frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m x_k - L)))]^{p_k} + D \frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(L)))]^{p_k} .$$

Then there exists an integer K_L such that $q(L) \le K_L$. Hence, we have

$$\frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m x_k)))]^{p_k}$$

$$\le D \frac{1}{h_r} \sum_{k \in I_r} [f_k (q (t_{kn}(\Delta_v^m x_k - L)))]^{p_k} + D \frac{1}{h_r} \max \left(1, [(K_L) f_k(1)]^H\right),$$

so $x \in [w, \theta, F, p, q]_1(\Delta_v^m)$. □

The proof of the following results are easy and thus omitted.

Theorem 2.4. *Let $F = (f_k)$ and $G = (g_k)$ be two sequences of moduli. For any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and seminorms q, q_1 and q_2, we have*

(i) $[w, \theta, F, p, q]_Z(\Delta_v^m) \subset [w, \theta, F \circ G, p, q]_Z(\Delta_v^m)$,

(ii) $[w, \theta, F, p, q]_Z(\Delta_v^m) \cap [w, \theta, G, p, q]_Z(\Delta_v^m) \subseteq [w, \theta, F + G, p, q]_Z(\Delta_v^m)$,

(iii) $[w, \theta, F, p, q_1]_Z(\Delta_v^m) \cap [w, \theta, F, p, q_2]_Z(\Delta_v^m) \subset [w, \theta, F, p, q_1 + q_2]_Z(\Delta_v^m)$,

(iv) If q_1 is stronger than q_2 then $[w, \theta, F, p, q_1]_Z(\Delta_v^m) \subset [w, \theta, F, p, q_2]_Z(\Delta_v^m)$,

(v) If q_1 is equivalent to q_2 then $[w, \theta, F, p, q_1]_Z(\Delta_v^m) = [w, \theta, F, p, q_2]_Z(\Delta_v^m)$,

(vi) $[w, \theta, F, p, q_1]_Z(\Delta_v^m) \cap [w, \theta, F, t, q_2]_Z(\Delta_v^m) \neq \varnothing$,

where $Z = 1$, 0 or ∞.

Lemma 2.5. *Let $F = (f_k)$ be a sequence of moduli and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f_k(x) \leq 2 f_k(1) \delta^{-1} x$ [14].*

Theorem 2.6. *Let $F = (f_k)$ be a sequence of moduli. Then*

$$[w, \theta, q]_Z(\Delta_v^m) \subset [w, \theta, F, q]_Z(\Delta_v^m).$$

Proof. We give the proof for $Z = 0$ only. Let $x \in [w, \theta, q]_0(\Delta_v^m)$. Then we have

$$\varphi_r = \frac{1}{h_r} \sum_{k \in I_r} q\left(t_{kn}\left(\Delta_v^m x_k\right)\right) \to 0$$

as $r \to \infty$ uniformly in n. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. Then we can write

$$\begin{aligned} &\frac{1}{h_r} \sum_{k \in I_r} f_k\left(q\left(\Delta_v^m x_k\right)\right) \\ &\quad = \frac{1}{h_r} \sum_{k \in I_r, q(t_{kn}(\Delta_v^m x_k)) \leq \delta} f_k\left(q\left(\Delta_v^m x_k\right)\right) + \frac{1}{h_r} \sum_{k \in I_r, q(t_{kn}(\Delta_v^m x_k)) > \delta} f_k\left(q\left(\Delta_v^m x_k\right)\right) \\ &\quad \leq \frac{1}{h_r}\left(h_r \varepsilon\right) + \frac{1}{h_r} 2 f_k(1) \delta^{-1} h_r \varphi_r. \end{aligned}$$

Therefore $x \in [w, \theta, F, q]_0(\Delta_v^m)$. The others can be treated similarly. □

Theorem 2.7. *Let $F = (f_k)$ be a sequence of moduli, if $\lim_{u \to \infty} \frac{f_k(u)}{u} = \beta > 0$, then*

$$[w, \theta, q]_1(\Delta_v^m) = [w, \theta, F, q]_1(\Delta_v^m).$$

Proof. By Theorem 2.6, we need only show that $[w, \theta, F, q]_1(\Delta_v^m) \subset [w, \theta, q]_1(\Delta_v^m)$. Let $\beta > 0$ and $x \in [w, \theta, F, q]_1(\Delta_v^m)$. Since $\beta > 0$, we have $f_k(u) \geq \beta u$ for all $u \geq 0$. Hence we have

$$\frac{1}{h_r} \sum_{k \in I_r} f_k\left(q\left(\Delta_v^m x_k - L\right)\right) \geq \frac{1}{h_r} \sum_{k \in I_r} \beta\left(q\left(\Delta_v^m x_k - L\right)\right) = \frac{1}{h_r} \beta \sum_{k \in I_r} \left(q\left(\Delta_v^m x_k - L\right)\right).$$

Therefore we have $x \in [w,\theta,q]_1(\Delta_v^m)$. □

Theorem 2.8. *Let* $0 < p_k \le r_k$ *and* $\left(\frac{r_k}{p_k}\right)$ *be bounded. Then* $[w,\theta,F,r,q]_Z(\Delta_v^m) \subset [w,\theta,F,p,q]_Z(\Delta_v^m)$ *where* $Z = 1,\ 0$ *or* ∞.

Proof. We shall prove only $[w,\theta,F,r,q]_0(\Delta_v^m) \subset [w,\theta,F,p,q]_0(\Delta_v^m)$. The other inclusions can be proved similarly. Let $x \in [w,\theta,F,r,q]_0(\Delta_v^m)$. Write $w_{k,n} = \left[f_k\left(q\left(t_{kn}(\Delta_v^m x_k)\right)\right)\right]^{r_k}$ and $\lambda_k = \frac{P_k}{r_k}$, so that $0 < \lambda < \lambda_k \le 1$ for each k.

We define the sequences $(u_{k,n})$ and $(s_{k,n})$ as follows:

Let $u_{k,n} = w_{k,n}$ and $s_{k,n} = 0$ if $w_{k,n} \ge 1$, and let $u_{k,n} = 0$ and $s_{k,n} = w_{k,n}$ if $w_{k,n} < 1$. Then it is clear that for all $k,n \in \mathbb{N}$, we have $w_{k,n} = u_{k,n} + s_{k,n}$, $w_{k,n}^{\lambda_k} = u_{k,n}^{\lambda_k} + s_{k,n}^{\lambda_k}$. Now it follows that $u_{k,n}^{\lambda_k} \le u_{k,n} \le w_{k,n}$ and $s_{k,n}^{\lambda_k} \le s_{k,n}^{\lambda}$. Now we have

$$\lim_k w_{k,n}^{\lambda_k} \le \lim_k w_{k,n} + \left(\lim_k s_{k,n}\right)^{\lambda}.$$

This implies that $x \in [w,\theta,F,p,q]_0(\Delta_v^m)$. □

Theorem 2.9. *Let* $m \ge 1$ *be a fixed integer, then* $[w,\theta,F,p,q]_Z(\Delta_v^{m-1}) \subset [w,\theta,F,p,q]_Z(\Delta_v^m)$.

Proof. The proof of the inclusions follows from the following inequality

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left[f_k\left(q\left(t_{kn}(\Delta_v^m x_k)\right)\right)\right]^{p_k} &\le D\frac{1}{h_r}\sum_{k\in I_r}\left[f_k\left(q\left(t_{kn}(\Delta_v^{m-1}x_k)\right)\right)\right]^{p_k}\\ &+D\frac{1}{h_r}\sum_{k\in I_r}\left[f_k\left(q\left(t_{kn}(\Delta_v^{m-1}x_{k+1})\right)\right)\right]^{p_k}.\end{aligned}$$

Theorem 2.10. *The sequence spaces* $[w,\theta,F,p,q]_0(\Delta_v^m)$ *and* $[w,\theta,F,p,q]_\infty(\Delta_v^m)$ *are not solid.*

Proof. We give the proof only for $[w,\theta,F,p,q]_0(\Delta_v^m)$.For this let $p_k = 1$ for all $k \in \mathbb{N}$, $\theta = (2^r)$, $f_k(x) = x$ for all $k \in \mathbb{N}$, $q(x) = |x|$ and $m = 0$.Consider the sequence $x_k = (-1)^k$ for all $k \in \mathbb{N}$ and (α_k) be defined as $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Then $(x_k) \in [w,\theta,F,p,q]_0(\Delta_v^m)$ but $(\alpha_k x_k) \notin [w,\theta,F,p,q]_0(\Delta_v^m)$. Hence $[w,\theta,F,p,q]_0(\Delta_v^m)$ is not solid. □

Theorem 2.11. *Let* $\theta = (k_r)$ *be a lacunary sequence. If* $1 < \liminf_r \rho_r < \limsup_r \rho_r < \infty$*, then we have* $[w,\sigma,F,p,q]_0(\Delta_v^m) = [w,\theta,F,p,q]_0(\Delta_v^m)$

$$[w,\sigma,F,p,q]_0(\Delta_v^m) = \left\{x \in w(X) : \lim_n \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(t_{kn}(\Delta_v^m x_k)\right)\right)\right]^{p_k} = 0\right\}.$$

Proof. Suppose $\liminf_r \rho_r > 1$ then there exists $\delta > 0$ such that $\rho_r = \frac{k_r}{k_{r-1}} \ge 1+\delta$

for all $r \geq 1$. Then for $x \in [w, \sigma, F, p, q]_0(\Delta_v^m)$, we write

$$\frac{1}{h_r}\sum_{i\in I_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}$$
$$= \frac{1}{h_r}\sum_{i=1}^{k_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}$$
$$= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{i=1}^{k_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}\right) - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{i=1}^{k_{r-1}}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}\right).$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \leq \frac{(1+\delta)}{\delta} \text{ and } \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

Since $k_r^{-1}\sum_{i=1}^{k_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} \to 0$ and $k_{r-1}^{-1}\sum_{i=1}^{k_{r-1}}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} \to 0$ uniformly in n, we have

$$\frac{1}{h_r}\sum_{i\in I_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} \to 0,$$

as $r \to \infty$ and uniformly in n, that is, $x \in [w, \theta, F, p, q]_0(\Delta_v^m)$.

If $\limsup_r \rho_r < \infty$, there exists $\beta > 0$ such that $\rho_r < \beta$ for all $r \geq 1$. Let $x \in [w, \theta, F, p, q]_0(\Delta_v^m)$ and $\varepsilon > 0$ be given. Then there exists $R > 0$ such that for every $j \geq R$ and all n

$$A_j = \frac{1}{h_j}\sum_{i\in I_j}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \ldots$. Now let u be any integer with $k_{r-1} < u \leq k_r$, where $r > R$. Then

$$\frac{1}{u}\sum_{i=1}^{u}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} \leq k_{r-1}^{-1}\sum_{i=1}^{k_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}$$
$$= k_{r-1}^{-1}\left\{\sum_{i\in I_1}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i} + \sum_{i\in I_2}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}\right.$$
$$\left. + \ldots + \sum_{i\in I_r}\left[f_i\left(q\left(t_{in}(\Delta_v^m x_i)\right)\right)\right]^{p_i}\right\}$$

$$= \frac{k_1}{k_{r-1}} k_1^{-1} \sum_{i \in I_1} [f_i(q(t_{in}(\Delta_v^m x_i)))]^{p_i} + \frac{k_2 - k_1}{k_{r-1}} (k_2 - k_1)^{-1} \sum_{i \in I_2} [f_i(q(t_{in}(\Delta_v^m x_i)))]^{p_i}$$

$$+ ... + \frac{k_R - k_{R-1}}{k_{r-1}} (k_R - k_{R-1})^{-1} \sum_{i \in I_R} [f_i(q(t_{in}(\Delta_v^m x_i)))]^{p_i}$$

$$+ ... + \frac{k_r - k_{r-1}}{k_{r-1}} (k_r - k_{r-1})^{-1} \sum_{i \in I_r} [f_i(q(t_{in}(\Delta_v^m x_i)))]^{p_i}$$

$$= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + ... + \frac{k_R - k_{R-1}}{k_{r-1}} A_R + \frac{k_{R+1} - k_R}{k_{r-1}} A_{R+1} + ... + \frac{k_r - k_{r-1}}{k_{r-1}} A_r$$

$$\leq (\sup_{j \geq 1} A_j) \frac{k_R}{k_{r-1}} + (\sup_{j \geq R} A_j) \frac{k_r - k_R}{k_{r-1}} < K \frac{k_R}{k_{r-1}} + \varepsilon\beta.$$

Since $k_{r-1} \to \infty$ as $u \to \infty$, it follows that $u^{-1} \sum_{i=1}^{u} [f_i(q(t_{in}(\Delta_v^m x_i)))]^{p_i} \to 0$ uniformly in n and consequently $x \in [w, \sigma, F, p, q]_0(\Delta_v^m)$. □

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