Baer and Quasi-Baer Modules over Some Classes of Rings

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ABSTRACT. We study Baer and quasi-Baer modules over some classes of rings. We also introduce a new class of modules called *AI*-modules, in which the kernel of every nonzero endomorphism is contained in a proper direct summand.

The main results obtained here are: (1) A module is Baer iff it is an *AI*-module and has *SSIP*. (2) For a perfect ring R, the direct sum of Baer modules is Baer iff R is primary decomposable. (3) Every injective R-module is quasi-Baer iff R is a QI-ring.

1. Introduction

Baer and quasi-Baer modules were introduced by Rizvi and Roman in [9], extending to modules, the same notions known for rings (see also[10] and [11]). These notions turn to be very useful and yield many interesting structure theorems.

In the present work, we study some questions relative to Baer and quasi-Baer modules. The work falls in the following theme: Given a class $\mathcal{C}$ of R-modules (injective, semisimple,...), find necessary and sufficient conditions on the ring R such that every Baer or quasi-Baer module is in $\mathcal{C}$, or conversely every module in $\mathcal{C}$ is Baer or quasi-Baer.

The material is divided into four sections. In section 2, we introduce a class of modules that we call *AI*-modules, in which every nonzero annihilator contains a nonzero idempotent. This class lies strictly between the class of $\mathcal{K}$-nonsingular modules and the class of Baer modules. Our scope, is to study the Baer modules via the *AI* property. In section 3, we give a characterization of Baer modules using the *AI* property and the *SSIP*, we show also that an *AI*-module with some chain condition on direct summands is Baer. In section 4, we study rings over which the direct sum of Baer modules is Baer. This provides us a characterization of perfect rings that are primary decomposable. In the last section, we characterize rings all of whose injective modules are quasi-Baer, it turn out that these are exactly the QI- rings.

Throughout this work R denotes an arbitrary ring, the Jacobson radical of R is denoted $J(R)$. All modules considered are left R-modules. We denote $\text{End}_R(M)$ the endomorphism ring of an R-module M. A submodule N of M is said to be fully invariant, if $u(N) \subset N$, for every $u \in \text{End}_R(M)$.

2. Definitions and basic properties

Definition 2.1([9]). An R-module M is said to be Baer (resp. quasi-Baer), if for every submodule (resp. fully invariant submodule) N of M, the left ideal $\{u \in \text{End}_R(M) : u(N) = 0\}$ of $\text{End}_R(M)$ is generated by an idempotent.

These are generalizations of the notion of Baer and quasi-Baer rings respectively. As noted in [9], a ring R is Baer (resp. quasi-Baer), if and only if, it is Baer (resp. quasi-Baer) as a left module, or right module over itself.

Note that Baer rings were introduced by Kaplansky [8], and quasi-Baer ring by Clark [4].

Definition 2.2. Let R be a ring. An R-module M is said to have the AI property, or that M is an AI-module, if for every nonzero endomorphism u of M, there exists a nonzero idempotent p in $\text{End}_R(M)$ such that $p(\text{Ker}(u)) = 0$.

As in [7], the term AI means that every **A**nihilator contains an **I**dempotent, since for $N = \text{Ker}(u)$, $\text{Ann}_S(N) = \{v \in S : v(N) = 0\}$, where $S = \text{End}_R(M)$, contains a nonzero idempotent.

Example 2.3. Every semisimple module and more generally, every module with von Neumann regular ring is AI.

Theorem 2.4. *Let M be an R-module and et $S = \text{End}_R(M)$. The following assertions are equivalent:*

(i) *M is an AI-module.*

(ii) *For every left ideal I of S, if $r_M(I) = \{m \in M : Im = 0\}$ is nonzero, then there exists an idempotent $p \in S$, such that $p \neq 1$ and $r_M(I) \subset p(M)$.*

Proof. (i)$\Rightarrow$(ii). Suppose that M is an AI-module, let I a left ideal of $S = \text{End}_R(M)$ such that $r_M(I) = N$ is nonzero. Then there exists a nonzero idempotent p of S such that $p(N) = 0$. This implies that $N \subset (1-p)(M)$.

Conversely, suppose that (ii) holds, if u is noninjective, put $N = \text{Ker}(u)$ and $I = \text{Ann}_S(N)$. We have $IN = 0$, thus $N \subset r_M(I) \subset p(M)$, where $p \neq 1$. This implies $(1-p)(N) = 0$. $\qquad\square$

Remark 2.5. Every Baer-module is AI, but the converse does not hold in general, as the following example shows, see Lam [12], Example 7.54.

Example 2.6. Let K be a field and R the ring of sequences $(a_1, a_2, \ldots, a_n, \ldots)$, of elements of K, which are stationnary. Then R is von Neumann regular, hence AI

as left R-module, but R is not Baer.

Example 2.7. Every module for which every nonzero endomorphism is monic, is AI.

Definition 2.8([9]). An R-module is said to be $\mathcal{K}$-nonsingular, if for every nonzero endomorphism u of M, $\mathrm{Ker}(u)$ is not essential in M.

Proposition 2.9. *Every AI-module is $\mathcal{K}$-nonsingular.*

Proof. Let M be an AI module and $u \in \mathrm{End}_R(M)$ be nonzero. Then there exists nonzero idempotent p in $\mathrm{End}_R(M)$, such that $p(\mathrm{Ker}(u)) = 0$. That is, $\mathrm{Ker}(u)$, is contained in a direct summand. Thus $\mathrm{Ker}(u)$ is not essential in M. □

Example 2.10. The converse of the preceding proposition is not true as we see by the following example. Let K be any field, consider the commutative K-algebra R generated by two elements a, b, such that $ab = 0$. Since R is noetherian and reduced, then it is nonsingular. If we take $u : R \to R$, defined by $u(x) = xa$, then $\mathrm{Ker}(u) = (b)$, the ideal generated by b, and there exists no nonzero idempotent p such that $p(\mathrm{Ker}(u)) = 0$.

As for Baer modules, the AI property is inherited by direct summands:

Proposition 2.11. *If M is an AI-module, then every summand of M is an AI-module.*

Proof. Let $M = N \oplus L$ be an AI-module, and $0 \neq u \in \mathrm{End}_R(N)$. Let $v = \left(\begin{smallmatrix} u & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{End}_R(M)$. Then $\mathrm{Ker}(v) = \mathrm{Ker}(u) \oplus L$. Since M is AI, there exists a nonzero idempotent $p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{End}_R(M)$ such that $p(\mathrm{Ker}(v)) = 0$. This implies that $b(L) = d(L) = 0$, hence $b = d = 0$. Now $p = \left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right)$, and $p^2 = \left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a^2 & 0 \\ ca & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right)$, thus $a^2 = a \neq 0$, and $a(\mathrm{Ker(u)}) = 0$. Consequently, N is AI. □

Example 2.12. In contrast with Proposition 2.11., the direct sum of AI-modules need not to be AI. As an example, take the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$, where p is a prime integer. These are AI-modules. Let $M = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, then M is not AI. For if we consider the endomorphism $u : M \to M$ defined by $u(x, y) = (0, \bar{x})$, then $\mathrm{Ker}(u) = p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ is essential in M. So, M is not $\mathcal{K}$-nonsingular, hence not AI.

3. AI-modules and Baer modules

In this section, we shall investigate some cases in which the AI-modules are Baer modules. First we recall a notion that was introduced by G. V. Wilson in [13].

Defiition 3.1. A module M is said to have the summand intersection property (SIP), if the intersection of any two direct summands of M is a direct summand. A module M is said to have the strong summand intersection property $(SSIP)$, if the intersection of any family of direct summands of M is a direct summand.

Clearly every semisimple module has $SSIP$. On the other hand, it was shown in [9] that every Baer module has $SSIP$.

The link between AI-modules and Baer modules is the $SSIP$ property, as shown by the next theorem:

Theorem 3.2. *Let M be an R-module then M is Baer, if and only if, M is AI and has SSIP.*

Proof. It remains only to show the sufficiency. Suppose that M is an AI-module with the $SSIP$ property. We shall show that M is Baer. Let N be a submodule of M, $S = \mathrm{End}_R(M)$, and $\mathrm{Ann}_S(N) = \{u \in S : u(N) = 0\}$. We must show that $\mathrm{Ann}_S(N) = Sp$, where p is an idempotent in $\mathrm{End}_R(M)$.

If $\mathrm{Ann}_S(N) = 0$, then we are done. Suppose that $\mathrm{Ann}_S(N) \neq 0$. Let $\mathcal{F} = \{q \in \mathrm{End}_R(M) : q^2 = q, \text{ and } q(N) = 0\}$. Since M is AI, then $\mathcal{F} \neq \varnothing$. Now M has SSIP, thus $\cap_{q \in \mathcal{F}} \mathrm{Ker}(q) = \mathrm{Ker}(p)$, for some idempotent p in S. We are going to show that $\mathrm{Ann}_S(N) = Sp$.

We have $Sp \subset \mathrm{Ann}_S(N)$. Now, let us show that $\mathrm{Ann}_S(N) \cap S(1-p) = \{0\}$. Suppose on the contrary that $\mathrm{Ann}_S(N) \cap S(1-p) \neq \{0\}$. Since $\mathrm{Ann}_S(N) \cap S(I-p) \subset \mathrm{Ann}_S(N + p(M))$, then $\mathrm{Ann}_S(N + p(M)) \neq 0$. The fact that M is AI, implies that there exists a nonzero idempotent f of S such that $f(N + p(M)) = 0$. Now $f(N) = 0$, thus $f \in \mathcal{F}$, so $\mathrm{Ker}(p) \subset \mathrm{Ker}(f)$, i.e. $f \cdot (I - p) = 0$, implying $fp = f$. But $f(p(M)) = 0$, hence $f = 0$ a contradiction. This means that $\mathrm{Ann}_S(N) \cap S(1-p) = \{0\}$. Now let $u \in \mathrm{Ann}_S(N)$, since $p \in \mathrm{Ann}_S(N)$, $u - up \in \mathrm{Ann}_S(N)$. Thus $u \cdot (1-p) \in \mathrm{Ann}_S(N)$. But, $u \cdot (1-p) \in S(1-p)$, hence $u \cdot (1-p) = 0$. i.e. $u = up \in Sp$. $\qquad \square$

Remark 3.3. In [9], Proposition 2.2, the authors showed that if: (1) M has $SSIP$ and (2) $\mathrm{Ker}(u)$ is a direct summand, for every $u \in \mathrm{End}_R(M)$, then M is Baer module. The condition of being AI in our theorem is weaker than the condition (2).

Theorem 3.4. *Let M be an R-module such that $\mathrm{End}_R(M)$ has no infinite orthogonal set of nonzero idempotents. Then M is Baer if and only if M is AI.*

Proof. (See also the proof of Theorem 7.55 in [12]). As in Theorem 3.2., let N be a submodule of M, such that and $I = \mathrm{Ann}_S(N)$ is nonzero. By [12], Proposition 6.59, S satisfies ACC on left direct summands. Take Sp maximal where p is taken among idempotents in I. We shall show that $I = Sp$. Again it suffices to show that $I \cap S(1-p) = \{0\}$. Otherwise, $\mathrm{Ann}(N + p(M))$ contains a nonzero idempotent f. We have $fp = 0$. Let $p' = p + (1-p)f$. Then p' is an idempotent in I. But $p'p = p = pp'$. It follows that $Sp \subset Sp'$. The maximality of Sp implies $Sp = Sp'$. Hence $p' = hp$, for some $h \in S$. Now $p'(1-p) = hp(1-p) = 0$, so $p'p = p'$. Thus $p' = p$, i.e. $(1-p)f = f - pf = 0$. Composing by f on the left, yields $f = 0$ a contradiction. $\qquad \square$

Corollary 3.5. *Every noetherian or artinian AI-module is Baer. In particular, it has SSIP.*

In general, artinian or noetherian modules need not have $SSIP$, as shown in [1], example 1. The authors gave there an example of finite dimensional algebra which does not have SIP as a left module over itself.

4. Direct sum of Baer modules

As shown in Example 2.12., direct sum of AI-modules need not be AI. This fact lead to the problem to know when this is true. In this section, we shall investigate this question. Namely, we shall study rings over which the direct sum of AI (resp. Baer) modules is AI (resp. Baer). This leads to the characterizations of some classes of rings.

Proposition 4.1. *Let M be an R-module, N a proper essential submodule and $S = M/N$. Then $M \oplus S$ is not $\mathcal{K}$-nonsingular.*

Proof. Let $u : M \oplus S \to M \oplus S$, defined by $u(x, y) = (0, \pi(x))$, where $\pi : M \to S$ is the canonical surjection. Then $\mathrm{Ker}(u) = N \oplus S$, is essential in $M \oplus S$. □

Proposition 4.2. *Let M be an R-module, such that every proper submodule of M is contained in a maximal submodule. If $M \oplus M/N$ is $\mathcal{K}$-nonsingular for every maximal submodule N, then M is semisimple.*

Proof. Suppose that M is not semisimple, then M contains a proper essential submodule which is contained in a maximal (essential) submodule N. Now, by Proposition 4.1., $M \oplus M/N$ is not $\mathcal{K}$-nonsingular. □.

Definition 4.3. Recall from [2], [5], that a ring R is said to be semilocal, if $R/J(R)$ is semisimple (artinian), where $J(R)$ is the Jacobson radical of R.

R is left (resp. right) perfect, if it is semilocal and every nonzero module has a maximal (resp. simple) submodule.

R is said to be perfect if it is left and right perfect. For example every left or right artinian ring is perfect.

A perfect ring R is called primary, if $R/J(R)$ is simple artinian. It is well known that a perfect ring is primary, if and only if, $R \cong \mathcal{M}_n(L)$, where L is a perfect local ring.

Finally, a perfect ring is called primary decomposable, if $R \cong \prod_{i=1}^{s} R_i$, where each R_i is primary.

The following theorem, gives a characterization of primary decomposable rings, using the direct sum of Baer or AI modules.

Theorem 4.4. *Let R be a perfect ring. Then the following assertions are equivalent:*

(i) *every K-nonsingular module is semi-simple.*

(ii) *every AI module is semisimple.*

(iii) *every Baer module is semisimple.*

(iv) *the direct sum of Baer modules is Baer.*

(v) *R is primary decomposable.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear.

$(iv) \Rightarrow (v)$. Suppose that (iv) holds, let M be an R-module such that $\mathrm{End}_R(M)$ is a division ring, then M is Baer. Since R is perfect, every proper submodule of M is contained in a maximal one. Now for every maximal submodule L, M/L is simple, thus Baer. By hypothesis, $M \oplus M/N$ is Baer, hence by Proposition 4.2., M is semisimple and then simple. It follows that R satisfies the converse of the Schur's Lemma. Then by [6], Theorem 1.2, R is primary decomposable.

$(v) \Rightarrow (i)$. Suppose that R is primary decomposable. Let M be $\mathcal{K}$-nonsingular. If M is non semisimple, then M contains a proper essential submodule N. Since R is primary decomposable perfect ring, there exists a nonzero endomorphism u of M such that $u(N) = 0$ (see[6]). This means that $\mathrm{Ker}(u)$ is essential. But M is $\mathcal{K}$-nonsingular. A contradiction. $\square$

Now we study the case when the base ring is left noetherian ring.

Theorem 4.5. *Let R be a left noetherian ring. Then the following assertions are equivalent:*

(i) *every $\mathcal{K}$-nonsingular module is semi-simple.*

(ii) *every AI module is semisimple.*

(iii) *every Baer module is semisimple.*

(iv) *the direct sum of Baer modules is Baer.*

(v) *R is artinian primary decomposable.*

Proof. It remains only to show that $(iv) \Rightarrow (v)$. Suppose first that R is semiprime left noetherian. Let Q the left total ring of fractions. Q is a semisimple ring. If e is any minimal idempotent of Q, then Re is an R-submodule of Q. Next we show that Re is Baer as R-module by showing that every endomorphism of Re is a monic. Let $u \in \mathrm{End}_R(Re)$ and $\lambda e \in \mathrm{Ker}(u)$. Take $v : Qe \to Qe$ defined by $v(xe) = xu(e)$. Then v is well defined and $v \in \mathrm{End}_Q(Qe)$ which is a division ring. We have $v(\lambda e) = \lambda u(e) = u(\lambda e) = 0$. Since v is an isomorphism, it follows that $\lambda e = 0$. Thus every $u \in \mathrm{End}_R(Re)$ is injective. Hence Re is a Baer module. On the other hand, Re is noetherian, consequentely it contains a maximal submodule N. By hypothesis, $Re \oplus Re/N$ is Baer. This implies that Re is semisimple. Now $R = Re_1 \oplus \ldots \oplus Re_n$, where $e_1, \ldots, e_n$ is a complete family of pairwise minimal orthogonal idempotents of Q. Since each Re_i is semisimple, it follows that R is semisimple.

Now let R be an arbitrary noetherian ring satisfying (iv). If P is a prime ideal of R, then clearly R/P satisfies the property (iv). Hence R/P is semisimple artinian. It follows that the Jacobson radical of R, $J(R)$, is contained in P. This implies that $J(R)$ is nil. But R is left noetherian, thus $J(R)$ is nilpotent. To conclude, R

is noetherian, $J(R)$ nilpotent and $R/J(R)$ semisimple, then R is artinian, and is primary decomposable by Theorem 4.4. □

In the case of commutative rings, we have a complete description, without any extra assumption on the ring R:

Theorem 4.6. *Let R be a commutative ring. Then the following are equivalent:*

(i) *Any direct sum of two Baer modules is Baer.*

(ii) *Every $\mathcal{K}$-nonsingular module is semisimple.*

(iii) *R is semilocal and $J(R)$ is a nilideal.*

Proof. (ii)⇒(i) is clear.

(i)⇒(iii). Suppose first that R is an integral domain. Let I be a maximal ideal. Then $R \oplus R/I$ is Baer. It follows that R is semisimple, hence R is a field. Now if R is arbitrary, then for every prime ideal P of R, R/P satisfies (i). Thus every prime ideal of R is maximal. This implies that $J(R)$ is nil and $T = R/J(R)$ is von Neumann regular. It remains to show that every VNR commutative ring T satisfying (i) is semisimple. Let $\hat{T}$ the injective envelope of T. Since $\hat{T}$ is nonsingular injective as T-module, it is a Baer T-module. On the other hand, by Kaplansky theorem (see [5]), and since T is a commutative von Neumann regular ring, then T is a V-ring (see Definition 5.1. below). Consequently, every T-module has a maximal submodule. Now if I is any maximal T-submodule of $\hat{T}$, then $\hat{T} \oplus \hat{T}/I$, is Baer. So by Proposition 4.2., $\hat{T}$ is semisimple as T-module, consequently, T is semisimple.

(iii)⇒(ii). Let M be $\mathcal{K}$-nonsingular. We shall show that $J(R)M = 0$. Let $a \in J(R)$. Consider $\lambda_a : M \to M$, $\lambda_a(x) = ax$. Since R is commutative, $\lambda_a \in \mathrm{End}_R(M)$. Suppose that $\lambda_a \neq 0$. If N is any nonzero submodule, take k the maximal integer such that $a^k N \neq 0$. Then $a^{k+1}N = 0$. This means that $\mathrm{Ker}(\lambda_a) \cap N \neq 0$. $\mathrm{Ker}(\lambda_a)$ is essential in M. A contradiction, since M is $\mathcal{K}$-nonsingular. Thus $\lambda_a = 0$. This implies $J(R)M = 0$. M is semisimple. □

5. When are injective modules quasi-Baer?

It has been shown in [10] Theorem 2.20, that if all injective R-modules are Baer, then R is semisimple artinian. This suggests the study of the same question replacing Baer by quasi-Baer. This will be the main objective of this section. We shall use extensively the following fact: Given any two R-modules M and N, then any $u \in \mathrm{End}_R(M \oplus N)$ can be considered under the form

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathrm{End}_R(M) & \mathrm{Hom}_R(N, M) \\ \mathrm{Hom}_R(M, N) & \mathrm{End}_R(N) \end{pmatrix}$$

Now, we recall some well known definitions.

Definition 5.1([5]). A ring R is called a QI-ring, see [5], if every quasi-injective

R-module is injective. Example of such ring was given by Cozzens. Any QI-ing is a V-ring, that is every simple R-module is injective, and is noetherian. It is not known if a QI-ring is hereditary (Boyle's conjecture).

Theorem 5.1. *Let R be a ring. The following assertions are equivalent:*

(i) *R is a QI-ring.*

(ii) *Every injective module is quasi-Baer.*

(iii) *For every injective R-module M and every fully invariant essential submodule N of M, $Ann_S(N) = 0$, where $S = End_R(M)$.*

Proof. (i)⇒(ii). Let R be a QI-ring, M an injective R-module and N a fully invariant submodule of M. Put $S = \mathrm{End}_R(M)$ and $I = \mathrm{Ann}_S(N)$. N is quasi-injective, hence injective since R is a QI-ring. It follows that N is a direct summand of M. Consequently, I is generated, as left ideal, by an idempotent.

(ii)⇒(iii) is clear.

(iii)⇒(i). First we show that R is a V-ring. Let H be a simple R-module, and $E(H)$ its injective envelope. Suppose that H is not injective, then $H \neq E(H)$. Let $x \in E(H)$ not in H and consider N a maximal submodule of M with respect to $x \notin N$. Then $E(H)/N$ is a uniform module with socle $H' = N + Rx/N$.

If $H' \cong H$, then there exists a nonzero morphism $f : H' \to E(H)$. This would imply the existence of a nonzero morphism $g : N + Rx \to E(H)$, such that $g(N) = 0$, and since $E(H)$ is injective, there exists a nonzero endomorphism u of $E(H)$, which extends g, and such that $v(N) = 0$. Hence $v(H) = 0$. A contradiction.

If $H' \not\cong H$, then every morphism of $E(H) \to E(H')$ is noninjective, otherwise $E(H)$ would be a nonzero direct summand of $E(H')$. Analogously every morphism $E(H') \to E(H)$ is non injective. Let $M = E(H) \oplus E(H')$ and $L = H \oplus H'$. Consider $\phi = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \mathrm{End}_R(M)$. By the fact that $\alpha(H) \subset H$ and $\delta(H') \subset H'$, $\beta(H') = 0$ and $\gamma(H) = 0$ (β and γ are not injective), we have $\phi(L) \subset L$. This means that L is a fully invariant submodule of M. On the other hand, Let $f = \imath \circ \pi$, where $\pi : E(H) \to E(H)/N$ is the canonical surjection, and $\imath : E(H)/N \to E(H')$ the canonical injection. If we put $\psi = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix} \right) \in \mathrm{End}_R(M)$, then $\psi(L) = 0$. A contradiction since L is essential in M.

In both cases, we obtain a contradiction. This implies that every simple module is injective so that R is a V-ring.

Now we show that R is Q.I. Let M be a quasi-injective R-module and P its injective hull. Suppose that M is not injective. Then $P \neq M$ and since R is a V-ring, there exists a maximal submodule T of P containing M. The factor module $H = P/T$ is simple and injective (again by the fact that R is a V-ring). If $\mathrm{Hom}_R(H, P) \neq 0$, then there exists a nonzero endomorphism u of P such that $u(T) = 0$, hence $u(M) = 0$ this leads to a contradiction, since M is fully invariant and essential in P. Thus $\mathrm{Hom}_R(H, M) = 0$. Now let $L = M \oplus H$, since $\mathrm{Hom}_R(H, M) = 0$ and M

is quasi-injective, then L is a fully invariant essential submodule of $P \oplus H$. If we put $\psi = \left(\begin{smallmatrix} 0 & 0 \\ \pi & 0 \end{smallmatrix} \right) \in \operatorname{End}_R(P \oplus H)$, where $\pi : P \to H$ is the canonical surjection, then $\psi(L) = 0$, a contradiction. □

Remark 5.2. As noted in [9], Theorem 4.1, if M is quasi-Baer, then $\operatorname{End}_R(M)$ is a quasi-Baer ring. The converse does not hold in general. If R is QI-ring, then for every injective R-module M, $\operatorname{End}_R(M)$ is a quasi-Baer ring. It is an interesting question to characterize rings over which every injective module has a quasi-Baer endomorphism ring.

In this direction, we note the following result concerning injective modules over hereditary noetherian rings:

Proposition 5.3. *Let M be an injective module over hereditary noetherian ring R. Then the ring $\operatorname{End}_R(M)$ is quasi-Baer.*

Proof. Let I be a two-sided ideal of $S = \operatorname{End}_R(M)$. Suppose that $\operatorname{Ann}_l I = \{u \in S : uI = 0\}$ is nonzero. Then $uI = 0$, for some nonzero $u \in S$. Consequently $I(M) \neq M$. But $I(M) = \sum_{h \in I} h(M)$. Since M is injective and $h(M)$ is isomorphic to a factor of M, then $h(M)$ is injective. Now over a hereditary noetherian ring the sum of injective submodules of an injective module is injective. This implies that $I(M)$ is injective. Consequently, $I(M)$ is a summand of M and $\operatorname{Ann}_S(I(M))$ is generated as left ideal by an idempotent p. Since $\operatorname{Ann}_S(I(M)) = \operatorname{Ann}_l I$, it follows that $\operatorname{Ann}_l I$ is generated by an idempotent. □

Corollary 5.4. *Let M be a divisible abelian group (a $\mathbb{Z}$-module), then the ring $\operatorname{End}_{\mathbb{Z}}(M)$ is quasi-Baer.*

Proof. This is clear since $\mathbb{Z}$ is hereditary noetherian and any divisible groupe is injective as $\mathbb{Z}$-module. □

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