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w-MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring and let M be a GV-torsionfree R-module. Then M is said to be a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$, and the w-envelope of M is defined by $M_w = \{x \in E(M) \mid Jx \subseteq M$ for some $J \in GV(R)\}$. In this paper, w-modules over commutative rings are considered, and the theory of w-operations is developed for arbitrary commutative rings. As applications, we give some characterizations of w-Noetherian rings and Krull rings.

0. Introduction

Let R be a domain with quotient field K, and let $F(R)$ be the set of nonzero fractional ideals of R. For $A \in F(R)$, set $A^{-1} = \{x \in K \mid xA \subseteq R\}$. Recall from [17] that for a domain R and a torsionfree R-module M, the w-envelope of M is defined by

$$M_w = \{x \in K \bigotimes_R M \mid Jx \subseteq M \text{ for some finitely generated ideal } J \text{ with } J^{-1} = R\}.$$

M is called a w-module if $M_w = M$, and M is said to be a w-ideal when M is an ideal of R with $M_w = M$. For $A \in F(R)$, the map $w : F(R) \to F(R)$, defined by $A \to A_w$, is a $*$-operation called the w-operation. One can see that the notion of a w-ideal coincides with the notion of a semi-divisorial ideal introduced by Glaz and Vasconcelos in 1977 [5] which may have some far reaching effects on the theory of $*$-operations. As a $*$-operation, the w-operation was briefly yet effectively touched on by Hedstrom and Houston in 1980 under the name of F_∞-operation [6]. Later, this $*$-operation was intensely studied by Wang and McCasland in a more general setting. In particular, Wang and McCasland showed that the w-envelope notion is a very useful tool in studying strong Mori domains [17, 18]. For the definition of a $*$-operation, the reader may consult [4].

There is a considerable amount of research devoted to extending multiplicative ideal theory to commutative rings containing zero divisors, see for example

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[7, 8, 10, 11, 12, 15]. Recently, the subject of the w-operation has generated considerable interest. For more information on the w-operation and strong Mori domains, the reader may consult Anderson and Cook [1], El Baghdadi and Gabelli [3], and Park [13, 14], etc. A natural problem is: how to extend the notion of w-modules to commutative rings with zero divisors. This is a motivation of our study. Our main purpose is to extend the notion of a w-module to commutative rings without any further regularity assumption. The methods employed in obtaining some results come from homological algebra, which are different from the methods used in the domain case. So we will see that the w-operation can bring in a lot of more homological algebra than the other $*$-operations on commutative rings. In addition, we prove enough results on w-modules which can be switched over to the w-envelopes of modules. As one might expect, some results on the w-operation on commutative rings coincide with the results obtained in the domain case. However, it is not to say that the proofs of some results are straightforward generalizations of the proofs for domains.

In Section 1, as a first step to the main goal, we introduce and study the concepts of GV-ideals and GV-torsionfree modules both of which constitute basic tools for subsequent considerations in this paper.

After preliminary studies of GV-ideals and GV-torsionfree modules, in Section 2, we devote to the study of w-modules over commutative rings. Let R be a commutative ring. A GV-torsionfree R-module M is said to be a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any GV-ideal J. We record some observations regarding w-modules.

In Section 3, we consider the w-envelope of a module. For a GV-torsionfree R-module M, the w-envelope of M is defined by $M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } GV\text{-ideal}\, J\}$, where $E(M)$ is the injective envelope of M. M is a w-module if and only if $M_w = M$. It will be seen later that the w-modules in the sense of [17, 18] are still w-modules. However, the notion of a w-module given in this article is more general. It is worth noting that different definitions of $*$-operations on arbitrary commutative rings appeared in the literatures [7], [8] and [15], but our "w-operation" satisfies all of them.

As applications, in Section 4, we give some new characterizations of Krull rings and display several w-Noetherian analogues of well-known results for Noetherian rings.

Throughout this paper, R will denote a commutative ring with identity $1 \neq 0$ and with total quotient ring $T(R)$. An element of R is regular if it is not a zero divisor. An ideal of R which contains a regular element is said to be a regular ideal.

1. *GV*-ideals and *GV*-torsionfree modules

Recall that if A, B, B_1 and C are R-modules, and $\alpha : B \to B_1$ is an R-homomorphism, then there exist induced R-homomorphisms $\alpha_* : \mathrm{Hom}_R(A, B)$

$\to \mathrm{Hom}_R(A, B_1)$ and $\alpha^* : \mathrm{Hom}_R(B_1, C) \to \mathrm{Hom}_R(B, C)$, which are defined by $\alpha_*(f) = \alpha f$ for all $f \in \mathrm{Hom}_R(A, B)$ and by $\alpha^*(g) = g\alpha$ for all $g \in \mathrm{Hom}_R(B_1, C)$, respectively.

For an R-module M, the dual module $\mathrm{Hom}_R(M, R)$ of M is denoted by M^*. There is a natural R-homomorphism φ from R into I^* given by $\varphi(r)(a) = ra$ for all $r \in R$ and $a \in I$, where I is an ideal of R. It is obvious that $R \overset{\varphi}{\cong} I^*$ if and only if $\mathrm{Hom}_R(R/I, R) = 0$ and $\mathrm{Ext}^1_R(R/I, R) = 0$.

Definition 1.1. An ideal J of a commutative ring R is called a Glaz-Vasconcelos ideal or a GV-ideal, denoted by $J \in GV(R)$, if J is finitely generated and the natural homomorphism $\varphi : R \to J^*$ is an isomorphism.

Proposition 1.2. *Let R be a commutative ring.*

(1) *$R \in GV(R)$.*

(2) *Let J_1 and J_2 be finitely generated ideals of R, and $J_1 \subseteq J_2$. If $J_1 \in GV(R)$, then $J_2 \in GV(R)$.*

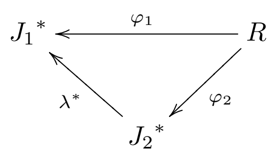
(3) *Let J_1 and J_2 be GV-ideals of R. Then $J_1 J_2 \in GV(R)$.*

(4) *If $J \in GV(R)$, then $J[X] \in GV(R[X])$.*

(5) *Let J_1, J_2 be ideals of commutative rings R_1, R_2, respectively. Assume that $R = R_1 \times R_2$. Then $J = J_1 \times J_2 \in GV(R)$ if and only if $J_i \in GV(R_i)$ for $i = 1, 2$.*

Proof. (1) is clear.

(2) It is easy to verify that the following diagram



is commutative, where φ_1 and φ_2 are defined as in the beginning of this section, and λ^* is induced by the inclusive map $\lambda : J_1 \to J_2$. To show that $J_2 \in GV(R)$, it suffices to prove λ^* is an isomorphism.

Here we consider an exact sequence $0 \to \mathrm{Hom}_R(J_2/J_1, R) \to \mathrm{Hom}_R(J_2, R) \overset{\lambda^*}{\to} \mathrm{Hom}_R(J_1, R)$. To conclude the proof, we only need to show that $\mathrm{Hom}_R(J_2/J_1, R) = 0$. Let $f \in \mathrm{Hom}_R(J_2/J_1, R)$ and $b \in J_2$. Then we have $af(\bar{b}) = f(a\bar{b}) = 0$ for any $a \in J_1$, where $\bar{b} = b + J_1$. Hence $J_1 \subseteq \mathrm{ann}(f(\bar{b}))$ (the annihilator of $f(\bar{b})$ in R). Define $g : R/J_1 \to R$ as follows: $g(\bar{r}) = rf(\bar{b})$, where $\bar{r} = r + J_1$. Clearly, g is a well-defined R-homomorphism. Since $\mathrm{Hom}_R(R/J_1, R) = 0$, $f(\bar{b}) = 0$, and so $f = 0$.

(3) By [16, Theorem 2.11], we have

$$\mathrm{Hom}_R(J_1 \bigotimes_R J_2, R) \cong \mathrm{Hom}_R(J_1, \mathrm{Hom}_R(J_2, R)) \cong \mathrm{Hom}_R(J_1, R) \cong R.$$

The epimorphism σ: $J_1 \bigotimes_R J_2 \to J_1 J_2$ induces a monomorphism

$$\sigma^* : \mathrm{Hom}_R(J_1J_2, R) \to \mathrm{Hom}_R(J_1 \bigotimes_R J_2, R),$$

where σ is defined by $\sigma(a \otimes b) = ab$ for all $a \in J_1$ and $b \in J_2$. Since the composite $R \overset{\varphi}{\to} \mathrm{Hom}_R(J_1J_2, R) \overset{\sigma^*}{\to} \mathrm{Hom}_R(J_1 \bigotimes_R J_2, R)$ is an isomorphism with φ defined as in the beginning of this section, σ^* is onto. It follows that φ is an isomorphism.

(4) For an R-module A, set $A[X] = A \bigotimes_R R[X]$. We have a canonical R-homomorphism

$$\theta_A : R[X] \bigotimes_R \mathrm{Hom}_R(A, R) \to \mathrm{Hom}_{R[X]}(A[X], R[X]),$$

which is defined by

$$\theta_A(f \otimes g)(\sum_{i=1}^{n} a_i \otimes f_i) = \sum_{i=1}^{n} g(a_i) f f_i,$$

where $f, f_i \in R[X]$, $a_i \in A$, and $g \in \mathrm{Hom}_R(A, R)$. It is easy to see that θ_A is monic, and θ_A is an isomorphism when A is a finitely generated free R-module.

Let $0 \to N \to F \to J \to 0$ be a short exact sequence, where F is a finitely generated free R-module. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & R[X] \bigotimes_R \mathrm{Hom}_R(J, R) & \longrightarrow & R[X] \bigotimes_R \mathrm{Hom}_R(F, R) & \longrightarrow & R[X] \bigotimes_R \mathrm{Hom}_R(N, R) \\
& \downarrow{\scriptstyle \theta_J} & & \downarrow{\scriptstyle \theta_F} & & \downarrow{\scriptstyle \theta_N} \\
0 \longrightarrow & \mathrm{Hom}_{R[X]}(J[X], R[X]) & \longrightarrow & \mathrm{Hom}_{R[X]}(F[X], R[X]) & \longrightarrow & \mathrm{Hom}_{R[X]}(N[X], R[X]).
\end{array}$$

Note that θ_F is an isomorphism, and θ_N, θ_J are monomorphisms. By diagram chasing, we have θ_J is epic and so is an isomorphism. Thus, $J[X] \in GV(R[X])$.

(5) Note that

$$\begin{array}{ccc}
R & = & R_1 \times R_2 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_R(J, R) & = & \mathrm{Hom}_{R_1}(J_1, R_1) \times \mathrm{Hom}_{R_2}(J_2, R_2)
\end{array}$$

is a commutative diagram, and so (5) holds. □

Definition 1.3. An R-module M is called a GV-torsionfree module if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, then $x = 0$.

The following theorem shows that a GV-torsionfree module has some homological properties, and provides a justification for the terminology.

Theorem 1.4. *For an R-module M, the following are equivalent*:

(1) *M is GV-torsionfree.*

(2) $\mathrm{Hom}_R(N, M) = 0$ *for any $J \in GV(R)$ and R/J-module N.*

(3) $\mathrm{Hom}_R(R/J, M) = 0$ *for any $J \in GV(R)$.*

Proof. (1) $\Rightarrow$ (2). Let $f \in \mathrm{Hom}_R(N, M)$. For any $x \in N$, $Jf(x) = f(Jx) = f(0) = 0$. It follows that $f(x) = 0$.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Let $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, and suppose $x \neq 0$. Define $g : R/J \to M$ by $g(\bar{r}) = rx$ for all $r \in R$, where $\bar{r} = r + J$. It is easy to verify that g is well-defined, and $g \neq 0$, a contradiction. □

Corollary 1.5. *R is a GV-torsionfree R-module.*

Corollary 1.6. *Let M be a GV-torsionfree R-module and F a flat R-module. Then $F \bigotimes_R M$ is GV-torsionfree. In particular, a flat R-module is GV-torsionfree, and so $T(R)$ is a GV-torsionfree R-module.*

Proof. For any $J \in GV(R)$, $\mathrm{Hom}_R(R/J, F \bigotimes_R M) \cong F \bigotimes_R \mathrm{Hom}_R(R/J, M)$ by [16, Lemma 3.83]. □

We use $E(M)$ to denote the injective envelope of an R-module M.

Proposition 1.7. (1) *Let M be a GV-torsionfree R-module with a submodule N. Then N is also GV-torsionfree.*

(2) *Let $\{M_i \mid i \in \Gamma\}$ be a family of GV-torsionfree R-modules. Then both $\prod_{i \in \Gamma} M_i$ and $\bigoplus_{i \in \Gamma} M_i$ are GV-torsionfree.*

(3) *Let M be an R-module and N a GV-torsionfree R-module. Then* Hom_R (M, N) *is a GV-torsionfree R-module. In particular, M^* and M^{**} are GV-torsionfree R-modules. Therefore, reflexive modules are GV-torsionfree.*

(4) *If M is a GV-torsionfree R-module, then so is $E(M)$.*

Proof. (1) and (2) are clear.

(3) Let $Jf = 0$ for some $J \in GV(R)$ and $f \in \mathrm{Hom}_R(M, N)$. For each $x \in M$, we have $Jf(x) = 0$. Since N is GV-torsionfree, $f(x) = 0$. The "In particular" statement comes from Corollary 1.5.

(4) Let $Jx = 0$ for some $J \in GV(R)$ and $x \in E(M)$, and suppose $x \neq 0$. Then there exists $r \in R$ such that $rx \neq 0$ and $rx \in M$. But we have $Jrx = 0$, and so $rx = 0$. This contradiction shows that $x = 0$. □

2. w-modules over commutative rings

We now introduce the notion of a w-module which comes from homological algebra.

Definition 2.1. A GV-torsionfree R-module M is said to be a w-module if, for any $J \in GV(R)$, $\mathrm{Ext}^1_R(R/J, M) = 0$.

It is clear that R is a w-module, and that, for a GV-torsionfree R-module M, $E(M)$ is a w-module.

Theorem 2.2. *Let M be a GV-torsionfree R-module. Then the following are equivalent*:

(1) *M is a w-module.*

(2) *Every R-homomorphism $f : J \to M$, where $J \in GV(R)$, can be extended to R.*

(3) *If $Jx \subseteq M$, where $J \in GV(R)$ and $x \in E(M)$, then $x \in M$.*

Proof. (1) $\Leftrightarrow$ (2). For each $J \in GV(R)$, we have an exact sequence

$$0 \to \mathrm{Hom}_R(R/J, M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(J, M) \to \mathrm{Ext}^1_R(R/J, M) \to 0.$$

The equivalence now follows from this consideration.

(2) $\Rightarrow$ (3). Suppose $Jx \subseteq M$ for some $J \in GV(R)$ and $x \in E(M)$. Define $f : J \to M$ by $f(r) = rx$ for all $r \in J$. It is easily seen that f is well-defined. Note that f can be extended to $g : R \to M$. Then $Jx = f(J) = g(J) = Jg(1)$. Since M is GV-torsionfree, so is $E(M)$ and hence $x = g(1) \in M$.

(3) $\Rightarrow$ (2). For each $f : J \to M$, where $J \in GV(R)$, there exists $g : R \to E(M)$ such that the diagram

$$\begin{array}{ccccc} & & M & \longrightarrow & E(M) \\ & & {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ 0 & \longrightarrow & J & \longrightarrow & R \end{array}$$

is commutative. Then $Jg(1) = g(J) = f(J) \subseteq M$. Thus $g(1) \in M$, as desired. $\square$

Proposition 2.3. *Let $\{M_i \mid i \in \Gamma\}$ be a family of GV-torsionfree R-modules. Then the following are equivalent:*

(1) *M_i is a w-module for each $i \in \Gamma$.*

(2) *$\prod_{i\in\Gamma} M_i$ is a w-module.*

(3) *$\bigoplus_{i\in\Gamma} M_i$ is a w-module.*

Proof. (1) $\Leftrightarrow$ (2). $\mathrm{Ext}^1_R(R/J, \prod_{i\in\Gamma} M_i) \cong \prod_{i\in\Gamma} \mathrm{Ext}^1_R(R/J, M_i)$ for any $J \in GV(R)$.

(1) $\Leftrightarrow$ (3). By [2, Exercise 16.3], we have the following commutative diagram:

$$\begin{array}{ccccccc} \bigoplus_{i\in\Gamma} \mathrm{Hom}_R(R, M_i) & \longrightarrow & \bigoplus_{i\in\Gamma} \mathrm{Hom}_R(J, M_i) & \longrightarrow & \bigoplus_{i\in\Gamma} \mathrm{Ext}^1_R(R/J, M_i) & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow\cong & & \downarrow\theta & & \\ \mathrm{Hom}_R(R, \bigoplus_{i\in\Gamma} M_i) & \longrightarrow & \mathrm{Hom}_R(J, \bigoplus_{i\in\Gamma} M_i) & \longrightarrow & \mathrm{Ext}^1_R(R/J, \bigoplus_{i\in\Gamma} M_i) & \longrightarrow & 0. \end{array}$$

By the Five Lemma, θ is an isomorphism. $\square$

As an immediate consequence of the above proposition, we have

Corollary 2.4. *Every projective module is a w-module.*

Proposition 2.5. *As an R-module, $T(R)$ is a w-module.*

Proof. By Theorem 2.2, we only need to show that if $Jx \subseteq T(R)$ for some $J \in GV(R)$ and $x \in E(T(R))$, then $x \in T(R)$. Since J is finitely generated, there exists a regular element s of R such that $Jsx \subseteq R$. Let $E(T(R)) = E(R) \bigoplus N$ for some R-module N. Set $x = y + z$, where $y \in E(R)$ and $z \in N$. Then we have $Jsz = Js(x - y) \subseteq N \bigcap E(R) = 0$. Since N is GV-torsionfree, $sz = 0$, and so $sx = sy \in E(R)$. Again by Theorem 2.2, $sx \in R$. Thus $x \in T(R)$. $\square$

Proposition 2.6. *Let M be a GV-torsionfree R-module, and let $\{M_i \mid i \in \Gamma\}$ be a directed family of w-submodules of M. Then $\bigcup_{i\in\Gamma} M_i$ is also a w-submodule of M.*

Proof. Since $\bigcup_{i\in\Gamma} M_i$ is a submodule of M, it is GV-torsionfree. Let $Jx \subseteq \bigcup_{i\in\Gamma} M_i$ for some $J \in GV(R)$ and $x \in E(\bigcup_{i\in\Gamma} M_i) \subseteq E(M)$. Since J is finitely generated, there exists $i \in \Gamma$ such that $Jx \subseteq M_i$. Let $E(M) = E(M_i) \bigoplus N$ for some R-module N. Set $x = y + z$, where $y \in E(M_i)$ and $z \in N$. Then we have $Jz = J(x - y) \subseteq N \bigcap E(M_i) = 0$, and so $z = 0$. Thus $x = y \in E(M_i)$. By Theorem 2.2, $x \in M_i$. Therefore, $\bigcup_{i\in\Gamma} M_i$ is a w-submodule of M. $\square$

Theorem 2.7. *Let M be a GV-torsionfree R-module. Then the following are equivalent*:

(1) *M is a w-module.*

(2) *If $0 \to M \to F \to N \to 0$ is an R-exact sequence, where F is a w-module, then N is a GV-torsionfree R-module.*

(3) *There exists an R-exact sequence $0 \to M \to F \to N \to 0$ such that F is a w-module and N is a GV-torsionfree R-module.*

Proof. (1) $\Rightarrow$ (2). For each $J \in GV(R)$, we have exactness of $\mathrm{Hom}_R(R/J, F) \to \mathrm{Hom}_R(R/J, N) \to \mathrm{Ext}^1_R(R/J, M)$. By Theorem 1.4 and Definition 2.1, we have $\mathrm{Hom}_R(R/J, F) = \mathrm{Ext}^1_R(R/J, M) = 0$, and so $\mathrm{Hom}_R(R/J, N) = 0$, as desired.

(2) $\Rightarrow$ (3). Choose an R-exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$.

(3) $\Rightarrow$ (1). For each $J \in GV(R)$, there exists an exact sequence $\mathrm{Hom}_R(R/J, N) \to \mathrm{Ext}^1_R(R/J, M) \to \mathrm{Ext}^1_R(R/J, F)$. Again by Theorem 1.4 and Definition 2.1, we have $\mathrm{Hom}_R(R/J, N) = \mathrm{Ext}^1_R(R/J, F) = 0$, and so $\mathrm{Ext}^1_R(R/J, M) = 0$. Then (1) holds. $\square$

Theorem 2.8. *Let A be an R-module and M a w-module. Then $\mathrm{Hom}_R(A, M)$ is a w-module. In particular, A^* and A^{**} are w-modules. Therefore, reflexive modules are w-modules.*

Proof. Let $F = \bigoplus R$ be a free R-module. Since $\mathrm{Hom}_R(F, M) \cong \prod \mathrm{Hom}_R(R, M) \cong \prod M$, $\mathrm{Hom}_R(F, M)$ is a w-module by Proposition 2.3. Let $0 \to B \to F \to A \to 0$ be an exact sequence with F free. Then there exists an exact sequence $0 \to \mathrm{Hom}_R(A, M) \to \mathrm{Hom}_R(F, M) \to X \to 0$, where X is a submodule of $\mathrm{Hom}_R(B, M)$. By Proposition 1.7 and Theorem 2.7, X is GV-torsionfree and so $\mathrm{Hom}_R(A, M)$ is a w-module. $\square$

3. The w-operation on commutative rings

We start with a study of the w-envelope of a GV-torsionfree module over a commutative ring R.

Definition 3.1. Let M be a GV-torsionfree R-module. Then the w-envelope of M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}.$$

By Theorem 2.2, we have M is a w-module if and only if $M_w = M$. So M is a w-ideal when M is an ideal of R with $M_w = M$. It is easy to see that M_w is a w-module and $0_w = 0$.

Proposition 3.2. *Let M be a GV-torsionfree R-module with submodules A and B. Then the following hold*:
(1) $cA_w \subseteq (cA)_w$ *for all* $c \in R$.
(2) $A \subseteq A_w$, *and* $A \subseteq B \Rightarrow A_w \subseteq B_w$.
(3) $(A_w)_w = A_w$.

Proof. (1) Let $x \in A_w$. Then $Jx \subseteq A$ for some $J \in GV(R)$, and so $Jcx \subseteq cA$. Let $E(A) = E(cA) \bigoplus N$ for some R-module N. Set $x = y+z$, where $y \in E(cA)$ and $z \in N$. Then we have $Jcz = Jc(x-y) \subseteq N \bigcap E(cA) = 0$, and so $cz = 0$. Thus $cx = cy \in E(cA)$. Therefore, $cx \in (cA)_w$.
(2) and (3) are straightforward. □

Recall that an element $a \in R$ is called a zero divisor for an R-module M if there exists $x \in M \backslash \{0\}$ such that $ax = 0$. a is regular if it is not a zero divisor.

Corollary 3.3. (1) *Let A be a GV-torsionfree R-module and $c \in R$. If c is a regular element for A, then $cA_w = (cA)_w$. In particular, if c is a regular element of R, then $(c)_w = (c)$.*
(2) *If $c \in T(R)$ and A is an R-submodule of $T(R)$, then $cA_w \subseteq (cA)_w$.*

Proof. (1) Clearly, c is also a regular element for A_w. Thus we have $cA_w \cong A_w$, and so cA_w is a w-module. Since $cA \subseteq cA_w$, $(cA)_w \subseteq (cA_w)_w = cA_w$.
(2) Set $c = \frac{r}{s}$, where $r, s \in R$, and s is a regular element of R. By Proposition 3.2, we have $rA_w \subseteq (rA)_w$. By (1), $s(\frac{r}{s}A)_w = (rA)_w$. Hence $cA_w = \frac{r}{s}A_w \subseteq \frac{1}{s}(rA)_w = (\frac{r}{s}A)_w = (cA)_w$. □

Remark 3.4. For a domain R with quotient field K, it is routine to verify that a torsionfree R-module is GV-torsionfree, and that $K \bigotimes_R M = E(M)$ for a torsionfree R-module M. Therefore, the w-modules in the sense of [17, 18] are also w-modules in the sense of Definition 2.1 but the converse does not hold in general. In fact, let R be a domain, and $a \in R \backslash \{0\}$. It is clear that $R/(a)$ is not a torsionfree R-module, but it is a GV-torsionfree R-module by Theorem 2.7 and Corollary 3.3. Therefore, we have $E(R/(a))$ is a w-module in this article.

Proposition 3.5. *Let J be a finitely generated ideal of R. Then $J \in GV(R)$ if and only if $J_w = R$.*

Proof. "Only if" part. By Proposition 3.2, we have $J_w \subseteq R_w = R$. On the other hand, it is clear that $J1 \subseteq J$. Let $E(R) = E(J) \bigoplus N$ for some R-module N. Set $1 = x + y$, where $x \in E(J)$ and $y \in N$. Then we have $Jy = J(1-x) \subseteq N \bigcap E(J) = 0$, and so $y = 0$. Thus $1 = x \in E(J)$. It follows that $1 \in J_w$.

"If" part. There exists $J_1 \in GV(R)$ such that $J_1 \subseteq J$. By Proposition 1.2, we have $J \in GV(R)$. □

The next theorem gives necessary and sufficient conditions for a GV-torsion-free module to be a w-module.

Theorem 3.6. *Let M be a GV-torsionfree R-module. Then the following are equivalent*:

(1) *M is a w-module.*

(2) $\mathrm{Ext}^1_R(N, M) = 0$ *for any $J \in GV(R)$ and R/J-module N.*

(3) $\mathrm{Ext}^1_R(A_w/A, M) = 0$ *for any GV-torsionfree R-module A.*

(4) *Every R-homomorphism $f : A \to M$, where A is GV-torsionfree, can be extended to A_w.*

Proof. (1) ⇒ (2). Let $F = \bigoplus R/J$ be a free R/J-module for $J \in GV(R)$. Then we have $\mathrm{Ext}^1_R(F, M) \cong \prod \mathrm{Ext}^1_R(R/J, M) = 0$.

Let $0 \to A \to F \to N \to 0$ be an R/J-exact sequence, where F is a free R/J-module. Then there exists an exact sequence $\mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(N, M) \to \mathrm{Ext}^1_R(F, M) = 0$. By Theorem 1.4, $\mathrm{Hom}_R(A, M) = 0$. Thus $\mathrm{Ext}^1_R(N, M) = 0$.

(2) ⇒ (1). Trivial.

(1) ⇒ (3). Let A_w/A be generated by the set $\{\bar{x}_i \mid i \in \Gamma\}$, where $\{x_i \mid i \in \Gamma\} \subseteq A_w$. Then there exists $J_i \in GV(R)$ such that $J_i x_i \subseteq A$ for each $i \in \Gamma$, and thus we have an epimorphism $\bigoplus_{i\in\Gamma} R/J_i \to A_w/A$. Let N be the kernel of this homomorphism. Then $\mathrm{Hom}_R(N, M) = 0$. In fact, suppose $f \in \mathrm{Hom}_R(N, M)$. For any $x \in N$, there is $J \in GV(R)$ such that $Jx = 0$. Hence $Jf(x) = f(Jx) = 0$. Since M is GV-torsionfree, we have $f(x) = 0$. Thus there exists an exact sequence

$$0 = \mathrm{Hom}_R(N, M) \to \mathrm{Ext}^1_R(A_w/A, M) \to \mathrm{Ext}^1_R(\bigoplus_{i\in\Gamma} R/J_i, M).$$

Since $\mathrm{Ext}^1_R(\bigoplus_{i\in\Gamma} R/J_i, M) \cong \prod_{i\in\Gamma} \mathrm{Ext}^1_R(R/J_i, M) = 0$, $\mathrm{Ext}^1_R(A_w/A, M) = 0$.

(3) ⇒ (4). It follows from the fact that

$$\mathrm{Hom}_R(A_w, M) \to \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(A_w/A, M)$$

is an exact sequence.

(4) ⇒ (1). By Theorem 2.2 and Proposition 3.5. □

Proposition 3.7. *Let M be a GV-torsionfree R-module. Then*

$M_w = \bigcup\{N_w | N$ *runs over all finitely generated R-submodules contained in M*$\}$.

Proof. Clearly, $\bigcup N_w \subseteq M_w$. Conversely, suppose $x \in M_w$. Then $Jx \subseteq M$ for some $J \in GV(R)$. Set $N = Jx$. Then we have $Jx \subseteq N$, and so $x \in N_w$. □

As the maximal submodules being prime, we have maximal w-submodules are prime. In this paper, we denote by w-max(R) the set of maximal w-ideals of R. Let M be an R-module with submodules A and B. Set $(A : B) = \{r \in R \,|\, rB \subseteq A\}$.

Proposition 3.8. *Let M be a w-module, and let A be a submodule of M which is maximal in the collection of proper w-submodules of M. Then A is prime. Therefore, a maximal w-ideal is prime.*

Proof. Let $rx \in A$ for some $r \in R$ and $x \in M$, and suppose $x \notin A$. Then $(A + Rx)_w = M$. By Proposition 3.2(1), $rM = r(A + Rx)_w \subseteq (rA + Rrx)_w \subseteq A_w = A$. Thus $r \in (A : M)$. □

We say that a GV-torsionfree module M is w-finite (or of finite type, when no confusion is likely) if $M_w = N_w$ for some finitely generated submodule N of M. By Proposition 2.6, it is easy to show that if M is a w-module of finite type, then any proper w-submodule of M is contained in a maximal w-submodule.

Theorem 3.9. *Let M be a GV-torsionfree R-module. Then $(M_w)_\mathfrak{p} = M_\mathfrak{p}$ for each prime w-ideal $\mathfrak{p}$ of R. Therefore, if M is w-finite, then $M_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for each prime w-ideal $\mathfrak{p}$ of R.*

Proof. Obviously, $M_\mathfrak{p} \subseteq (M_w)_\mathfrak{p}$. Conversely, let $x \in (M_w)_\mathfrak{p}$. Then there exists $s \in R\backslash\mathfrak{p}$ with $sx \in M_w$. Thus $Jsx \subseteq M$ for some $J \in GV(R)$. By Proposition 3.5, we have $J \not\subseteq \mathfrak{p}$. It follows that $J_\mathfrak{p} = R_\mathfrak{p}$, and so $sx \in J_\mathfrak{p}sx \subseteq M_\mathfrak{p}$. Hence $x \in M_\mathfrak{p}$, and then $(M_w)_\mathfrak{p} \subseteq M_\mathfrak{p}$. □

Corollary 3.10. *Let M be a GV-torsionfree R-module with submodules A and B. Then $A_w = B_w$ if and only if $A_\mathfrak{m} = B_\mathfrak{m}$ for any $\mathfrak{m} \in w\text{-}\max(R)$.*

Proof. Let $w\text{-}\max(R) = \emptyset$, and suppose that c is a regular element of R. Then $(c) = (c)_w = R$, and so c is a unit. Therefore $R = T(R)$. Here we consider the case $w\text{-}\max(R) \neq \emptyset$.

"Only if" part is clear by Theorem 3.9.

"If" part. Suppose $x \in A_w$. Set $I = (B_w : Rx)$. Then I is a w-ideal of R. By Theorem 3.9, we have $(B_w)_\mathfrak{m} = B_\mathfrak{m} = A_\mathfrak{m} = (A_w)_\mathfrak{m}$. It follows that

$$I_\mathfrak{m} = \{\, a \in R_\mathfrak{m} \,|\, a\frac{x}{1} \in (B_w)_\mathfrak{m}\} = \{a \in R_\mathfrak{m} \,|\, a\frac{x}{1} \in (A_w)_\mathfrak{m}\,\} = R_\mathfrak{m}.$$

Thus $I \not\subseteq \mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$, and so $I = R$. Therefore, $x \in B_w$. It follows that $A_w \subseteq B_w$. The inverse can be proved similarly. □

Before moving to another topic, we should note that different definitions of $*$-operation on arbitrary commutative rings appeared in the literatures [7], [8] and [15], but our "w-operation" satisfies all of them.

Let $\mathfrak{F}(R)$ be the set of R-submodules of $T(R)$. For $A \in \mathfrak{F}(R)$, set $A^{-1} = \{x \in T(R) \,|\, xA \subseteq R\}$, $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup B_v$, where B runs over all finitely generated R-submodules of A. It is easy to see that for an ideal I of R, $I^{-1} \cong I^*$ if I contains a regular element. Thus a finitely generated regular ideal J of R is a GV-ideal if and only if $J^{-1} = R$. A star operation $*$ on R is a mapping $A \to A_*$ from $\mathfrak{F}(R)$ to $\mathfrak{F}(R)$ which satisfies the following conditions for all $c \in T(R)$ and $A, B \in \mathfrak{F}(R)$:

(1) $cA_* \subseteq (cA)_*$.
(2) $A \subseteq A_*$, and $A \subseteq B$ implies $A_* \subseteq B_*$.
(3) $(A_*)_* = A_*$.
(4) $R_* = R$.

It is routine to see that the v-operation and the t-operation on R are $*$-operations. An $A \in \mathfrak{F}(R)$ is called a $*$-module if $A_* = A$, and is called a $*$-ideal when A is an ideal of R with $A_* = A$. A star operation $*$ is said to have finite character if for any $A \in \mathfrak{F}(R)$,

$$A_* = \bigcup\{B_* \mid B \text{ runs over all finitely generated } R\text{-submodules contained in } A\}.$$

We define the w-operation by $A \to A_w$ for all $A \in \mathfrak{F}(R)$. Then, by Proposition 3.7, the w-operation on R has finite character.

Proposition 3.11. *Let A and B be R-submodules of $T(R)$, and let $\{B_i\}$ be a family of R-submodules of $T(R)$. Then the following hold*:

(1) $(\sum_i B_i)_* = (\sum_i (B_i)_*)_*$.
(2) $\bigcap_i (B_i)_* = (\bigcap_i (B_i)_*)_*$.
(3) $(AB)_* = (A_*B)_* = (A_*B_*)_*$.
(4) $(A^{-1})_* = A^{-1}$.
(5) $(A_*)^{-1} = A^{-1}$. *Therefore, if $A_* = B_*$, then $A^{-1} = B^{-1}$.*
(6) $A_* \subseteq A_v$. *Therefore, $A_* \subseteq A_t$ provided that $*$ has finite character.*

Proof. The proofs of all parts are straightforward. □

For $A \in \mathfrak{F}(R)$, we say that A is $*$-invertible if $(AA^{-1})_* = R$. If A is w-invertible, then A is w-finite, and $A_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for any $\mathfrak{m} \in w\text{-}\max(R)$. In the domain case, Anderson and Cook [1] and Park [13] have independently shown that a nonzero ideal of R is t-invertible if and only if it is w-invertible, and that an ideal of R is a maximal t-ideal if and only if it is a maximal w-ideal. As in the domain case, there are also nice relations between the t-operation and the w-operation on arbitrary commutative rings, which will be useful in our further study.

Theorem 3.12. *Let A be a regular ideal of R. Then $A_t = R$ if and only if $A_w = R$.*

Proof. If $A_w = R$, then $A_t = R$ because of $A_w \subseteq A_t$. Conversely, suppose $A_t = R$. Then there exists a finitely generated subideal B of A such that $B_v = R$. Without loss of generality, we can assume that B is regular. In this case, $B \in GV(R)$. Consequently, $R = B_w \subseteq A_w \subseteq R$ implies $A_w = R$. □

Corollary 3.13. *Let A be a regular ideal of R. Then A is t-invertible if and only if A is w-invertible.*

Proof. By Theorem 3.12, $(AA^{-1})_t = R$ if and only if $(AA^{-1})_w = R$. □

Corollary 3.14. *Let $\mathfrak{m}$ be a regular ideal of R. Then $\mathfrak{m}$ is a maximal t-ideal if and only if $\mathfrak{m}$ is a maximal w-ideal.*

Proof. "Only if" part. Suppose that I is a w-ideal of R properly containing $\mathfrak{m}$. Then $I_t = R$, and so $I_w = R$. Thus $\mathfrak{m}$ is a maximal w-ideal.

"If" part. Clearly, $\mathfrak{m}_t \neq R$. Since $\mathfrak{m}_t$ is a w-ideal, $\mathfrak{m}_t = \mathfrak{m}$. Thus $\mathfrak{m}$ is a maximal t-ideal. □

Proposition 3.15. *Let $\mathfrak{p}$ be a prime ideal of R. Then either $\mathfrak{p}_w = \mathfrak{p}$ or $\mathfrak{p}_w = R$.*

Proof. Suppose $\mathfrak{p}_w \neq R$. For $x \in \mathfrak{p}_w$, there exists $J \in GV(R)$ such that $Jx \subseteq \mathfrak{p}$. Since $J \nsubseteq \mathfrak{p}$, $x \in \mathfrak{p}$. Therefore, $\mathfrak{p}_w = \mathfrak{p}$. □

Proposition 3.16. *Let $\mathfrak{p}$ be a w-invertible regular prime w-ideal of R. Then $\mathfrak{p}$ is a maximal w-ideal.*

Proof. Suppose that I is an ideal of R properly containing $\mathfrak{p}$. Choose $c \in I \backslash \mathfrak{p}$, and let $\mathfrak{p} = B_w$, where B is a finitely generated subideal of $\mathfrak{p}$. Without loss of generality, we may assume that B is regular. Set $J = (B, c)$. For $x \in J^{-1}$, we have $xcB \subseteq B \subseteq \mathfrak{p}$. Since $c \notin \mathfrak{p}$, $xB \subseteq \mathfrak{p}$. Then $x\mathfrak{p} = xB_w \subseteq (xB)_w \subseteq \mathfrak{p}$. So $x\mathfrak{p}\mathfrak{p}^{-1} \subseteq \mathfrak{p}\mathfrak{p}^{-1}$. Since $(\mathfrak{p}\mathfrak{p}^{-1})_w = R$, $x \in R$. Thus $J^{-1} = R$, and so $J \in GV(R)$. Since $J \subseteq I$, $I_w = R$, as required. □

4. Characterizations of w-Noetherian rings and Krull rings

In this section, we will give some new characterizations of Krull rings through the w-operation and display several w-Noetherian analogues of well-known results for Noetherian rings. But first we have to look at the w-Noetherian ring which is an extension of the notion of a strong Mori domain introduced by Wang and McCasland (see [17, 18]).

Definition 4.1. A w-module M is called a w-Noetherian module if M satisfies the ACC on its w-submodules. R is said to be a w-Noetherian ring if R is a w-Noetherian module.

By the above definition, it is clear that every w-submodule of a w-Noetherian module is a w-Noetherian module. The proofs of the next two results are routine, therefore they will be omitted.

Proposition 4.2. *For a w-module M, the following are equivalent*:
(1) *M is a w-Noetherian module.*
(2) *Every w-submodule of M is of finite type.*
(3) *Every non-empty set of w-submodules of M has a maximal element.*
(4) *Every submodule of M is of finite type.*

When $M = R$, we have

Proposition 4.3. *For a commutative ring R, the following are equivalent*:
(1) *R is a w-Noetherian ring.*
(2) *Every w-ideal of R is of finite type.*

(3) *Every non-empty set of w-ideals of R has a maximal element.*
(4) *Every ideal of R is of finite type.*

Corollary 4.4. *If R is a w-Noetherian ring, then $R_{\mathfrak{p}}$ is Noetherian for each prime w-ideal $\mathfrak{p}$ of R.*

Proof. It follows from Proposition 4.3 and Theorem 3.9. □

It is easy to verify that, for any two submodules A and B of a GV-torsionfree R-module M, $(A+B)_w = (A_w + B_w)_w$. Here we have:

Proposition 4.5. *Let $M_1, M_2, \dots, M_n$ be w-modules. Then $\bigoplus_{i=1}^n M_i$ is a w-Noetherian module if and only if M_i is a w-Noetherian module for each $1 \leqslant i \leqslant n$.*

Proof. "Only if" part is trivial.

"If" part. It suffices to prove the case $n = 2$. Let $M = M_1 \bigoplus M_2$, and let N be a w-submodule of M. Set $B = N \bigcap M_1$ and $C = \pi(N)$, where $\pi : M \to M_2$ is a projective map. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & N & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M & \xrightarrow{\pi} & M_2 & \longrightarrow & 0 \end{array}$$

Since M_1 and M_2 are both w-Noetherian modules, we have $B = (B_1)_w$ and $C_w = \pi(N_1)_w$, where B_1 and N_1 are finitely generated submodules of B and N, respectively. Next we show $N = (B_1 + N_1)_w$.

Let $x \in N$. Then $\pi(x) \in C$. Thus $J\pi(x) \subseteq \pi(N_1)$ for some $J \in GV(R)$. It follows that $Jx \subseteq (B + N_1)$. Hence $x \in (B + N_1)_w = ((B_1)_w + (N_1)_w)_w = (B_1 + N_1)_w$, and so $N = (B_1 + N_1)_w$. Therefore, M is a w-Noetherian module by Proposition 4.2. □

We adopt Kennedy's definition of a Krull ring. Recall from [10] that a ring R is called a Krull ring if there exists a family $\{(V_\alpha, P_\alpha) \mid \alpha \in \Gamma\}$ of discrete rank one valuation pairs of $T(R)$ with associated valuations $\{\nu_\alpha \mid \alpha \in \Gamma\}$ such that:

(1) $R = \bigcap\{V_\alpha \mid \alpha \in \Gamma\}$.

(2) $\nu_\alpha(a) = 0$ almost everywhere on Γ for each regular element $a \in T(R)$, and each P_α is a regular ideal of V_α.

So we do not assume that Krull rings are Marot rings. Recall that a ring R is said to be a Marot ring if every regular ideal can be generated by a set of regular elements. There is another definition of a Krull ring (see [7, 15]), which is precisely a Marot Krull ring. In [10] Kennedy showed that a Krull ring is completely integral closed and satisfies the ACC on regular v-ideals. In response to Kennedy's question, Matsuda [12] proved that the converse is also true. We will see below that R is a Krull ring if and only if R is completely integrally closed and satisfies the ACC on regular w-ideals.

Theorem 4.6. *Let R be a commutative ring. Then the following are equivalent*:

(1) *R is a Krull ring.*

(2) *Every regular ideal is w-invertible.*

(3) *Every regular w-ideal is w-invertible.*

(4) *Every regular prime ideal is w-invertible.*

(5) *Every regular prime w-ideal is w-invertible.*

(6) *R is completely integrally closed and satisfies the ACC on regular w-ideals.*

(7) *R is completely integrally closed and satisfies the ACC on regular v-ideals.*

(8) *R is completely integrally closed and every regular t-ideal is a v-ideal.*

(9) *R is completely integrally closed and every regular maximal w-ideal is a v-ideal.*

(10) *R is completely integrally closed and every regular w-ideal is a v-ideal.*

Proof. (1) $\Leftrightarrow$ (7) follows from [12].

(2) $\Leftrightarrow$ (3) is clear.

(2) $\Rightarrow$ (4) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (3) is similar to the proof of $(vi) \Rightarrow (v)$ of [17, Theorem 5.4].

(2) $\Rightarrow$ (6). Let I be a regular ideal of R. Then $(II^{-1})_w = R$, and so $(II^{-1})_v = R$. Thus, by [10, Proposition 1.1], R is completely integrally closed. On the other hand, since every regular ideal of R is w-finite, every non-empty set of regular w-ideals of R has a maximal element. Therefore, R satisfies the ACC on regular w-ideals.

(6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) are obvious.

(9) $\Rightarrow$ (2). Let I be a regular ideal of R. Then $(II^{-1})_v = R$ by [10, Proposition 1.1]. Suppose $(II^{-1})_w \neq R$. Then there exists a maximal w-ideal $\mathfrak{m}$ such that $(II^{-1})_w \subseteq \mathfrak{m}$. Thus $(II^{-1})_v \subseteq \mathfrak{m}_v = \mathfrak{m}$, a contradiction.

(3) + (6) $\Rightarrow$ (10). Note that every w-invertible regular w-ideal is a v-ideal.

(10) $\Rightarrow$ (9) is trivial. □

One can borrow the techniques from [17, 18] to obtain easily the following results, so the proofs are omitted.

Theorem 4.7. *Let R be a commutative ring.*

(1) (*The Cohen Theorem for w-Noetherian rings*) *R is a w-Noetherian ring if and only if each prime w-ideal of R is of finite type.*

(2) (*The Krull Intersection Theorem for w-Noetherian rings*) *Let R be a w-Noetherian ring and M a w-Noetherian module. If $B = \bigcap_{n=1}^{\infty}(I^n M)_w$, where I is an ideal of R, then $B = (IB)_w$.*

(3) (*The Generalized PIT for w-Noetherian rings*) *Let R be a w-Noetherian ring, and let $I = (a_1, a_2, \dots, a_n)_w$ be a w-ideal of R. If $\mathfrak{p}$ is a prime ideal of R minimal over I, then* $\mathrm{ht}\mathfrak{p} \leqslant n$.

It is worth noting that for a w-Noetherian ring R, if $\mathfrak{p}$ is a prime ideal of R minimal over $a \in R$ which is a regular element, then $\mathrm{ht}\mathfrak{p} = 1$.

Proposition 4.8. *A direct product of finitely many w-Noetherian rings is a w-Noetherian ring.*

Proof. Let $R_1, R_2, \dots, R_n$ be w-Noetherian rings. Set $R = R_1 \times R_2 \times \cdots \times R_n$. It is enough to prove the case $n = 2$. Let I be a w-ideal of R. Then $I = I_1 \times I_2$, where I_i is an ideal of R_i for $i = 1, 2$. By Proposition 4.3, $(I_i)_w = (B_i)_w$ for a finitely generated subideal B_i of I_i, where $i = 1, 2$. To complete the proof, we only need to show that $I = (B_1 \times B_2)_w$. Obviously, $(B_1 \times B_2)_w \subseteq I$. Conversely, let $a = (a_1, a_2) \in I$, where $a_i \in I_i$ for $i = 1, 2$. Then there exists $J_i \in GV(R_i)$ such that $J_i a_i \subseteq B_i$ for $i = 1, 2$. Thus $(J_1 \times J_2)(a_1, a_2) \subseteq B_1 \times B_2$. Hence $a \in (B_1 \times B_2)_w$ by Proposition 1.2(5), and so $I \subseteq (B_1 \times B_2)_w$. □

Theorem 4.9 (The Hilbert Basis Theorem for w-Noetherian rings)**.** *If R is a w-Noetherian ring, then $R[X]$ is likewise a w-Noetherian ring.*

Proof. Let H be a w-ideal of $R[X]$. Suppose that I_s is the ideal of R generated by leading coefficients of polynomials of degree s in H, where $s = 0, 1, \dots$. Then $I_s \subseteq I_{s+1}$, and thus there exists a nonnegative integer m such that $(I_m)_w = (I_{m+1})_w = \cdots$ and $I_0, I_1, \dots, I_m$ are w-finite. Let $(I_s)_w = (a_{s1}, \dots, a_{sn_s})_w$, where $a_{s1}, \dots, a_{sn_s} \in I_s$ for $s = 0, 1, \dots, m$. Then there exists polynomial f_{si} in H whose leading coefficient is a_{si}, where $i = 1, \dots, n_s$ and $s = 0, 1, \dots, m$. Set $A = \sum_{s=0}^{m} \sum_{i=1}^{n_s} R[X] f_{si}$. To show that $R[X]$ is a w-Noetherian ring, it suffices by Proposition 4.3 to show that $H = A_W$, where A_W denotes the w-envelope of A as an $R[X]$-module. Obviously, $A_W \subseteq H$. On the other hand, let $f \in H$. First, $0 \in A_W$. Now let $f = ax^s + \cdots$ have degree s. Then $a \in I_s$.

We prove by induction that $f \in A_W$ for every $s \geqslant 0$. For $s \leqslant m$ the assertion is clear. Let $s > m$ and assume the statement holds for all $\deg(f) < s$. Then $a \in (I_s)_w = (I_m)_w$. Thus there exists $J = (d_1, \dots, d_t) \in GV(R)$ such that $Ja \subseteq (a_{m1}, \dots, a_{mn_m})$, and so $d_j a = \sum_{i=1}^{n_m} b_i a_{mi}$ for $1 \leqslant j \leqslant t$, where $b_i \in R$. Set $g_j = d_j f - \sum_{i=1}^{n_m} b_i x^{s-m} f_{mi}$ for each $1 \leqslant j \leqslant t$. Then g_j has degree less than s, and hence $g_j \in A_W$ by the induction hypothesis. Consequently, $d_j f \in A_W$ for each $1 \leqslant j \leqslant t$, and so $J[X] f \subseteq A_W$. By Proposition 1.2, $J[X] \in GV(R[X])$ and thus $f \in A_W$, as required. □

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References

[1] D. D. Anderson and S. J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), no. 5, 2461–2475.

[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Second Edition, Springer-Verlag, New York, 1992.

[3] S. El Baghdadi and S. Gabelli, *w-divisorial domains*, J. Algebra **285** (2005), no. 1, 335–355.

[4] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
[5] S. Glaz and W. V. Vasconcelos, *Flat ideals. II*, Manuscripta Math. **22** (1977), no. 4, 325–341.
[6] J. R. Hedstrom and E. G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), no. 1, 37–44.
[7] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York, 1988.
[8] B. G. Kang, *Characterizations of Krull rings with zero divisors*, J. Pure Appl. Algebra **146** (2000), no. 3, 283–290.
[9] I. Kaplansky, *Commutative Rings*, Revised Edition, Univ. Chicago Press, Chicago, 1974.
[10] R. E. Kennedy, *Krull rings*, Pacific J. Math. **89** (1980), no. 1, 131–136.
[11] T. G. Lucas, *The Mori property in rings with zero divisors*, Rings, modules, algebras, and abelian groups, 379–400, Lecture Notes in Pure and Appl. Math., 236, Dekker, New York, 2004.
[12] R. Matsuda, *On Kennedy's problems*, Comment. Math. Univ. St. Paul. **31** (1982), no. 2, 143–145.
[13] M. H. Park, *Group rings and semigroup rings over strong Mori domains*, J. Pure Appl. Algebra **163** (2001), no. 3, 301–318.
[14] ———, *On overrings of strong Mori domains*, J. Pure Appl. Algebra **172** (2002), no. 1, 79–85.
[15] D. Portelli and W. Spangher, *Krull rings with zero divisors*, Comm. Algebra **11** (1983), no. 16, 1817–1851.
[16] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.
[17] F. G. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.
[18] ———, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165.

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