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SOME NOTES ON EXTENSIONS OF BASIC UNIVALENCE CRITERIA

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Abstract. The object of the present paper is to obtain a more general condition for univalence of analytic functions in the open unit disk U. The significant relationships and relevance with other results are also given. A number of known univalent conditions would follow upon specializing the parameters involved in our main results.

1. Introduction

We denote by U_r the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$, by $U = U_1$ the open unit disk of the complex plane and by I the interval $[0, \infty)$.

Let A denote the class of analytic functions in the open unit disk U which satisfy the usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Three of the most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Nehari [4], Ozaki-Nunokawa [7] and Becker [1]. Some extensions of these three criteria were given by (see [6, 9, 10, 11, 12, 13] and [14]). During the time, unlike there were obtained a lot of univalence criteria (see also [2], [3] and [5]).

Our univalence conditions contain as special cases, Tudor's results and other results obtained by some of the authors cited in references.

Theorem 1.1 (see [1]). *Let $f \in A$. If for all $z \in U$*

$$(1.1) \qquad (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

then the function f is univalent in U.

Theorem 1.2 (see [7]). *Let $f \in A$. If for all $z \in U$*

$$(1.2) \qquad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1,$$

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then the function f is univalent in U.

Theorem 1.3 (see [4]). *Let $f \in A$. If for all $z \in U$*

$$|\{f, z\}| \leq \frac{2}{(1-|z|^2)^2}, \tag{1.3}$$

where

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1.4}$$

Then the function f is univalent in U.

In the present paper we consider the analyticity and univalence of functions $f(z)$ belonging to the class A. Our considerations are based on the theory of Loewner chains. A function $L : U \times I \to \mathbb{C}$ is called a Loewner chain if it is analytic and univalent in U and $L(z, s)$ is subordinate to $L(z, t)$ for all $0 \leq s \leq t < \infty$. Consider f and g analytic functions in U. We say that f is subordinate to g, written $f \prec g$, if there exists a function w analytic in U which satisfies $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ for all $z \in U$.

2. Preliminaries

In proving our results, we will need the following theorem due to Ch. Pommerenke [8].

Theorem 2.1. *Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \cdots$, $a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I, and locally uniform with respect to U_r. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \ \forall z \in U_r,$$

where $p(z, t)$ is analytic in U and satisfies the condition $\Re(p(z, t)) > 0$ for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z, t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk U.

3. Main results

Making use of Theorem 2.1 we can prove now, our main results.

Theorem 3.1. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. Let g and h be two analytic functions in U, $g(z) = 1 + b_1z + \cdots$, $h(z) = c_0 + c_1z + \cdots$. If the following inequalities*

$$\left|\left(\frac{1}{\alpha}\frac{f'(z)}{g(z)} - 1\right) - \frac{m-1}{2}\right| < \frac{m+1}{2} \tag{3.1}$$

and

(3.2)

$$\left|\left(\frac{1}{\alpha}\frac{f'(z)}{g(z)}-1\right)|z|^{2(m+1)} + z^2\left(1-|z|^{m+1}\right)^2\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{f'(z)h(z)}{f(z)}+\frac{1}{\alpha}\frac{f'(z)h^2(z)}{g(z)}+\frac{g'(z)h(z)}{g(z)}-h'(z)\right] + z|z|^{m+1}\left(1-|z|^{m+1}\right)\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{f'(z)}{f(z)}+\frac{2}{\alpha}\frac{f'(z)h(z)}{g(z)}+\frac{g'(z)}{g(z)}\right] -\frac{m-1}{2}|z|^{m+1}\right| \leq \frac{m+1}{2}|z|^{m+1}$$

are satisfied for all $z \in U$, then the function f is univalent in U where the principal branch is intended.

Proof. Let a and b be two positive real numbers such that $m = \frac{b}{a}$. We prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times I \to \mathbb{C}$, defined formally by

$$L(z,t) = f^{1-\alpha}(e^{-at}z)\left[f(e^{-at}z)+\frac{(e^{bt}-e^{-at})zg(e^{-at}z)}{1+(e^{bt}-e^{-at})zh(e^{-at}z)}\right]^{\alpha} \tag{3.3}$$

is analytic in U_r for all $t \in I$.
Let us consider the function $\varphi_1(z,t)$ given by

$$\varphi_1(z,t) = 1+(e^{bt}-e^{-at})zh(e^{-at}z). \tag{3.4}$$

For all $t \in I$ and $z \in U$ we have $e^{-at}z \in U$ and because h analytic, the function $\varphi_1(z,t)$ is analytic in U and $\varphi_1(0,t) = 1$. Then there exist a disc U_{r_1}, $0 < r_1 < 1$, in which $\varphi_1(z,t) \neq 0$ for all $t \in I$ and $z \in U_{r_1}$.

For the function

$$\varphi_2(z,t) = \left[f(e^{-at}z)+\frac{(e^{bt}-e^{-at})zg(e^{-at}z)}{\varphi_1(z,t)}\right]^{\alpha}, \tag{3.5}$$

$\varphi_2(z,t) = z^{\alpha}\varphi_3(z,t)$, it can be easily shown that $\varphi_3(z,t)$ is analytic in U_{r_1} and $\varphi_3(0,t) = e^{\alpha bt}$ for all $t \in I$. From these considerations it follows that the function

$$L(z,t) = f^{1-\alpha}(z,t)\varphi_2(z,t) \tag{3.6}$$

is analytic in U_{r_1} for all $t \in I$ and has the following form

$$L(z,t) = a_1(t)z + \cdots .$$

We have

$$a_1(t) = e^{[\alpha(a+b)-a]t} \tag{3.7}$$

for which we consider the uniform branch equal to 1 at the origin. Because $\Re(\alpha) > \frac{1}{m+1}$ is equivalent with $\Re(\alpha) > \frac{a}{a+b}$, we have that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

Moreover, $a_1(t) \neq 0$ for all $t \in I$.

From the analyticity of $L(z,t)$ in U_{r_1}, it follows that there exists a number r_2, $0 < r_2 < r_1$, and a constant $K = K(r_2)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K,\ \forall z \in U_{r_2},\ t \in I.$$

Then, by Montel's Theorem, $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\in I}$ is a normal family in U_{r_2}. From the analyticity of $\frac{\partial L(z,t)}{\partial t}$, we obtain that for all fixed numbers $T > 0$ and $r_3,\ 0 < r_3 < r_2$, there exists a constant $K_1 > 0$ (that depends on T and r_3) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1,\ \forall z \in U_{r_3},\ t \in [0,T].$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_3}.

Let $p : U_r \times I \to \mathbb{C}$ be the function in $U_r,\ 0 < r < r_3$ for all $t \in I$, defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}.$$

If the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} = \frac{\frac{z\partial L(z,t)}{\partial z} - \frac{\partial L(z,t)}{\partial t}}{\frac{z\partial L(z,t)}{\partial z} + \frac{\partial L(z,t)}{\partial t}} \tag{3.8}$$

is analytic in $U \times I$ and $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$, then $p(z,t)$ has an analytic extension with positive real part in U for all $t \in I$. From equality (3.8) we have

$$w(z,t) = \frac{(1+a)A_\alpha(z,t) + (1-b)}{(1-a)A_\alpha(z,t) + (1+b)}, \tag{3.9}$$

where

(3.10)

$$\begin{aligned} &A_\alpha(z,t) \\ &= e^{-(a+b)t}\left\{\frac{f'(e^{-at}z)}{\alpha g(e^{-at}z)} + (e^{bt}-e^{-at})^2 z^2\left[\frac{(1-\alpha)f'(e^{-at}z)h(e^{-at}z)}{\alpha f(e^{-at}z)}\right.\right. \\ &\quad \left. + \frac{f'(e^{-at}z)h^2(e^{-at}z)}{\alpha g(e^{-at}z)} + \frac{g'(e^{-at}z)h(e^{-at}z)}{g(e^{-at}z)} - h'(e^{-at}z)\right] \\ &\quad \left. + (e^{bt}-e^{-at})z\left[\frac{(1-\alpha)f'(e^{-at}z)}{\alpha f(e^{-at}z)} + 2\frac{f'(e^{-at}z)h(e^{-at})}{\alpha g(e^{-at}z)} + \frac{g'(e^{-at}z)}{g(e^{-at}z)}\right] - 1\right\} \end{aligned}$$

for $z \in U$ and $t \in I$.

The inequality $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$, where $w(z,t)$ is defined by (3.9), is equivalent to

$$\left| A_\alpha(z,t) - \frac{b-a}{2a} \right| < \frac{b+a}{2a}, \qquad \forall z \in U,\ t \in I. \tag{3.11}$$

Define

$$B_\alpha(z,t) = A_\alpha(z,t) - \frac{m-1}{2}, \quad \forall z \in U,\ t \in I. \tag{3.12}$$

From (3.1), (3.10) and $\Re(\alpha) > \frac{1}{m+1}$ we have

$$|B_\alpha(z,0)| = \left| \left(\frac{f'(z)}{\alpha g(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{3.13}$$

and

$$|B_\alpha(0,t)| = \left| \left(\frac{1}{\alpha} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2}. \tag{3.14}$$

Since $|e^{-at}z| \le |e^{-at}| = e^{-at} < 1$ for all $z \in \bar{U} = \{z \in \mathbb{C} : |z| \le 1\}$ and $t > 0$, we find that $B_\alpha(z,t)$ is an analytic function in $\bar{U}$. Using the maximum modulus principle it follows that for all $z \in U - \{0\}$ and each $t > 0$ arbitrarily fixed there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|B_\alpha(z,t)| < \max_{|z|=1} |B_\alpha(z,t)| = \left| B_\alpha(e^{i\theta},t) \right| \tag{3.15}$$

for all $z \in U$ and $t \in I$.
Denote $u = e^{-at}e^{i\theta}$. Then $|u| = e^{-at}$, $e^{-(a+b)t} = (e^{-at})^{m+1} = |u|^{m+1}$ and from (3.10) we have

(3.16)

$$\begin{aligned} &\left| B_\alpha(e^{i\theta},t) \right| \\ &= \Bigg| \left(\frac{f'(u)}{\alpha g(u)} - 1 \right) |u|^{m+1} \\ &\quad + \frac{u^2 \left(1 - |u|^{m+1}\right)^2}{|u|^{m+1}} \left[\frac{(1-\alpha) f'(u) h(u)}{\alpha f(u)} + \frac{f'(u) h^2(u)}{\alpha g(u)} + \frac{g'(u) h(u)}{g(u)} - h'(u) \right] \\ &\quad + u \left(1 - |u|^{m+1}\right) \left[\frac{(1-\alpha) f'(u)}{\alpha f(u)} + 2 \frac{f'(u) h(u)}{\alpha g(u)} + \frac{g'(u)}{g(u)} \right] - \frac{m-1}{2} \Bigg|. \end{aligned}$$

Because $u \in U$, the inequality (3.2) implies that

$$\left| B_\alpha(e^{i\theta},t) \right| \le \frac{m+1}{2},$$

and from (3.13), (3.14) and (3.15), we conclude that

$$|B_\alpha(z,t)| = \left| A_\alpha(z,t) - \frac{m-1}{2} \right| \le \frac{m+1}{2}$$

for all $z \in U$ and $t \in I$. Therefore $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $L(z,t)$ has an analytic and univalent extension to the whole unit disk U for all $t \in I$. For $t = 0$ we have $L(z,0) = f(z)$ for $z \in U$and therefore the function f is analytic and univalent in U. □

Remark 3.1. (1) By putting $m = 1$ in Theorem 3.1 we obtain all Tudor's results in [14].

(2) The univalence criteria which results from Theorem 3.1 when $m = 1$ and $\alpha = 1$ is due to Ovesea-Tudor and Owa in [6].

Corollary 3.1. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. Suppose that there exists an analytic function h in U, $h(z) = c_0 + c_1 z + \cdots$. If the following inequality*

(3.17)

$$\begin{aligned}&\left|\left(\frac{1}{\alpha}-1\right)|z|^{2(m+1)}\right.\\&+z^2\left(1-|z|^{m+1}\right)^2\left[\frac{(1-\alpha)f'(z)h(z)}{\alpha f(z)}+\frac{h^2(z)}{\alpha}+\frac{f''(z)h(z)}{f'(z)}-h'(z)\right]\\&\left.+z|z|^{m+1}\left(1-|z|^{m+1}\right)\left[\frac{(1-\alpha)f'(z)}{\alpha f(z)}+\frac{2h(z)}{\alpha}+\frac{f''(z)}{f'(z)}\right]-\frac{m+1}{2}|z|^{m+1}\right|\\&\le\frac{m+1}{2}|z|^{m+1}\end{aligned}$$

holds true for all $z \in U$, then the function f is univalent in U where the principal branch is intended.

Proof. It results from Theorem 3.1 with $g = f'$. □

If we choose $g = f'$ and $h = -\frac{1}{2}\frac{f''}{f'}$ in Theorem 3.1 we obtain the following univalence criterion.

Corollary 3.2. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. If the following inequality*

(3.18)

$$\begin{aligned}&\left|\left(\frac{1}{\alpha}-1\right)|z|^{2(m+1)}\right.\\&+\left(1-|z|^{m+1}\right)^2\left\{\frac{1}{2}z^2\{f;z\}+\frac{1}{2}(\frac{1-\alpha}{\alpha})\left[\frac{1}{2}\left(\frac{zf''(z)}{f(z)}\right)^2-\frac{z^2f''(z)}{f(z)}\right]\right\}\\&\left.+|z|^{m+1}\left(1-|z|^{m+1}\right)(\frac{1-\alpha}{\alpha})\left[\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right]-\frac{m-1}{2}|z|^{m+1}\right|\\&\le\frac{m+1}{2}|z|^{m+1}\end{aligned}$$

holds true for all $z \in U$, then the function f is univalent in U, where the principal branch is intended.

Remark 3.2. (1) If we consider $m = 1$ and $\alpha = 1$ in Corollary 3.2, the inequality (3.18) becomes (1.3) and then we obtain the univalence criterion due to Nehari [4].

(2) Setting $m = 1$ in Corollary 3.2, we obtain the univalence criterion due to Raducanu [9].

Corollary 3.3. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. Suppose there exists an analytic function $h(z)$ in U, $h(z) = c_0 + c_1 z + \cdots$. If the following inequalities*

$$\left| \frac{z^2 f'(z)}{\alpha f^2(z)} - \frac{m+1}{2} \right| < \frac{m+1}{2} \tag{3.19}$$

and

(3.20)
$$\left| \left(\frac{z^2 f'(z)}{\alpha f^2(z)} - 1 \right) |z|^{2(m+1)} + z^2 \left(1 - |z|^{m+1}\right)^2 \left[\frac{(1+\alpha) f'(z) h(z)}{\alpha f(z)} + \frac{z^2 f'(z) h^2(z)}{\alpha f^2(z)} - \frac{2h(z)}{z} - h'(z) \right] + z |z|^{m+1} \left(1 - |z|^{m+1}\right) \left[\frac{(1+\alpha) f'(z)}{\alpha f(z)} + \frac{2z^2 f'(z) h(z)}{\alpha f^2(z)} - \frac{2}{z} \right] - \frac{m-1}{2} |z|^{m+1} \right| \le \frac{m+1}{2} |z|^{m+1}$$

are satisfied for all $z \in U$, then the function f is univalent in U where the principal branch is intended.

Proof. It results from Theorem 3.1 with $g(z) = \left(\frac{f(z)}{z}\right)^2$. □

If we choose $g(z) = \left(\frac{f(z)}{z}\right)^2$ and $h(z) = \frac{1}{z} - \frac{f(z)}{z^2}$ in Theorem 3.1 we obtain the following corollary.

Corollary 3.4. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. If the following inequalities*

$$\left| \left(\frac{z^2 f'(z)}{\alpha f^2(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{3.21}$$

and

$$\left| \left(\frac{z^2 f'(z)}{\alpha f^2(z)} - 1 \right) + \left(\frac{\alpha - 1}{\alpha}\right) \left(1 - |z|^{m+1}\right) \left(\frac{z f'(z)}{f(z)} \right) - \frac{m-1}{2} |z|^{m+1} \right| \le \frac{m+1}{2} |z|^{m+1} \tag{3.22}$$

are satisfied for all $z \in U$, then the function f is univalent in U, where the principal branch is intended.

Remark 3.3. (1) If we consider $\alpha = 1$ in Corollary 3.4 we obtain the univalence criterion due to Raducanu et al. [13].

(2) Putting $m = 1$ in Corollary 3.4 we obtain the univalence criterion due to Raducanu [10].

(3) If we consider $m = 1$ and $\alpha = 1$ in Corollary 3.4, the inequalities (3.21) and (3.22) becomes (1.2) and then we obtain the univalence criterion due to Ozaki-Nunokawa [7].

Corollary 3.5. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. If the following inequality*

$$\left| \left(\frac{1}{\alpha} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2},$$

$$\begin{aligned} (3.23) \qquad & \left| \frac{1-\alpha}{\alpha} \left[1 - (1 - |z|^{m+1}) \frac{zf'(z)}{f(z)} \right] \right. \\ & \left. +(1 - |z|^{m+1}) z \frac{d}{dz} \left[\log \frac{z^2 f'(z)}{f^2(z)} \right] - \frac{m-1}{2} |z|^{m+1} \right| \\ \leq\ & \frac{m+1}{2} |z|^{m+1} \end{aligned}$$

is satisfied for all $z \in U$, then the function f is univalent in U where the principal branch is intended.

Proof. It results from Theorem 3.1 with $g(z) = f'(z)$ and $h(z) = \frac{1}{z} - \frac{f'(z)}{f(z)}$. □

Remark 3.4. (1) If we consider $m = 1$ in Corollary 3.5 we obtain the univalence criterion due to Raducanu [11].

(2) For $m = 1$ and $\alpha = 1$ in Corollary 3.5 we obtain Goluzin's criterion for univalence [3].

Corollary 3.6. *Let m be a positive real number and let α be a complex number such that $\Re(\alpha) > \frac{1}{m+1}$ and $f \in A$. If the following inequality*

(3.24)
$$\left| \left(\frac{1}{\alpha} - 1 \right) |z|^{m+1} + z(1 - |z|^{m+1}) \left[(\frac{1-\alpha}{\alpha}) \frac{f'(z)}{f(z)} + \frac{f''(z)}{f'(z)} \right] - \frac{m-1}{2} \right| \leq \frac{m+1}{2}$$

is satisfied for all $z \in U$, then the function f is univalent in U where the principal branch is intended.

Proof. It results from Theorem 3.1 with $g(z) = f'(z)$ and $h(z) = 0$. □

Remark 3.5. If we consider $\alpha = m = 1$ in Corollary 3.6, the inequality (3.24) becomes (1.1) and then we obtain the univalence criterion due to Becker [1].

Finally, if we take $\alpha \to \infty$ in Corollary 3.6 ($z \in U$) we obtain another univalence criterion as follows.

Corollary 3.7. *Let m be a positive real number and $f \in A$. If the following inequality*

$$\left| z(1-|z|^{m+1})\left[\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right] - |z|^{m+1} - \frac{m-1}{2}\right| \leq \frac{m+1}{2} \tag{3.25}$$

holds true for all $z \in U$, then the function f is univalent in U where the principal branch is intended.

Remark 3.6. If we consider $\alpha \to \infty$, $z \in U$ in the Corollaries 3.1 and 3.2 we can obtain other new univalence criteria.

Remark 3.7. The famous univalence criteria obtained by Nehari, Ozaki-Nunokawa and Becker contain $|z|^2$ in their expressions. From Theorem 3.1 we obtain new and more general results with $|z|^{m+1}$ $(m > 0)$ instead of $|z|^2$.

Example 3.1. The function

$$f(z) = \frac{z}{1 - \frac{z^{m+1}}{m+1}}, \quad (m \geq 1) \tag{3.26}$$

is analytic and univalent in U.

Proof. From equality (3.26) we have

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{m}{m+1} z^{m+1}. \tag{3.27}$$

Taking into account (3.27), $\alpha = 1$ and $m \geq 1$, the conditions (3.21) and (3.22) in Corollary 3.4 becomes, respectively,

$$\begin{aligned} \left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \frac{m-1}{2}\right| &= \left|\frac{m}{m+1} z^{m+1} - \frac{m-1}{2}\right| \\ &< \frac{m}{m+1} + \frac{m-1}{2} = \frac{m^2+2m-1}{2(m+1)} < \frac{m+1}{2} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{|z|^{m+1}}\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \frac{m-1}{2}|z|^{m+1}\right| \\ = \; &\frac{1}{|z|^{m+1}}\left|\frac{m}{m+1} z^{m+1} - \frac{m-1}{2}|z|^{m+1}\right| < \frac{m}{m+1} + \frac{m-1}{2} \\ = \; &\frac{m^2+2m-1}{2(m+1)} < \frac{(m+1)^2}{2(m+1)} < \frac{m+1}{2}, \end{aligned}$$

which are satisfied the conditions (3.21) and (3.22) of Corollary 3.4. It follows that the function f defined by (3.26) is analytic and univalent in U. By using the Mathematica 7.0 program, for $m = 5$, we can obtain the graphic of $f(z) = \frac{z}{1-z^6/6}$ (see Figure 1). □

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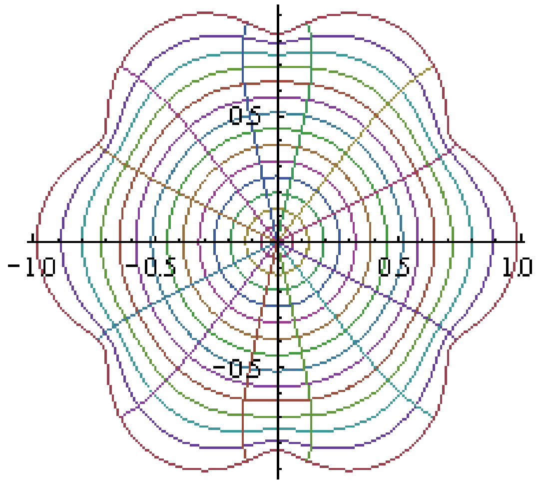

FIGURE 1

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