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CHARACTER ANALOGUES OF INFINITE SERIES FROM A CERTAIN MODULAR TRANSFORMATION FORMULA

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Abstract. In this paper, we find character analogues of infinite series identities which come from a certain modular transformation formula given by B. C. Berndt.

1. Introduction

A special case of the character analogues of Ramanujan-Guinand formula [5, 6] shows the following beautiful symmetric identity [3];

For α, $\beta > 0$ with $\alpha\beta = \pi^2$,

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \sigma_{-1}(n) e^{-2\alpha n/5} = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \sigma_{-1}(n) e^{-2\beta n/5}, \tag{1.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ means the Legendre symbol and

$$\sigma_s(n) = \sum_{d|n} d^s.$$

This type of identities can be derived most naturally from the functional equation of the associated zeta-function, for example, see [4]. In this paper, we obtain this type of identities from character analogues of infinite series identities which come from Theorem 2.1 in [2].

The followings are necessary notation. We choose the branch of the argument as $-\pi \leq \arg w < \pi$ for a complex w. Let $e(w) = e^{2\pi i w}$ and let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$, the upper half-plane. For every complex $\tau \in \mathbb{H}$, $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ denotes a modular transformation with $c > 0$ and $c \equiv 0 \pmod N$. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and define the associated vectors R and H by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

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and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

For a positive integer N, let λ_N denote the characteristic function of the integers modulo N, i.e.,

$$\lambda_N(m) = \begin{cases} 1, & \text{if } m \equiv 0 \pmod N, \\ 0, & \text{otherwise.} \end{cases}$$

For real x, α and $\mathrm{Re}(s) > 1$, let

$$\psi(x, \alpha, s) := \sum_{n+\alpha>0} \frac{e(nx)}{(n+\alpha)^s}.$$

We see that the function $\psi(x, \alpha, s)$ can be analytically continued to the entire s-plane [1]. Let

$$\Psi^{\pm}(x, \alpha, s) := \psi(x, \alpha, s) + e\left(\pm\frac{s}{2}\right)\psi(-x, -\alpha, s).$$

For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. For $\tau \in \mathbb{H}$ and an arbitrary complex number s, we define

$$A_N(\tau, s; r, h) := \sum_{Nm+r_1>0} \sum_{n-h_2>0} \frac{e\left(Nmh_1 + ((Nm + r_1)\tau + r_2)(n - h_2)\right)}{(n - h_2)^{1-s}}.$$

Let

$$H_N(\tau, s; r, h) := A_N(\tau, s; r, h) + e\left(\frac{s}{2}\right) A_N(\tau, s; -r, -h).$$

We now state the principal theorem which shall be used to obtain main results.

Theorem 1.1 ([2]). *Let* $Q = \{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > -d/c\}$, $\varrho_N = c\{R_2\} - Nd\{R_1/N\}$ *and* $c = Nc'$. *Then for* $\tau \in Q$ *and all* s,

$$\begin{aligned}
&(c\tau + d)^{-s} H_N(V\tau, s; r, h) \\
&= H_N(\tau, s; R, H) - \lambda_N(r_1)e(-r_1h_1)(c\tau + d)^{-s}\Gamma(s)(-2\pi i)^{-s}\Psi^{+}(h_2, r_2, s) \\
&\quad + \lambda_N(R_1)e(-R_1H_1)\Gamma(s)(-2\pi i)^{-s}\Psi^{-}(H_2, R_2, s) \\
&\quad + (2\pi i)^{-s} L_N(\tau, s; R, H),
\end{aligned}$$

where

$$\begin{aligned}
&L_N(\tau, s; R, H) \\
&:= \sum_{j=1}^{c'} e(-H_1(Nj + N[R_1/N] - c) - H_2([R_2] + 1 + [(Njd + \varrho_N)/c] - d)) \\
&\qquad \cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(Nj - N\{R_1/N\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(Njd+\varrho_N)/c\}u}}{e^u - e(-H_2)} du,
\end{aligned}$$

where C *is a loop beginning at* $+\infty$, *proceeding in the upper half-plane, encircling the origin in the positive direction so that* $u = 0$ *is the only zero of*

$$\left(e^{-(c\tau+d)u} - e(cH_1 + dH_2)\right)\left(e^u - e(-H_2)\right)$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Remark 1.2. Note that after evaluation of $L_N(\tau, s; R, H)$ for an integer s, the transformation formula in Theorem 1.1 will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

Let $B_n(x)$ denote Bernoulli polynomials, $n \geq 0$, which are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi).$$

The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\}),\ n \geq 0$.

2. A transformation formula for $\prod_{n=1}^{\infty} (1 - q^n)^{-\chi(n)}$

Let χ be a Dirichlet character of modulus N. Let

$$f(q) := \prod_{n=1}^{\infty} (1 - q^n)^{-\chi(n)},$$

where $q = e(\tau)$ and $\tau \in \mathbb{H}$. Let $Vq := e(V\tau)$. In this section, we take a modular transformation V as

$$V\tau = \frac{\tau - 1}{N\tau - N + 1}.$$

Theorem 2.1. *Let χ be an even Dirichlet character of modulus N and let $z = N\tau - N + 1$. Then, for any $z \in \mathbb{H}$,*

$$\log f(Vq) = \log f(q) - \pi i \left(z + z^{-1}\right) \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) B_2\left(\frac{k}{N}\right) + 2\pi i \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) B_1\left(\frac{k}{N}\right)^2.$$

Proof. Since χ is even,

$$\begin{aligned}
\log f(q) &= -\sum_{n=1}^{\infty} \chi(n) \log\left(1 - q^n\right) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \chi(n) \frac{q^{nm}}{m} \\
&= \sum_{k=1}^{N-1} \chi(k) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^{(nN+k)m}}{m} \\
&= \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) \left(\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^{(nN+k)m}}{m} + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^{(nN+N-k)m}}{m} \right)
\end{aligned}$$

$$= \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) H_N(\tau, 0; r_k, 0),$$

where $r_k = (k, 0)$. Then, by Theorem 1.1, we have

$$\begin{aligned} \log f(Vq) &= \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) H_N(V\tau, 0; r_k, 0) \\ &= \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) \left(H_N(\tau, 0; R_k, 0) + L_N(\tau, 0; R_k, 0) \right), \end{aligned} \tag{2.1}$$

where $R_k = (k, -k)$. It is easy to see that

$$\begin{aligned} H_N(\tau, 0; R_k, 0) &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^{(nN+k)m}}{m} + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^{(nN+N-k)m}}{m} \\ &= H_N(\tau, 0; r_k, 0). \end{aligned} \tag{2.2}$$

For $L_N(\tau, 0; R_k, 0)$, using the residue theorem, we obtain that

$$\begin{aligned} L_N(\tau, 0; R_k, 0) &= 2\pi i \sum_{j=0}^{2} \frac{B_j \left(1 - \left\{\frac{k}{N}\right\}\right) \bar{B}_{2-j} \left(1 + \frac{Nk-k}{N}\right)}{j!(2-j)!} (-z)^{j-1} \\ &= 2\pi i \sum_{j=0}^{2} \frac{B_j \left(\frac{k}{N}\right) B_{2-j} \left(\frac{k}{N}\right)}{j!(2-j)!} (-z)^{j-1} \\ &= \pi i B_2 \left(\frac{k}{N}\right) \left(z + z^{-1}\right) + 2\pi i B_1 \left(\frac{k}{N}\right)^2, \end{aligned} \tag{2.3}$$

where $z = N\tau - N + 1$. Thus by combining (2.1), (2.2), (2.3) and applying Remark 1.2, we obtain the desired result. □

Let $\zeta_N = e^{2\pi i/N}$ and let

$$\sigma_s(\chi, n) := \sum_{d|n} \chi(d) d^s.$$

Theorem 2.2. *Let χ be an even Dirichlet character of modulus N. Then, for any $z \in \mathbb{H}$,*

$$\begin{aligned} &\sum_{n=1}^{\infty} \chi(n) \zeta_N^n \sigma_{-1}(\bar{\chi}, n) e(-nz^{-1}/N) \\ &= \sum_{n=1}^{\infty} \chi(n) \zeta_N^{-n} \sigma_{-1}(\bar{\chi}, n) e(nz/N) - \pi i \left(z + z^{-1}\right) \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) B_2 \left(\frac{k}{N}\right) \\ &\quad + 2\pi i \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) B_1 \left(\frac{k}{N}\right)^2. \end{aligned}$$

Proof. Since

$$V\tau = \frac{1}{N}(1 - z^{-1}),\ \tau = \frac{1}{N}(z-1)+1,$$

$$e((nN+k)mV\tau) = \zeta_N^{km} e(-(nN+k)mz^{-1}/N), \tag{2.4}$$

and

$$e((nN+k)m\tau) = \zeta_N^{-km} e((nN+k)mz/N), \tag{2.5}$$

we have

$$\begin{aligned} \log f(Vq) &= \sum_{k=1}^{N-1} \chi(k) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \zeta_N^{km} \frac{e(-(nN+k)mz^{-1}/N)}{m} \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \chi(n)\zeta_N^{mn} e(-mnz^{-1}/N) \\ &= \sum_{n=1}^{\infty} \chi(n)\zeta_N^{n} \sigma_{-1}(\bar{\chi}, n) e(-nz^{-1}/N). \end{aligned} \tag{2.6}$$

Similarly,

$$\log f(q) = \sum_{n=1}^{\infty} \chi(n)\zeta_N^{-n} \sigma_{-1}(\bar{\chi}, n) e(nz/N). \tag{2.7}$$

Now, applying (2.6), (2.7) to Theorem 2.1, the proof is complete. □

Theorem 2.3. *Let* $\alpha,\ \beta > 0$ *with* $\alpha\beta = \pi^2$. *Then*

$$\begin{aligned} &\sum_{n=1}^{\infty} \chi(n)\zeta_N^{n} \sigma_{-1}(\bar{\chi}, n) e^{-2\alpha n/N} \\ &= \sum_{n=1}^{\infty} \chi(n)\zeta_N^{-n} \sigma_{-1}(\bar{\chi}, n) e^{-2\beta n/N} + (\beta - \alpha) \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) B_2\left(\frac{k}{N}\right) \\ &\quad + 2\pi i \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) B_1\left(\frac{k}{N}\right)^2. \end{aligned}$$

Proof. Put $z = \frac{\pi i}{\alpha}$ in Theorem 2.2. □

If p is a prime with $p \equiv 1 \pmod 4$, then we can put $\chi = \left(\frac{\cdot}{p}\right)$ in Theorem 2.3, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

Corollary 2.4. *Let* p *be a prime with* $p \equiv 1 \pmod 4$. *Then*

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(\left(\frac{\cdot}{p}\right), n) \cos(2\pi n/p) e^{-2\alpha n/p}$$

$$= \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(\left(\frac{\cdot}{p}\right), n) \cos(2\pi n/p) e^{-2\beta n/p} + (\beta - \alpha) \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{k}{p}\right) B_2(k/p).$$

Proof. Let $\chi = \left(\frac{\cdot}{p}\right)$ in Theorem 2.3 and equate the real parts. □

Corollary 2.5. *Let p be a prime with $p \equiv 1 \pmod 4$. Then*

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(\left(\frac{\cdot}{p}\right), n) \sin(2\pi n/p) e^{-2\alpha n/p}$$

$$= -\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{-1}(\left(\frac{\cdot}{p}\right), n) \sin(2\pi n/p) e^{-2\beta n/p} + 2\pi \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{k}{p}\right) B_1(k/p)^2.$$

Proof. Let $\chi = \left(\frac{\cdot}{p}\right)$ in Theorem 2.3 and equate the imaginary parts. □

Corollary 2.4 and Corollary 2.5 do not show symmetric identities when they are compared with (1.1). However we obtain a generalized version of Theorem 2.3 and have some symmetric identities in next section.

3. Generalized identities

In this section, we obtain more generalized character analogues than Theorem 2.1.

Theorem 3.1. *Let χ be an even Dirichlet character of modulus N. Then for any integer ℓ and for any $z \in \mathbb{H}$,*

$$z^{2\ell} \sum_{n=1}^{\infty} \chi(n) \zeta_N^n \sigma_{-2\ell-1}(\bar{\chi}, n) e(-nz^{-1}/N)$$

$$= \sum_{n=1}^{\infty} \chi(n) \zeta_N^{-n} \sigma_{-2\ell-1}(\bar{\chi}, n) e(nz/N)$$

$$+ (2\pi i)^{2\ell+1} \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) \sum_{j=0}^{2\ell+2} \frac{B_j\left(\frac{k}{N}\right) B_{2\ell+2-j}\left(\frac{k}{N}\right)}{j!(2\ell+2-j)!} (-z)^{j-1}.$$

Proof. Let $r_k = (k, 0)$ and let

$$V\tau = \frac{\tau - 1}{N\tau - N + 1}.$$

We see that

(3.1)

$$H_N(\tau, -2\ell; r_k, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((mN + k)n\tau)}{n^{1+2\ell}} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((mN + N - k)n\tau)}{n^{1+2\ell}}.$$

By Theorem 1.1,

$$
\begin{aligned}
&(N\tau - N + 1)^{2\ell} H_N(V\tau, -2\ell; r_k, 0) \\
&= H_N(\tau, -2\ell; R_k, 0) + (2\pi i)^{2\ell} L_N(\tau, -2\ell; R_k, 0),
\end{aligned} \tag{3.2}
$$

where $R_k = (k, -k)$. Multiplying both sides in (3.2) by $\chi(k)$ and summing over k, we find that

$$
\begin{aligned}
&(N\tau - N + 1)^{2\ell} \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) H_N(V\tau, -2\ell; r_k, 0) \\
&= \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) H_N(\tau, -2\ell; R_k, 0) + (2\pi i)^{2\ell} \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) L_N(\tau, -2\ell; R_k, 0).
\end{aligned} \tag{3.3}
$$

Plugging (3.1) into (3.3),

$$
\begin{aligned}
&(N\tau - N + 1)^{2\ell} \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((mN+k)nV\tau)}{n^{1+2\ell}} \\
&= \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e((mN+k)n\tau)}{n^{1+2\ell}} \\
&\quad + (2\pi i)^{2\ell} \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) L_N(\tau, -2\ell; R_k, 0).
\end{aligned} \tag{3.4}
$$

It comes from the residue theorem that

$$
L_N(\tau, -2\ell; R_k, 0) = 2\pi i \sum_{j=0}^{2\ell+2} \frac{B_k\left(\frac{k}{N}\right) B_{2\ell+k}\left(\frac{k}{N}\right)}{j!(2\ell+2-j)!} (-N\tau + N - 1)^{j-1}. \tag{3.5}
$$

Let $z = N\tau - N + 1$. Apply (2.4), (2.5), (3.5) to (3.4) to obtain

$$
\begin{aligned}
&z^{2\ell} \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \zeta_N^{(mN+k)n} \frac{e(-(mN+k)nz^{-1}/N)}{n^{1+2\ell}} \\
&= \sum_{k=1}^{N-1} \chi(k) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \zeta_N^{-(mN+k)n} \frac{e((mN+k)nz/N)}{n^{1+2\ell}} \\
&\quad + (2\pi i)^{2\ell+1} \sum_{k=1}^{[\frac{N-1}{2}]} \chi(k) \sum_{j=0}^{2\ell+2} \frac{B_j\left(\frac{k}{N}\right) B_{2\ell+2-j}\left(\frac{k}{N}\right)}{j!(2\ell+2-j)!} (-z)^{j-1}.
\end{aligned} \tag{3.6}
$$

Employing Remark 1.2, we obtain the desired results by the similar manner in (2.6). □

If we put $\ell = 0$ in Theorem 3.1, then Theorem 2.2 follows.

Theorem 3.2. *Let χ be an even Dirichlet character of modulus N. Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then for any integer ℓ,*

$$\begin{aligned}
&\alpha^{-\ell}\sum_{n=1}^{\infty}\chi(n)\zeta_N^n\sigma_{-2\ell-1}(\bar{\chi},n)e^{-2\alpha n/N}\\
&= (-\beta)^{-\ell}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{-2\ell-1}(\bar{\chi},n)e^{-2\beta n/N}\\
&\quad - 2^{2\ell+1}\sum_{k=1}^{[\frac{N-1}{2}]}\chi(k)\sum_{j=0}^{2\ell+2}\frac{B_j\left(\frac{k}{N}\right)B_{2\ell+2-j}\left(\frac{k}{N}\right)}{j!(2\ell+2-j)!}(-\pi i)^j\alpha^{1+\ell-j}.
\end{aligned}$$

Proof. Put $z = \frac{\pi i}{\alpha}$ in Theorem 3.1. □

Corollary 3.3. *For any integer $M > 0$,*

$$\begin{aligned}
&\alpha^{M}\sum_{n=1}^{\infty}\chi(n)\zeta_N^n\sigma_{2M-1}(\bar{\chi},n)e^{-2\alpha n/N}\\
&= (-\beta)^{M}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{2M-1}(\bar{\chi},n)e^{-2\beta n/N}.
\end{aligned}$$

Proof. Let $\ell = -M$ for any integer $M > 0$ in Theorem 3.2. □

Corollary 3.3 does not look to show symmetric identities. But if we put $\chi = \left(\frac{\cdot}{p}\right)$ for a prime p with $p \equiv 1 \pmod 4$, then we obtain moderately good symmetric identities.

Corollary 3.4. *Let p be a prime with $p \equiv 1 \pmod 4$. Then for any integer $M > 0$,*

$$\begin{aligned}
&\alpha^{2M}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M-1}(\left(\frac{\cdot}{p}\right),n)\cos(2\pi n/p)e^{-2\alpha n/p}\\
&= \beta^{2M}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M-1}(\left(\frac{\cdot}{p}\right),n)\cos(2\pi n/p)e^{-2\beta n/p}.
\end{aligned}$$

Proof. Let $\chi = \left(\frac{\cdot}{p}\right)$ in Corollary 3.3. Replace M by $2M$ and equate the real parts. □

Corollary 3.5. *Let p be a prime with $p \equiv 1 \pmod 4$. Then for any integer $M > 0$,*

$$\begin{aligned}
&\alpha^{2M-1}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M-3}(\left(\frac{\cdot}{p}\right),n)\sin(2\pi n/p)e^{-2\alpha n/p}\\
&= \beta^{2M-1}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M-3}(\left(\frac{\cdot}{p}\right),n)\sin(2\pi n/p)e^{-2\beta n/p}.
\end{aligned}$$

Proof. Let $\chi = \left(\frac{\cdot}{p}\right)$ in Corollary 3.3. Replace M by $2M-1$ and equate the imaginary parts. □

For an odd character χ, by the same manner in case of even character, we have the following theorems and corollaries.

Theorem 3.6. *Let $\chi(n)$ be an odd Dirichlet character of modulus N. Then for any integer ℓ and any $z \in \mathbb{H}$,*

$$\begin{aligned} &z^{2\ell+1}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{n}\sigma_{-2\ell-2}(\bar{\chi},n)e(-nz^{-1}/N)\\ =&\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{-2\ell-2}(\bar{\chi},n)e(nz/N)\\ &+(2\pi i)^{2\ell+2}\sum_{k=1}^{[\frac{N-1}{2}]}\chi(k)\sum_{j=0}^{2\ell+3}\frac{B_j\left(\frac{k}{N}\right)B_{2\ell+3-j}\left(\frac{k}{N}\right)}{j!(2\ell+3-j)!}(-z)^{j-1}.\end{aligned}$$

Theorem 3.7. *Let $\chi(n)$ be an odd Dirichlet character of modulus N. Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. Then for any integer ℓ,*

$$\begin{aligned} &\alpha^{-\ell-\frac{1}{2}}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{n}\sigma_{-2\ell-2}(\bar{\chi},n)e^{-2\alpha n/N}\\ =&(-\beta)^{-\ell-\frac{1}{2}}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{-2\ell-2}(\bar{\chi},n)e^{-2\beta n/N}\\ &+2^{2\ell+2}\sum_{k=1}^{[\frac{N-1}{2}]}\chi(k)\sum_{j=0}^{2\ell+3}\frac{B_j\left(\frac{k}{N}\right)B_{2\ell+3-j}\left(\frac{k}{N}\right)}{j!(2\ell+3-j)!}(-\pi i)^j\alpha^{\frac{3}{2}+\ell-j}.\end{aligned}$$

Corollary 3.8. *For any integer $M>0$,*

$$\begin{aligned} &\alpha^{M+\frac{1}{2}}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{n}\sigma_{2M}(\bar{\chi},n)e^{-2\alpha n/N}\\ =&(-\beta)^{M+\frac{1}{2}}\sum_{n=1}^{\infty}\chi(n)\zeta_N^{-n}\sigma_{2M}(\bar{\chi},n)e^{-2\beta n/N}.\end{aligned}$$

If p is a prime with $p\equiv 3 \pmod 4$, then we can put $\chi=\left(\frac{\cdot}{p}\right)$ in Theorem 3.7.

Corollary 3.9. *Let p be a prime with $p\equiv 3 \pmod 4$. Then for any integer $M>0$,*

$$\alpha^{2M+\frac{1}{2}}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\sigma_{4M}(\left(\frac{\cdot}{p}\right),n)\cos(2\pi n/p)e^{-2\alpha n/p}$$

$$= \beta^{2M+\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{4M}(\left(\frac{\cdot}{p}\right), n) \sin(2\pi n/p) e^{-2\beta n/p}.$$

Corollary 3.10. *Let p be a prime with $p \equiv 3 \pmod 4$. Then for any integer $M > 0$,*

$$\alpha^{2M-\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{4M-2}(\left(\frac{\cdot}{p}\right), n) \cos(2\pi n/p) e^{-2\alpha n/p}$$
$$= -\beta^{2M-\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \sigma_{4M-2}(\left(\frac{\cdot}{p}\right), n) \sin(2\pi n/p) e^{-2\beta n/p}.$$

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