

SLANT HELICES IN MINKOWSKI SPACE \mathbf{E}_1^3

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ABSTRACT. We consider a curve $\alpha = \alpha(s)$ in Minkowski 3-space \mathbf{E}_1^3 and denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the Frenet frame of α . We say that α is a slant helix if there exists a fixed direction U of \mathbf{E}_1^3 such that the function $\langle \mathbf{N}(s), U \rangle$ is constant. In this work we give characterizations of slant helices in terms of the curvature and torsion of α . Finally, we discuss the tangent and binormal indicatrices of slant curves, proving that they are helices in \mathbf{E}_1^3 .

1. Introduction and statement of results

Let \mathbf{E}_1^3 be the Minkowski 3-space, that is, \mathbf{E}_1^3 is the real vector space \mathbb{R}^3 endowed with the standard flat metric

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of \mathbf{E}_1^3 . An arbitrary vector $v \in \mathbf{E}_1^3$ is said spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$, and lightlike (or null) if $\langle v, v \rangle = 0$ and $v \neq 0$. The norm (length) of a vector v is given by $|v| = \sqrt{|\langle v, v \rangle|}$.

Given a regular (smooth) curve $\alpha : I \subset \mathbb{R} \rightarrow \mathbf{E}_1^3$, we say that α is spacelike (resp. timelike, lightlike) if $\alpha'(t)$ is spacelike (resp. timelike, lightlike) at any $t \in I$, where $\alpha'(t) = d\alpha/dt$. If α is spacelike or timelike we say that α is a non-null curve. In such case, we can reparametrize α by the arc-length $s = s(t)$, that is, $|\alpha'(s)| = 1$. We say then that α is arc-length parametrized. If the curve α is lightlike, the acceleration vector $\alpha''(t)$ must be spacelike for all t . We change the parameter t by $s = s(t)$ in such way that $|\alpha''(s)| = 1$ and we say that α is pseudo arc-length parametrized. In any of the above cases, we say that α is a unit speed curve.

Given a unit speed curve α in Minkowski space \mathbf{E}_1^3 it is possible to define a Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ associated for each point s [5, 7, 10]. Here \mathbf{T} , \mathbf{N} and \mathbf{B} are the tangent, normal and binormal vector field, respectively. The

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geometry of the curve α can be described by the differentiation of the Frenet frame, which leads to the corresponding Frenet equations.

In Euclidean space \mathbf{E}^3 a helix is a curve where the tangent lines make a constant angle with a fixed direction. Helices are characterized by the fact that the ratio τ/κ is constant along the curve, where τ and κ stands for the torsion and curvature of α , respectively [1]. In Minkowski space \mathbf{E}_1^3 , one defines a helix in Minkowski space in a similar way. Although different expressions of the Frenet equations appear depending on the causal character of the Frenet trihedron (see the next sections below), in many cases, a helix is characterized by the constancy of the function τ/κ again [2, 4, 9].

Recently, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean space. A *slant helix* is a curve whose normal lines make a constant angle with a fixed direction [3]. They characterize a slant helix if and only if the function

$$(1) \quad \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant. See also [6, 8]. Motivated by what happens in Euclidean ambient space, we give the next:

Definition 1.1. A unit speed curve α is called a slant helix if there exists a non-zero constant vector field U in \mathbf{E}_1^3 such that the function $\langle \mathbf{N}(s), U \rangle$ is constant.

It is important to point out that, in contrast to what happens in Euclidean space, in Minkowski ambient space we can not define the angle between two vectors (except that both vectors are of timelike type). For this reason, we avoid to say about the angle between the vector fields $\mathbf{N}(s)$ and U .

It is the aim of this paper to give the following characterization of slant helices in the spirit of the one given in Equation (1). We will assume throughout this work that the curvature and torsion functions do not equal zero.

Theorem 1.2. Let α be a unit speed timelike curve in \mathbf{E}_1^3 . Then α is a slant helix if and only if either one the next two functions

$$(2) \quad \frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \quad \text{or} \quad \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish.

Theorem 1.3. Let α be a unit speed spacelike curve in \mathbf{E}_1^3 .

- (a) If the normal vector of α is spacelike, then α is a slant helix if and only if either one the next two functions

$$(3) \quad \frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \quad \text{or} \quad \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish.

- (b) If the normal vector of α is timelike, then α is a slant helix if and only if the function

$$(4) \quad \frac{\kappa^2}{(\tau^2 + \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant.

- (c) Any spacelike curve with lightlike normal vector is a slant curve.

In the case that α is a lightlike curve, we have:

Theorem 1.4. Let α be a unit speed lightlike curve in \mathbf{E}_1^3 . Then α is a slant helix if and only if the torsion is

$$(5) \quad \tau(s) = \frac{a}{(bs + c)^2},$$

where a , b and c are constants, with $bs + c \neq 0$.

The proof of above theorems is carried in Section 2. In Section 3, we discuss the tangent and the binormal indicatrix of a slant helix, showing that they are helices in \mathbf{E}_1^3 . Also, we give characterizations of slant helices in terms of its involutes.

2. The proofs

In this section, we prove Theorems 1.2, 1.3 and 1.4 by distinguishing the causal character of the curve α .

2.1. Timelike slant helices

Let α be a unit speed timelike curve in \mathbf{E}_1^3 . The Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of α is given by

$$\mathbf{T}(s) = \alpha'(s), \quad \mathbf{N}(s) = \frac{\alpha''(s)}{|\alpha''(s)|}, \quad \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s),$$

where \times is the Lorentzian cross-product [5, 7]. The Frenet equations are

$$(6) \quad \begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix},$$

where κ and τ stand for the curvature and torsion of the curve, respectively. In order to prove Theorem 1.2, we first assume that α is a slant helix. Let U be the vector field such that the function $\langle \mathbf{N}(s), U \rangle = c$ is constant. There exist smooth functions a_1 and a_3 such that

$$(7) \quad U = a_1(s)\mathbf{T}(s) + c\mathbf{N}(s) + a_3(s)\mathbf{B}(s), \quad s \in I.$$

As U is constant, a differentiation in (7) together (6) gives

$$(8) \quad \begin{cases} a_1' + c\kappa & = 0 \\ \kappa a_1 - \tau a_3 & = 0 \\ a_3' + c\tau & = 0 \end{cases}$$

From the second equation in (8), we infer

$$(9) \quad a_1 = a_3 \left(\frac{\tau}{\kappa} \right).$$

Moreover

$$(10) \quad \langle U, U \rangle = -a_1^2 + c^2 + a_3^2 = \text{constant}.$$

We point out that this constraint, together the second and third equation of (8) is equivalent to the very system (8). Combining (9) and (10), let m be the constant given by

$$\epsilon m^2 := a_3^2 \left(\left(\frac{\tau}{\kappa} \right)^2 - 1 \right), \quad m > 0, \epsilon \in \{-1, 0, 1\}.$$

If $\epsilon = 0$, then $a_3 = 0$ and from (8) we have $a_1 = c = 0$. This means that $U = 0$: contradiction. Thus $\epsilon = 1$ or $\epsilon = -1$ and so $\tau^2 - \kappa^2 \neq 0$. This gives

$$a_3 = \pm \frac{m}{\sqrt{\left(\frac{\tau}{\kappa}\right)^2 - 1}} \quad \text{or} \quad a_3 = \pm \frac{m}{\sqrt{1 - \left(\frac{\tau}{\kappa}\right)^2}}$$

on I . The third equation in (8) yields

$$\frac{d}{ds} \left[\pm \frac{m}{\sqrt{\left(\frac{\tau}{\kappa}\right)^2 - 1}} \right] = c\tau \quad \text{or} \quad \frac{d}{ds} \left[\pm \frac{m}{\sqrt{1 - \left(\frac{\tau}{\kappa}\right)^2}} \right] = -c\tau$$

on I . This can be written as

$$\frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' = \pm \frac{c}{m} \quad \text{or} \quad \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' = \mp \frac{c}{m}.$$

This shows (2). Conversely, assume that the condition (2) is satisfied. In order to simplify the computations, we assume that the first function in (2) is a constant, namely, c (the other case is analogous). Define

$$(11) \quad U = \frac{\tau}{\sqrt{\tau^2 - \kappa^2}} \mathbf{T} + c\mathbf{N} + \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}} \mathbf{B}.$$

A differentiation of (11) together the Frenet equations gives $\frac{dU}{ds} = 0$, that is, U is a constant vector. On the other hand, $\langle \mathbf{N}(s), U \rangle = 1$ and this means that α is a slant helix. This concludes the proof of Theorem 1.2.

Remark 2.1. In Theorem 1.2 we need to assure that the function $\tau^2 - \kappa^2$ does not vanish everywhere. We do not know what happens if it vanishes at some points. On the other hand, any timelike curve that satisfies $\tau(s)^2 - \kappa(s)^2 = 0$ is a slant curve. The reasoning is the following. For simplicity, we only consider the case that $\tau = \kappa$. We define $U = \mathbf{T}(s) + \mathbf{B}(s)$, which is constant using the Frenet equations (6). Moreover, $\langle \mathbf{N}, U \rangle = 0$, that is, α is a slant curve. Finally, we point that there exist curves in \mathbf{E}_1^3 such that $\tau = \kappa = \text{const}$: the fundamental theorem of the theory of curves assures the existence of a timelike curve α with constant curvature and torsion.

2.2. Spacelike slant helices

Let α be a unit speed spacelike curve in \mathbf{E}_1^3 . In the case that the normal vector $\mathbf{N}(s)$ of α is spacelike or timelike, the proof of Theorem 1.3 is similar to the one given for Theorem 1.2. We omit the details.

The case that remains to study is that the normal vector $\mathbf{N}(s)$ of the curve is a lightlike vector for any $s \in I$. Now the Frenet trihedron is $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \mathbf{T}'(s)$ and $\mathbf{B}(s)$ is the unique lightlike vector orthogonal to $\mathbf{T}(s)$ such that $\langle \mathbf{N}(s), \mathbf{B}(s) \rangle = 1$. Then the Frenet equations as

$$(12) \quad \begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \tau(s) & 0 \\ -1 & 0 & \tau(s) \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}.$$

Here τ is the torsion of the curve (recall that $\tau(s) \neq 0$ for any $s \in I$). We show that *any* such curve is a slant helix. Let $a_2(s)$ be any non-trivial solution of the O.D.E. $y'(s) + \tau(s)y(s) = 0$ and define $U = a_2(s)\mathbf{N}(s)$. By using (12), $dU(s)/ds = 0$, that is, U is a (non-zero) constant vector field of \mathbf{E}_1^3 and, obviously, $\langle \mathbf{N}(s), U \rangle \equiv 0$: this shows that α is a slant helix.

2.3. Lightlike slant helices

In this section we show Theorem 1.4. Let α be a unit lightlike curve in \mathbf{E}_1^3 . The Frenet frame of α is $\mathbf{T}(s) = \alpha'(s)$; the vector normal $\mathbf{N}(s) = \mathbf{T}'(s)$ is spacelike and let $\mathbf{B}(s)$ be the unique lightlike vector orthogonal to $\mathbf{N}(s)$ such that $\langle \mathbf{T}(s), \mathbf{B}(s) \rangle = 1$. The Frenet equations are

$$(13) \quad \begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \tau(s) & 0 & -1 \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}.$$

Here $\tau(s)$ is the torsion of α , which is assumed with the property $\tau(s) \neq 0$ for any $s \in I$.

Assume that α is a slant helix. Let U be the constant vector field such that the function $\langle \mathbf{N}(s), U \rangle = c$ is constant. As in the above cases

$$U = a_1(s)\mathbf{T}(s) + c\mathbf{N}(s) + a_3(s)\mathbf{B}(s), \quad s \in I,$$

and

$$(14) \quad \begin{cases} a_1' + c\tau & = 0 \\ a_1 - \tau a_3 & = 0 \\ a_3' - c & = 0. \end{cases}$$

Then $a_3(s) = cs + m$, $m \in \mathbb{R}$ and $a_1 = (cs + m)\tau$. Using the first equation of (14), we have $(cs + m)\tau' + 2c\tau = 0$. The solution of this equation is

$$\tau(s) = \frac{n}{(cs + m)^2},$$

where m and n are constant. This proves (5) in Theorem 1.4.

Conversely, if the condition (5) is satisfied, we define

$$U = \frac{a}{bs+c} \mathbf{T}(s) + b\mathbf{N}(s) + (bs+c)\mathbf{B}(s).$$

Using the Frenet equations (13) we obtain that $dU(s)/ds = 0$, that is, U is a constant vector field of \mathbf{E}_1^3 . Finally, $\langle \mathbf{N}(s), U \rangle = b$ and this shows that α is a slant helix.

Corollary 2.2. *Let $\alpha : I \rightarrow \mathbf{E}_1^3$ be a unit speed curve.*

- (a) *Let α be a timelike curve or a spacelike curve with spacelike normal vector and $\kappa^2 - \tau^2 \neq 0$. Then α is a slant helix if and only if the curve $\beta : I \rightarrow \mathbf{E}_1^2$ defined by $\beta(s) = (\int \kappa ds, \int \tau ds)$ is a planar curve with constant curvature in the Lorentz-Minkowski plane \mathbf{E}_1^2 .*
- (b) *Let α be a spacelike curve with timelike normal vector and $\kappa^2 - \tau^2 \neq 0$. Then α is a slant helix if and only if the curve $\beta : I \rightarrow \mathbf{E}^2$ defined by $\beta(s) = (\int \kappa ds, \int \tau ds)$ is a circle in Euclidean plane \mathbf{E}^2 .*

Proof. In all case, the proof is similar. We do only the case that α is timelike and, without loss of generality, we assume that $\kappa^2 > \tau^2$. The curvature κ_β of β satisfies

$$-\kappa_\beta^2(s) = \frac{1}{|\beta'(s)|^6} \left(|\beta'(s)|^2 |\beta''(s)|^2 - \langle \beta'(s), \beta''(s) \rangle^2 \right).$$

Using the Lorentzian metric of \mathbf{E}_1^2 , we have the next computations:

$$|\beta'(s)|^2 = \kappa^2 - \tau^2, \quad |\beta''(s)|^2 = \kappa'^2 - \tau'^2, \quad \langle \beta'(s), \beta''(s) \rangle^2 = (\kappa\kappa' - \tau\tau')^2.$$

Then

$$\kappa_\beta(s) = \pm \frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'.$$

As conclusion of Theorem 1.2, the curvature function κ_β is constant if and only if α is a slant curve. \square

3. Indicatrices and involutes of a non-null slant helices

In this section, we investigate the tangent indicatrix and binormal indicatrix of a slant helix as well as its involutes. We restrict to non-null curves whose normal vector \mathbf{N} is spacelike or timelike. Thus we are considering timelike curves or spacelike curves with spacelike or timelike normal vector. Given a unit speed such curve $\alpha : I \rightarrow \mathbf{E}_1^3$, the indicatrix (resp. binormal indicatrix) is the curve $\mathbf{T} : I \rightarrow \mathbf{E}_1^3$ (resp. the curve $\mathbf{B} : I \rightarrow \mathbf{E}_1^3$), where \mathbf{T} (resp. \mathbf{B}) is the tangent vector (resp. binormal vector) to α . Here we follow ideas from [6]. Denote by $\mathbb{H}^2 = \{x \in \mathbf{E}_1^3; \langle x, x \rangle = -1\}$ the hyperbolic space and by $\mathbb{S}_1^2 = \{x \in \mathbf{E}_1^3; \langle x, x \rangle = 1\}$ the De Sitter space. If the image of a curve lies in \mathbb{H}^2 or \mathbb{S}_1^2 we say that the curve is *spherical*. In particular, the tangent indicatrix and the binormal indicatrix are spherical.

Theorem 3.1. *Let α be a unit speed timelike curve or a spacelike (with spacelike or timelike normal vector) curve. If α is a slant helix in \mathbf{E}_1^3 , then the tangent indicatrix \mathbf{T} of α is a (spherical) helix.*

Proof. We denote the curvature and the torsion of \mathbf{T} by κ_T and τ_T respectively. We are going to prove that the ratio τ_T/κ_T is constant, which shows that \mathbf{T} is a (spherical) helix in \mathbf{E}_1^3 . In order to compute κ_T and τ_T , we point out that the tangent indicatrix is not arc-length parametrized. In general and if $\beta(t)$ is non-parametrized by the arc-length curve, the corresponding formulae of the curvature and the torsion are (see, for example, [7]):

$$\kappa_\beta^2(t) = \epsilon \frac{|\beta'(t)|^2|\beta''(t)|^2 - \langle \beta'(t), \beta''(t) \rangle^2}{|\beta'(t)|^6}, \quad \tau_\beta(t) = -\epsilon \frac{\det(\beta'(t), \beta''(t), \beta'''(t))}{\kappa_\beta(t)^2|\beta'(t)|^6},$$

where ϵ is 1 or -1 depending on $\beta''(t)$ is a spacelike or timelike vector, respectively.

Consider that α is a curve with normal vector \mathbf{N} spacelike or timelike. Denote $\epsilon_0 = 1$ or -1 depending if \mathbf{N} is spacelike or timelike, respectively. Then the tangent indicatrix \mathbf{T} is a spacelike curve or a timelike curve. For both cases,

$$\kappa_T^2 = \frac{1}{\kappa^2}(\kappa^2 - \epsilon_0\tau^2), \quad \det(\mathbf{T}', \mathbf{T}'', \mathbf{T}''') = \epsilon_0\kappa^5\left(\frac{\tau}{\kappa}\right)', \quad \tau_T = \epsilon_0 \frac{\kappa\left(\frac{\tau}{\kappa}\right)'}{\kappa^2 - \epsilon_0\tau^2}.$$

In the case that α is a timelike curve, then \mathbf{T} is a spacelike curve and

$$\kappa_T^2 = -\frac{1}{\kappa^2}(\kappa^2 - \tau^2), \quad \det(\mathbf{T}', \mathbf{T}'', \mathbf{T}''') = -\kappa^5\left(\frac{\tau}{\kappa}\right)', \quad \tau_T = \frac{\kappa\left(\frac{\tau}{\kappa}\right)'}{\kappa^2 - \tau^2}.$$

As a conclusion,

$$\frac{\tau_T}{\kappa_T} = \epsilon_0 \frac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\left(\kappa^2 - \epsilon_0\tau^2\right)^{3/2}}, \quad \text{or} \quad \frac{\tau_T}{\kappa_T} = \frac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\left(\tau^2 - \kappa^2\right)^{3/2}}.$$

Taking into account Theorems 1.2 and 1.3 (formulae (2), (3) and (4)), the ratio τ_T/κ_T is constant. This shows that \mathbf{T} is a helix and we conclude the proof. \square

Theorem 3.2. *Let α be a unit speed timelike curve or a spacelike (with spacelike or timelike normal vector) curve. If α is a slant helix in \mathbf{E}_1^3 , then the binormal indicatrix \mathbf{B} of α is a (spherical) helix.*

Proof. The computations are similar as in Theorem 3.1. Again, we write κ_B and τ_B the curvature and torsion of the curve \mathbf{B} respectively. Consider α a spacelike curve. Then the binormal indicatrix \mathbf{B} is a timelike or a spacelike curve, depending if \mathbf{N} is spacelike or timelike, respectively.

$$\kappa_B^2 = \frac{\kappa^2 - \epsilon_0\tau^2}{\tau^2}, \quad \det(\mathbf{B}', \mathbf{B}'', \mathbf{B}''') = \epsilon_0\kappa^2\tau^3\left(\frac{\tau}{\kappa}\right)', \quad \tau_B = \frac{\kappa^2}{\tau(\kappa^2 - \epsilon_0\tau^2)}\left(\frac{\tau}{\kappa}\right)',$$

where $\epsilon_0 = 1$ or -1 depending if \mathbf{N} is spacelike or timelike, respectively.

If α is timelike, then \mathbf{B} is a spacelike curve. We have

$$\kappa_B^2 = \frac{\tau^2 - \kappa^2}{\tau^2}, \quad \det(\mathbf{B}', \mathbf{B}'', \mathbf{B}''') = \kappa^2 \tau^3 \left(\frac{\tau}{\kappa}\right)', \quad \tau_B = \frac{\kappa^2}{\tau(\tau^2 - \kappa^2)} \left(\frac{\tau}{\kappa}\right)'.$$

Therefore we have

$$\frac{\tau_B}{\kappa_B} = \delta(\tau) \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{\left(\kappa^2 - \epsilon_0 \tau^2\right)^{3/2}}, \quad \text{or} \quad \frac{\tau_B}{\kappa_B} = \delta(\tau) \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{\left(\kappa^2 - \tau^2\right)^{3/2}},$$

where δ is 1 or -1 depending if τ is positive or negative, respectively. Anyway, and by using (2), (3) and (4), both expressions are constants, which proves that the binormal indicatrix \mathbf{B} is a helix in \mathbf{E}_1^3 . \square

We end this section giving a characterization of a slant helix in terms of its involutes. Recall that if $\alpha : I \rightarrow \mathbf{E}_1^3$ is a curve, an involute of α is a curve $\beta : I \rightarrow \mathbf{E}_1^3$ such that for each $s \in I$ the point $\beta(s)$ lies on the tangent line to α at s and $\langle \alpha'(s), \beta'(s) \rangle = 0$. If α is a non-null curve, the equation of an involute is $\beta(s) = \alpha(s) + (c - s)\mathbf{T}(s)$, where c is a constant and \mathbf{T} is the unit tangent vector of α .

Theorem 3.3. *Let α be a unit speed timelike curve or a spacelike (with spacelike or timelike normal vector) curve. Let β be an involute of α . Then α is a slant helix if and only if β is a helix.*

Proof. Denote by κ_β and τ_β the curvature and the torsion of β , respectively.

If α is a timelike curve,

$$\kappa_\beta^2 = \frac{\tau^2 - \kappa^2}{\kappa^2(c - s)^2}, \quad \tau_\beta = -\frac{\kappa}{(c - s)(\tau^2 - \kappa^2)} \left(\frac{\tau}{\kappa}\right)'.$$

Then

$$(15) \quad \frac{\tau_\beta}{\kappa_\beta} = -\frac{\kappa^2}{(\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'.$$

If α is a spacelike curve,

$$\kappa_\beta^2 = \frac{\kappa^2 - \epsilon_0 \tau^2}{\kappa^2(c - s)^2}, \quad \tau_\beta = \frac{\kappa}{(c - s)(\kappa^2 - \epsilon_0 \tau^2)} \left(\frac{\tau}{\kappa}\right)'$$

and

$$(16) \quad \frac{\tau_\beta}{\kappa_\beta} = \frac{\kappa^2}{(\kappa^2 - \epsilon_0 \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'.$$

Here ϵ_0 is 1 or -1 depending if \mathbf{N} is a spacelike or a timelike vector, respectively. The proof finishes using (15), (16) and Theorems 1.2 and 1.3. \square

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