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SOME LIMIT THEOREMS RELATED TO MULTI-DIMENSIONAL DIFFUSIONS IN A RANDOM ENVIRONMENT

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Abstract. In this paper, we consider a multi-dimensional diffusion process in a self-similar random environment and prove a limit theorem for the shape of the full trajectory of the diffusion by using the localization phenomenon.

1. Introduction

Let $\mathcal{W}$ be the space of continuous functions on $\mathbf{R}^n$ vanishing at the origin and let Q be a probability measure on it. We call an element of $\mathcal{W}$ an environment. For given an environment w, let $\mathbf{P}_x^w$ be the probability measure on Ω, the canonical path space of real-valued continuous functions from $[0,\infty)$ to $\mathbf{R}^n$, such that $\{X(t), \mathbf{P}_x^w, x \in \mathbf{R}^n\}$ is a diffusion process with generator

$$\mathcal{L}^w = \frac{1}{2} e^w \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(e^{-w}\frac{\partial}{\partial x_i}\right). \tag{1}$$

It is well known that such a process $X(t)$ can be constructed from a Brownian motion by a drift transformation ([2]).

Let $\mathcal{P}_x$ be the probability measure on $\mathcal{W}\times\Omega$ defined by $\mathcal{P}_x(dwd\omega) = Q(dw)\mathbf{P}_x^w(d\omega)$. Then the diffusion $X(t)$ can be regarded as a stochastic process defined on the probability space $(\mathcal{W}\times\Omega, \mathcal{P}_x)$ and it is called as a *diffusion process in a random environment.* This process has a close connection with the model of Sinai's random walk and exhibits some interesting features ([1], [5], [8], [9], [10]). Among those, Brox [1] showed when $n = 1$ and $w(x)$ is a Brownian environment that there exists a nontrivial measurable function $b_1 : \mathcal{W} \mapsto \mathbf{R}$ such that for any $\varepsilon > 0$,

$$\mathcal{P}_x\left(\left|\alpha^{-2}X\left(e^\alpha\right) - b_1(w)\right| > \varepsilon\right) \longrightarrow 0 \quad \text{as } \alpha\to\infty \tag{2}$$

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which is the so-called subdiffusivity. This result was a consequence of a localization phenomenon, the diffusion is trapped in some valleys of its potential w, and was extended to a large class of random environments ([5]). However, all their methods are heavily rely on the one dimensional situation.

The first analogue result of (2) in higher dimensional cases was obtained by Mathieu [8]. The novelty of his work was the introduction of some analytic approaches such as asymptotics of the first non-zero eigenvalue of the generator (1) with a small noise and the related Dirichlet form theory. Motivated by this work, Tanaka [13] also proved that $\{X(t), \mathcal{P}_x, x \in \mathbf{R}^n\}$ is to be recurrent for all dimensions.

We are interested in the long time asymptotics for the shape of the full trajectory of the multi-dimensional diffusion $X(t)$ in a random environment. More precisely, let us consider the random set of trajectory of the diffusion $X(t)$ up to time t:

$$\mathcal{X}(t) := \overline{\{X(s) : 0 \leq s \leq t\}},$$

where $\overline{A}$ stands for the closure of a set A. We may call the process $\mathcal{X}(t)$ on the probability space $(\mathcal{W} \times \Omega, \mathcal{P}_x, x \in \mathbf{R}^n)$ a *set valued diffusion process in a random environment*. Our purpose in this article is to extend the result (2) of Brox to the case of set valued diffusion $\mathcal{X}(t)$.

To state our main theorem precisely, let $\mathcal{K}$ be a family of non-empty compact sets of $\mathbf{R}^n$ and $D_r(w)$ the connected component containing the origin of the sub-level domain $\{x \in \mathbf{R}^n : w(x) < r\}$. Let d_H be the Hausdorff distance on $\mathcal{K}$, that is, for $K_1, K_2 \in \mathcal{K}$

$$d_H(K_1, K_2) = \inf \{\varepsilon > 0 : U_\varepsilon(K_1) \supset K_2,\ U_\varepsilon(K_2) \supset K_1\},$$

where $U_\varepsilon(K)$ denotes the ε-neighborhood of K. Our result is based on some assumptions for the random characteristics of the environment. The main result of this article is stated as follows:

Theorem 1.1. *Let f be a probability density on $\mathbf{R}^n$ with compact support. Suppose that*

(**A.1**) *For any $\alpha > 0$ and any fixed $\lambda > 0$, the environment*

$$\{w_{\alpha,\lambda}(x) := \alpha^{-1} w(\alpha^\lambda x), x \in \mathbf{R}^n\}$$

has the same law as $\{w(x), x \in \mathbf{R}^n\}$ under Q, namely $(\mathcal{W}, Q)$ is a λ^{-1}-self-similar random environment.

(**A.2**) *For Q-a.s. $D_r(w)$ is bounded for any $r > 0$ and for Q-a.s. $w(x) \geq 0$ for all $x \in \mathbf{R}^n$.*

Then for any $r > 0$, $\varepsilon > 0$ and any fixed $\lambda > 0$,

$$\mathcal{P}_f \left(d_H \left(\alpha^{-\lambda} \mathcal{X}\left(e^{\alpha r}\right), \overline{D_r\left(w_{\alpha,\lambda}\right)} \right) > \varepsilon \right) \longrightarrow 0 \quad \text{as } \alpha \to \infty.$$

That is, the law of the random set $\alpha^{-\lambda}\mathcal{X}(e^{\alpha r})$ under $\mathcal{P}_f$ converges to the law of the compact set $\overline{D_r(w)}$ under Q as $\alpha \to \infty$.

The natural and typical examples of the random environment satisfying the assumptions (**A.1**) and (**A.2**) can be provided by $\mathcal{W} = \{w : w(x) = |B(x)|, x \in \mathbf{R}^n\}$, where $B(x)$ is a Lévy's Brownian motion with a n-dimensional time (cf. [8]) and $\mathcal{W} = \{w : w(x) = \sum_{i=1}^n |B(x_i)|, x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n\}$, where $\{B_i(x)\}_{1 \le i \le n}$ is a family of one dimensional Brownian motions which are mutually independent (cf. [12]).

Our theorem tells us that the shape of the $\mathcal{K}$-valued stochastic process $r^\lambda \mathcal{X}(t)$ converges very slowly to that of the connected component containing the origin of $\{x \in \mathbf{R}^n : w(x) < r\}$ by the effect of environment.

To prove our theorem, we address in Section 2 some notations and facts on a one parameter family of diffusion processes and their associated Dirichlet forms (for the general theory of Dirichlet forms, we refer readers to [2]). In Section 3, we obtain some asymptotic results on the first exit times of parameterized diffusions originally due to Mathieu [8] with some modifications (cf. [6], [7]). The proof of the main theorem will be given in Section 4. A key point of the proof is to use a modified environment to show that the parameterized diffusions enter any ball before leaving the domain $D_r(w)$.

2. Preliminaries on parameterized diffusions

For a given $w \in \mathcal{W}$, consider a symmetric closable form on $L^2(\mathbf{R}^n; e^{-\alpha w}dx)$ (energy form) parameterized by $\alpha > 0$:

$$\mathcal{E}^{\alpha w}(u, v) = \frac{1}{2} \int_{\mathbf{R}^n} \nabla u(x) \cdot \nabla v(x)\, e^{-\alpha w(x)} dx, \quad u, v \in C_0^1(\mathbf{R}^n), \tag{3}$$

where $C_0^1(\mathbf{R}^n)$ is the space of continuously differentiable functions with compact support in $\mathbf{R}^n$. Then $(\mathcal{E}^{\alpha w}, H^1(\mathbf{R}^n))$ becomes a regular (local) Dirichlet form defined by the smallest closed extension of (3) (see [2]). Here $H^1(\mathbf{R}^n) = \{u \in L^2(\mathbf{R}^n) : \partial_i u \in L^2(\mathbf{R}^n), 1 \le i \le n\}$. By a general theory of Dirichlet forms, there exists a diffusion process associated with $(\mathcal{E}^{\alpha w}, H^1(\mathbf{R}^n))$ and we denote it by $\{X(t), \mathbf{P}_x^{\alpha w}, x \in \mathbf{R}^n\}$. Let f be a probability density function on $\mathbf{R}^n$ and set

$$\mathbf{P}_f(\cdot) := \int_{\mathbf{R}^n} \mathbf{P}_x(\cdot) f(x) dx$$

for a probability distribution $\mathbf{P}_x$. Then $\{X(t), \mathbf{P}_f^{\alpha w}\}$ can be regarded as the diffusion process with an initial distribution f of the generator (1) replaced w by αw.

For a fixed $\lambda > 0$, we simply write as

$$w_\alpha(x) := \alpha^{-1} w(\alpha^\lambda x), \quad f_\alpha(x) := \alpha^{\lambda n} f(\alpha^\lambda x),\ x \in \mathbf{R}^n. \tag{4}$$

Let $\{R_\beta^w\}_{\beta > 0}$ be the resolvent of $\{X(t), \mathbf{P}_f^w\}$. Then for any $\varphi \in L^2(\mathbf{R}^n; e^{-w}dx)$ and $v \in H^1(\mathbf{R}^n)$,

$$\mathcal{E}_\beta^w \left(R_\beta^w \varphi, v\right) = \int_{\mathbf{R}^n} \varphi(x) v(x) e^{-w(x)} dx, \tag{5}$$

where $\mathcal{E}^w_\beta(\phi,\psi) := \mathcal{E}^w(\phi,\psi) + \beta\int_{\mathbf{R}^n}\phi(x)\psi(x)e^{-w(x)}dx$ for $\phi,\psi\in H^1(\mathbf{R}^n)$. By the change of variable with $x=\alpha^\lambda y$, the equation (5) yields that

$$\begin{aligned}\mathcal{E}^{\alpha w_\alpha}_{\alpha^{2\lambda}\beta}\left(R^w_\beta\varphi(\alpha^\lambda\cdot), v(\alpha^\lambda\cdot)\right) &= \int_{\mathbf{R}^n}\alpha^{2\lambda}\varphi(\alpha^\lambda y)v(\alpha^\lambda y)e^{-\alpha w_\alpha(y)}dy \\ &= \mathcal{E}^{\alpha w_\alpha}_{\alpha^{2\lambda}\beta}\left(R^{\alpha w_\alpha}_{\alpha^{2\lambda}\beta}(\alpha^{2\lambda}\varphi)(\alpha^\lambda\cdot), v(\alpha^\lambda\cdot)\right).\end{aligned}$$

So the fact that $\{\alpha^{2\lambda}R^{\alpha w_\alpha}_{\alpha^{2\lambda}\beta}\}_{\beta>0}$ is the resolvent of $\{\alpha^\lambda X(\alpha^{-2\lambda}t), \mathbf{P}^{\alpha w_\alpha}_{f_\alpha}\}$ implies the following lemma.

Lemma 2.1. *For $w\in\mathcal{W}$ and $\alpha>0$,*

$$\left\{\alpha^{-\lambda}X(t), \mathbf{P}^w_f\right\} \overset{d}{=} \left\{X(\alpha^{-2\lambda}t), \mathbf{P}^{\alpha w_\alpha}_{f_\alpha}\right\}, \tag{6}$$

where $\overset{d}{=}$ means the equality in distribution.

For any $r>0$, let $r(\alpha) := r-(2\lambda/\alpha)\log\alpha$. Clearly $r(\alpha)\to r$ as $\alpha\to\infty$. Then (6) can be rewritten as

$$\left\{\alpha^{-\lambda}X(e^{\alpha r}), \mathbf{P}^w_f\right\} \overset{d}{=} \left\{X(e^{\alpha r(\alpha)}), \mathbf{P}^{\alpha w_\alpha}_{f_\alpha}\right\} \tag{7}$$

by taking $t=e^{\alpha r}$.

Remark 2.2. Under the assumption (**A.1**), $\mathbf{P}^{\alpha w_\alpha}_\cdot$ and $\mathbf{P}^{\alpha w}_\cdot$ are same for Q-a.s.. Therefore we see from (7) that a subdiffusivity problem of the diffusion $X(t)$ is reformulated as an α-asymptotic problem of the parameterized diffusion $\{X(e^{\alpha r(\alpha)}), \mathcal{P}^\alpha_{f_\alpha}\}$. Here $\mathcal{P}^\alpha_x(dwd\omega) := Q(dw)\mathbf{P}^{\alpha w}_x(d\omega)$.

Let D be an arbitrary bounded domain in $\mathbf{R}^n$. We introduce the so-called 3-*depths of* D relative to an environment (cf. [4], [8]): the depths of D,

$$d = d(D,w) = \inf_{\partial D} w - \inf_D w,\quad d' = d'(D,w) = \sup_D w - \inf_D w$$

and the critical depth (or elevation) of D,

$$c = c(D,w) = \sup_{x,y\in D}\left\{\inf_\phi \sup_{t\in[0,1]} w(\phi(t)) - w(x) - w(y)\right\} + \inf_D w,$$

where ϕ is a continuous path from $[0,1]$ to D such that $\phi(0)=x, \phi(1)=y$. Note that the quantities d', d and c depend only on the landscape of w and do not depend on any boundary conditions (such as smoothness) of ∂D. Moreover, it is easy to check that if $D\equiv D_r(w)$ $(r>0)$, then $d'=d$ and $d>c$.

Let $H^1(D) = \{u\in L^2(D) : \partial_i u\in L^2(D), 1\le i\le n\}$ and denotes $H^1_0(D)$ by the closure of $C^1_0(D)$ in $H^1(D)$. It is well known that the absorbing process of the diffusion $\{X(t), \mathbf{P}^{\alpha w}_x, x\in\mathbf{R}^n\}$ on D is by definition the diffusion process $\{X(t), \mathbf{P}^{\alpha w, D}_x, x\in D\}$ whose regular Dirichlet form on $L^2(D; e^{-\alpha w}dx)$ is coincided with $(\mathcal{E}^{\alpha w, D}, H^1_0(D))$, where

$$\mathcal{E}^{\alpha w, D}(u,v) = \frac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)\, e^{-\alpha w(x)}dx$$

([2]). On the other hand, by defining $H^1_*(D) := (H^1_0(D) + \text{constant})$ as a new Sobolev space, $(\mathcal{E}^{\alpha w,D}, H^1_*(D))$ becomes a regular Dirichlet form on $L^2(D^*; e^{-\alpha w}dx)$ as well ([6]). In fact, this form is the smallest one in the family of Dirichlet forms $\mathcal{M} = \{(\mathcal{E}^{\alpha w,D}, \mathcal{F}^D) : H^1_0(D) \subset \mathcal{F}^D \subseteq H^1(D),\ 1 \in \mathcal{F}^D\}$. Here D^* denotes the one-point compactification of D. Let us denote by $\{X(t), \mathbf{P}^{\alpha w,*}_x, x \in D^*\}$ the diffusion process associated with $(\mathcal{E}^{\alpha w,D}, H^1_*(D))$. This process is irreducible and recurrent on D^*. More general profound properties on this kind of process and its associated Dirichlet form can be found in [3].

For a notational convenience, we set

$$u(\alpha) \succ a \text{ if } \liminf_{\alpha\to\infty} \frac{1}{\alpha} \log u(\alpha) \geq a, \quad u(\alpha) \prec a \text{ if } \limsup_{\alpha\to\infty} \frac{1}{\alpha} \log u(\alpha) \leq a.$$

Let $m_\alpha(dx)$ be the normalized underlying measure defined by

$$m_\alpha(dx) = I_D Z_\alpha^{-1} e^{-\alpha w(x)} dx, \quad Z_\alpha = \int_D e^{-\alpha w(x)} dx.$$

Note that $(1/\alpha)\log Z_\alpha \to -\inf_D w$ as $\alpha \to \infty$. Let $\gamma(\alpha)$ be the first non-zero eigenvalue of $(\mathcal{E}^{\alpha w,D}, H^1_*(D))$, the so-called spectral gap, defined by

$$\gamma(\alpha) = \inf_{u \in H^1_*(D)} \frac{\mathcal{E}^{\alpha w,D}(u,u)}{\int_D (u - \int_D u\, dm_\alpha)^2 e^{-\alpha w} dx}.$$

The following asymptotic lower bound of $\gamma(\alpha)$ for an arbitrary bounded domain D of $\mathbf{R}^n$ was obtained by [6].

Lemma 2.3. $\gamma(\alpha) \succ -c$.

3. Lemmas on exit times

Let $\tau_D^{\alpha w}$ be the first exit time of the diffusion $X(t)$ out of D, that is, $\tau_D^{\alpha w} := \inf\{t > 0 : X(t) \notin D\}$. For any $\beta > 0$, set $h_\beta^{\alpha w}(x) := \mathbf{E}_x^{\alpha w,*}(\exp(-\beta \tau_D^{\alpha w}))$ the Laplace transform of $\tau_D^{\alpha w}$, where $\mathbf{E}_x^{\alpha w,*}$ denotes the expectation relative to $\{X(t), \mathbf{P}^{\alpha w,*}_x, x \in D^*\}$.

In this section, we consider the asymptotics in α of the distribution of $\tau_D^{\alpha w}$ studied in [6], [8] under some modifications. We note that no additional condition is imposed (smoothness as like in [8]) on a domain $D \subset \mathbf{R}^n$, and it makes no difference to replace the diffusion $\{X(t), \mathbf{P}^{\alpha w}_x, x \in \mathbf{R}^n\}$ by $\{X(t), \mathbf{P}^{\alpha w,*}_x, x \in D^*\}$ (or $\{X(t), \mathbf{P}^{\alpha w,D}_x, x \in D\}$) as far as the exit time distribution from D is concerned.

Lemma 3.1. *Let $\theta(\alpha)$ be the function such that $\int_D h^{\alpha w}_{\theta(\alpha)} dm_\alpha = 1/2$.*

(i) $-d' \prec \theta(\alpha) \prec -d$.

(ii) *Assume $d > c$. Then for any $k > 0$, $\int_D h^{\alpha w}_{k\theta(\alpha)} dm_\alpha \to 1/(1+k)$ as $\alpha \to \infty$. In particular, set $D \equiv D_r(w)$ $(r > 0)$. Then for any open ball*

B such that $\overline{B} \subset D$,

$$\left\| h^{\alpha w}_{k\theta(\alpha)} - \frac{1}{1+k} \right\|_{L^2(B)} \longrightarrow 0 \tag{8}$$

as $\alpha \to \infty$.

Proof. We may assume that $d > 0$ (otherwise the present lemma is trivial). Note that it follows from the proof of Theorem 3.1 in [6] that for any $\varepsilon > 0$,

$$\int_D h^{\alpha w}_{e^{-\alpha(d-\varepsilon)}} dm_\alpha \longrightarrow 0 \quad \text{as } \alpha \to \infty. \tag{9}$$

On the other hand, for the absorbing diffusion process $\{X(t), \mathbf{P}^{\alpha w, D}_x, x \in D\}$, put $u^D_\alpha(t,x) = \mathbf{P}^{\alpha w, D}_x(t < \tau^{\alpha w}_D)$. Then by definition,

$$\begin{aligned} \mathcal{E}^{\alpha w, D}\left(u^D_\alpha(t,\cdot), u^D_\alpha(t,\cdot)\right) &= \lim_{s \to 0} \left(\frac{u^D_\alpha(t,\cdot) - u^D_\alpha(s+t,\cdot)}{s}, u^D_\alpha(t,\cdot) \right)_{e^{-\alpha w} dx} \\ &= -\frac{1}{2} \int_D \frac{\partial}{\partial t} \left| u^D_\alpha(t,x) \right|^2 e^{-\alpha w(x)} dx \\ &= -\frac{1}{2} \frac{d}{dt} H^D_\alpha(t), \end{aligned} \tag{10}$$

where $(\cdot,\cdot)_\mu$ denotes the inner product on $L^2(D;\mu)$ and

$$H^D_\alpha(t) = \int_D \left| u^D_\alpha(t,x) \right|^2 e^{-\alpha w(x)} dx.$$

Let denote $(\frac{1}{2}\mathbf{D}, H^1_0(D))$ by the Dirichlet form $(\mathcal{E}^{\alpha w, D}, H^1_0(D))$ when $\alpha = 0$. By the boundedness of D, it is transient (Example 1.5.3 in [2]) and thus for any $u \in H^1_0(D)$, it holds that

$$\int_D u(x)^2 dx \leq 2 \| R^D 1 \|_\infty \mathbf{D}(u,u),$$

where R^D is the Green operator of $(\frac{1}{2}\mathbf{D}, H^1_0(D))$ ([11]). Using this, we have

$$H^D_\alpha(t) \leq 2 \| R^D 1 \|_\infty e^{\alpha d'} \mathcal{E}^{\alpha w, D}\left(u^D_\alpha(t,\cdot), u^D_\alpha(t,\cdot)\right). \tag{11}$$

By combining (10) and (11),

$$e^{-\alpha d'} \| R^D 1 \|_\infty^{-1} \leq -\frac{d}{dt} \log H^D_\alpha(t)$$

that implies

$$H^D_\alpha(t) \leq H^D_\alpha(0) \exp\left(-\| R^D 1 \|_\infty^{-1} e^{-\alpha d'} t \right). \tag{12}$$

Take $r' > 0$ such that $r' \in (d', d'+\varepsilon)$ for any $\varepsilon > 0$. Applying $t = e^{\alpha r'}$ to (12), we see then

$$\begin{aligned} \mathbf{P}^{\alpha w,*}_{m_\alpha}\left(e^{\alpha r'} < \tau^{\alpha w}_D\right) &= \mathbf{P}^{\alpha w,D}_{m_\alpha}\left(e^{\alpha r'} < \tau^{\alpha w}_D\right) \\ &= Z_\alpha^{-1}\int_D u^D_\alpha\left(e^{\alpha r'}, x\right)e^{-\alpha w(x)}dx \\ &\le e^{\alpha(\inf_D w+\varepsilon)}H^D_\alpha\left(e^{\alpha r'}\right)^{1/2} \longrightarrow 0 \quad \text{as } \alpha\to\infty, \end{aligned}$$

which implies that

$$\int_D h^{\alpha w}_{e^{-\alpha(d'+\varepsilon)}}dm_\alpha \longrightarrow 1 \quad \text{as } \alpha\to\infty. \tag{13}$$

Now, (i) of the lemma is a consequence of (9) and (13).

The first assertion of (ii) was proved in Theorem [6] (also, in Theorem II.1 in [8] related to the reflected Dirichlet form $(\mathcal{E}^{\alpha w,D}, H^1(D))$ under the smoothness of D). Finally, we prove the last assertion of (ii). The idea of the proof is originally due to [8], but we need to improve some technical tools in our settings (D is not smooth). Let B be an open ball such that $\overline{B}\subset D\equiv D_r(w)$ $(r>0)$ and $r_B := \max_{\overline{B}} w \ \in (0,d)$. Let B_1 be an open ball such that $B_1\subset B$ and $r_{B_1} = \max_{\overline{B}_1} w \in (0, d-c)$. Then for any $\varepsilon\in(0, d-c-r_{B_1})$,

$$\begin{aligned} &\left\| h^{\alpha w}_{k\theta(\alpha)} - \int_D h^{\alpha w}_{k\theta(\alpha)}dm_\alpha\right\|^2_{L^2(B_1)} \\ &\le e^{\alpha r_{B_1}}\left\| h^{\alpha w}_{k\theta(\alpha)} - \int_D h^{\alpha w}_{k\theta(\alpha)}dm_\alpha\right\|^2_{L^2(B_1; e^{-\alpha w}dx)} \\ &\le \gamma(\alpha)^{-1}e^{\alpha r_{B_1}}\mathcal{E}^{\alpha w,D}_{k\theta(\alpha)}\left(h^{\alpha w}_{k\theta(\alpha)}, h^{\alpha w}_{k\theta(\alpha)}\right) \\ &\le \ \prec\ -(d-c-r_{B_1}-\varepsilon) \end{aligned} \tag{14}$$

by virtue of Lemma 2.3, Lemma 3.1(i) and the fact that for $\beta > 0$

$$\mathcal{E}^{\alpha w,D}_\beta(h^{\alpha w}_\beta, h^{\alpha w}_\beta) = \mathcal{E}^{\alpha w,D}_\beta(h^{\alpha w}_\beta, 1) = \beta\int_D h^{\alpha w}_\beta e^{-\alpha w}dx \ \le\ \beta.$$

Therefore we see that (8) holds for the open ball B_1 by combining (14) and the first assertion of (ii). On the other hand, by Poincaré inequality related to $H^1(B)$ for the open ball B, there exists a constant γ_B^{-1} such that for any $\varepsilon\in(0, d-r_B)$,

$$\begin{aligned} \left\| h^{\alpha w}_{k\theta(\alpha)} - \left\langle h^{\alpha w}_{k\theta(\alpha)}\right\rangle_B\right\|^2_{L^2(B)} &\le \gamma_B^{-1}\int_B\left|\nabla h^{\alpha w}_{k\theta(\alpha)}\right|^2dx \\ &\le 2\gamma_B^{-1}e^{\alpha r_B}\mathcal{E}^{\alpha w,D}_{k\theta(\alpha)}\left(h^{\alpha w}_{k\theta(\alpha)}, h^{\alpha w}_{k\theta(\alpha)}\right) \\ &\prec -(d-r_B-\varepsilon), \end{aligned} \tag{15}$$

where $\langle h\rangle_B := \int_B h\,dx/\int_B dx$. Hence, for the open balls B_1 and B,

$$\left\| h^{\alpha w}_{k\theta(\alpha)} - \frac{1}{1+k} \right\|_{L^2(B_1)} \longrightarrow 0, \quad \left\| h^{\alpha w}_{k\theta(\alpha)} - \left\langle h^{\alpha w}_{k\theta(\alpha)} \right\rangle_B \right\|_{L^2(B_1)} \longrightarrow 0$$

as $\alpha \to \infty$ and thus

$$\left\langle h^{\alpha w}_{k\theta(\alpha)} \right\rangle_B \longrightarrow \frac{1}{1+k} \quad \text{as } \alpha \to \infty. \tag{16}$$

Now, applying (16) to (15), we conclude that (8) also holds for any open ball B such that $\overline{B} \subset D$. □

Lemma 3.2. *Let f be a probability density function on $D \equiv D_r(w)$ $(r > 0)$ and let f_α be the scaled function of f defined in* (4)*. Then for any $k > 0$,*

$$\int_D h^{\alpha w}_{k\theta(\alpha)} df_\alpha \longrightarrow \frac{1}{1+k} \quad \text{as } \alpha \to \infty. \tag{17}$$

Proof. Let B_0 be an open ball centered at 0 satisfying $\overline{B}_0 \subset D$. In view of the proof of the last assertion of Lemma 3.1(ii), we see that (8) holds for B_0 and its speed of decay is exponential. Note that the support of f_α is contained in B_0 for large enough α. So by the similar argument in the proof of Theorem II.3 in [8], we have

$$\begin{aligned} &\left\| h^{\alpha w}_{k\theta(\alpha)} - \frac{1}{1+k} \right\|_{L^2(D; f_\alpha dx)} \\ &\le 4 \int_{\{f \ge N\}} f(x)\,dx + \alpha^{\lambda n} N \left\| h^{\alpha w}_{k\theta(\alpha)} - \frac{1}{1+k} \right\|_{L^2(B_0)} \longrightarrow 0 \end{aligned}$$

by letting $\alpha \to \infty$ and $N \to \infty$. This ends the proof of the lemma. □

4. Proof of Theorem 1.1

In what follows, let d_H be the Hausdorff distance on a family of non-empty compact sets $\mathcal{K}$ of $\mathbf{R}^n$. Clearly, $\overline{D_r(w)}$ (for fixed w) is a non-decreasing set valued function and is an element of $\mathcal{K}$ for any $r > 0$.

Lemma 4.1. *For Q-a.s., $\overline{D_\cdot(w)}$ is continuous on $(0, \infty)$ with respect to d_H.*

Proof. First, we prove that (for fixed w) $\overline{D_\cdot(w)}$ is left continuous on $(0,\infty)$ with respect to d_H. To do this, it suffices to show that for any $\varepsilon > 0$, there exists $s \in (0, r)$ such that $U_\varepsilon(\overline{D_s(w)}) \supset \overline{D_r(w)}$. Set

$$\ell_s(x) := \inf_{y \in \overline{D_s(w)}} |x - y|, \quad x \in \overline{D_r(w)}.$$

Then $\ell_s(\cdot)$ is a continuous function on $\overline{D_r(w)}$. Moreover, $\lim_{s\uparrow r} \ell_s(x) = 0$ for any $x \in \overline{D_r(w)}$. Indeed, by the connectedness of $D_r(w)$, there exists a continuous path $\phi : [0,1] \to D_r(w)$ such that $\phi(0) = 0$ and $\phi(1) = x$ for $x \in D_r(w)$. For this ϕ, we can choose $s > 0$ such that $s \in (\sup_{t\in[0,1]} w(\phi(t)), r)$

and $x \in D_s(w)$. Therefore we see that $\ell_s(x)$ converges uniformly to 0 on $\overline{D_r(w)}$ as $s \uparrow r$ and consequently, there exists $s \in (0, r)$ such that $\ell_s(x) < \varepsilon$ ($\varepsilon > 0$) for all $x \in \overline{D_r(w)}$. Now, we prove the assertion of the lemma. Let $J(w)$ be the set of discontinuous points of $\overline{D_r(w)}$. By the left continuity of $\overline{D_\cdot(w)}$, $J(w)$ is a denumerable set. Therefore $\int_{J(w)} dr = 0$ and

$$\int_0^\infty Q\,(r \in J(w))\,dr = \mathbf{E}^Q \left(\int_{J(w)} dr \right) = 0, \tag{18}$$

where $\mathbf{E}^Q$ denotes the expectation related to $(\mathcal{W}, Q)$. Since for any $\alpha > 0$ and $r > 0$

$$D_r(w_\alpha) = \alpha^{-\lambda} D_{\alpha r}(w),$$

the λ^{-1}-self-similarity of the environment implies that

$$Q(r \in J(w)) = Q(r \in J(w_\alpha)) = Q(\alpha r \in J(w)) = 0.$$

Hence we see that $Q(r \in J(w)) = 0$ does not depend on $r > 0$ and we obtain the desired result. □

Lemma 4.2. *Let f, f_α and D be the same as in Lemma 3.2. Let $r(\alpha)$ be a function such that $r(\alpha) \to r$ $(r > 0)$ as $\alpha \to \infty$. Then*

$$\left| \mathbf{P}_{f_\alpha}^{\alpha w,*} \left(X(e^{\alpha r(\alpha)}) \in B \right) - m_\alpha(B) \right| \longrightarrow 0 \quad \text{as } \alpha \to \infty$$

for any open ball B such that $\overline{B} \subset D$.

Proof. Note that it makes no difference to replace an environment w by $w - \inf_D w$ as far as the critical depth c, diffusion $\{X(t), \mathbf{P}_x^{\alpha w,*}, x \in D^*\}$ and the associated spectral gap $\gamma(\alpha)$ are considered. So we may assume $\inf_D w = 0$ without loss of generality. Let $\{p_t^{\alpha w,*}\}_{t>0}$ be the $L^2(D; m_\alpha)$-semigroup of $\{X(t), \mathbf{P}_x^{\alpha w,*}, x \in D^*\}$ and $\{F_\gamma^{\alpha w,*}\}$ the associated spectral family of $\{p_t^{\alpha w,*}\}_{t>0}$, that is, $p_t^{\alpha w,*} = \int_0^\infty e^{-\gamma t} dF_\gamma^{\alpha w,*}$. Take a large enough $\alpha > 0$ satisfying $c < r(\alpha)$. For an open ball B such that $\overline{B} \subset D$, define the function g on D by $g(x) = I_B(x) - m_\alpha(B)$. Then

$$\begin{aligned} &\left| \mathbf{P}_{f_\alpha}^{\alpha w,*}(X(e^{\alpha r(\alpha)}) \in B) - m_\alpha(B) \right| \\ &= \int_D \left| \mathbf{E}_x^{\alpha w,*} \left(g(X(e^{\alpha r(\alpha)})) \right) \right| f_\alpha(x)\,dx \\ &\le \int_{\{f \ge N\}} f(x)\,dx + \alpha^{\lambda n} N \int_D \left| p_{e^{\alpha r(\alpha)}}^{\alpha w,*} g(x) \right| dx \\ &\le \int_{\{f \ge N\}} f(x)\,dx + \alpha^{\lambda n} N Z_\alpha e^{\alpha r} \left\| p_{e^{\alpha r(\alpha)}}^{\alpha w,*} g \right\|_{L^2(D; m_\alpha)}. \end{aligned} \tag{19}$$

On the other hand, by Lemma 2.3 and the fact that $\|g\|_{L^1(D;m_\alpha)} = 0$,

$$\begin{aligned}\left\|p^{\alpha w,*}_{e^{\alpha r(\alpha)}} g\right\|^2_{L^2(D;m_\alpha)} &= \int_{\gamma(\alpha)}^{\infty} \exp\left(-2\gamma e^{\alpha r(\alpha)}\right) d\left(F^{\alpha w,*}_{\gamma} g, g\right)_{L^2(D;m_\alpha)} \\ &\leq \exp(-2\gamma(\alpha)e^{\alpha r(\alpha)}\|g\|^2_{L^2(D;m_\alpha)} \\ &\leq \exp\left(-2e^{\alpha(r(\alpha)-c-\varepsilon)}\right)\end{aligned}$$

for any $\varepsilon \in (0, r(\alpha) - c)$. Applying this relation to (19), we obtain the lemma by letting $\alpha \to \infty$ and $N \to \infty$. □

Now, we are prepared to prove our main theorem.

Proof of Theorem 1.1. In view of (7) and its related remark mentioned right after, it holds that

$$\left\{\alpha^{-2}\mathcal{X}(e^{\alpha r}), \mathcal{P}_f\right\} \stackrel{d}{=} \left\{\mathcal{X}(e^{\alpha r(\alpha)}), \mathcal{P}^{\alpha}_{f_\alpha}\right\},$$

where $r(\alpha) = r - (2\lambda/\alpha)\log\alpha$, $\mathcal{P}^{\alpha}_{x}(dwd\omega) = Q(dw)\mathbf{P}^{\alpha w}_{x}(d\omega)$ and f_α is the scaled function of f defined in (4). Using this, we shall prove that for any $\varepsilon > 0$

$$\mathcal{P}^{\alpha}_{f_\alpha}\left(d_H\left(\mathcal{X}(e^{\alpha r(\alpha)}), \overline{D_r(w)}\right) > \varepsilon\right) \longrightarrow 0 \quad \text{as } \alpha \to \infty.$$

First, take ε' such that $\varepsilon' \in (0, r - r(\alpha))$ and apply $k = e^{\alpha\varepsilon'}$ to (17). Then by Lemma 3.1(i), we have

$$\begin{aligned}\mathbf{P}^{\alpha w}_{f_\alpha}\left(\tau^{\alpha w}_{D_r(w)} < e^{\alpha r(\alpha)}\right) &= \mathbf{P}^{\alpha w,*}_{f_\alpha}\left(\tau^{\alpha w}_{D_r(w)} < e^{\alpha r(\alpha)}\right) \\ &\leq e\int_{D_r(w)} h^{\alpha w}_{e^{\alpha\varepsilon'} e^{-\alpha(r(\alpha)+\varepsilon')}} f_\alpha dx \\ &\leq e\int_{D_r(w)} h^{\alpha w}_{e^{\alpha\varepsilon'}\theta(\alpha)} f_\alpha dx \longrightarrow 0 \quad \text{as } \alpha \to \infty\end{aligned}$$

which implies that for any $\varepsilon > 0$, $\mathcal{X}(e^{\alpha r(\alpha)}) \subset U_\varepsilon(\overline{D_r(w)})$, $\mathcal{P}^{\alpha}_{f_\alpha}$-a.s. as $\alpha \to \infty$. Now, it remains to prove that under $\mathcal{P}^{\alpha}_{f_\alpha}$,

$$\overline{D_r(w)} \subset U_\varepsilon(\mathcal{X}(e^{\alpha r(\alpha)})) \quad \text{as } \alpha \to \infty. \tag{20}$$

To this end, let $B(x)$ be an open ball of $D_r(w)$ centered at x with radius $\varepsilon''/2$, where $x \in \overline{D_{r-\varepsilon''/2}(w)}$ and $\varepsilon'' \in (0, \varepsilon)$. By the same reason in the proof of Lemma 4.2, we may assume that $\inf_{\overline{D_r(w)}} w = 0$. Consider a modified environment $\widetilde{w}(x)$ of $w(x)$ on $\overline{D_r(w)}$ relative to $B(x)$ defined as follows:

$$\widetilde{w}(x) = w(x) \text{ on } B(x)^c, \quad \widetilde{w}(x) \leq w(x) \text{ on } \overline{B(x)}$$

and

$$\inf_{x\in\overline{B(x)}} \widetilde{w}(x) = -\delta \ (\delta > 0).$$

For this $\widetilde{w}$, let consider the Dirichlet form $(\mathcal{E}^{\alpha\widetilde{w}}, H^1_*(D_r(w)))$, the associated diffusion $\{X(t), \mathbf{P}_x^{\alpha\widetilde{w},*}, x \in D_r(w)^*\}$, the normalized underlying measure $\widetilde{m}_\alpha$ on $D_r(w)$, the depth $\widetilde{d}$ and the critical depth $\widetilde{c}$ of $D_r(w)$:

$$\widetilde{d} = d(D_r(w), \widetilde{w}), \qquad \widetilde{c} = c(D_r(w), \widetilde{w}) = \sup_{x,y\in D_r(w)} c_{x,y}(\widetilde{w})$$

with

$$c_{x,y}(\widetilde{w}) = \inf_{\phi} \sup_{t\in[0,1]} \widetilde{w}(\phi(t)) - \widetilde{w}(x) - \widetilde{w}(y) - \delta,$$

in a similar way of Section 2. Then since $D_r(w)$ is also a sub-level domain of $\widetilde{w}$, it is easy to check that $\widetilde{m}_\alpha(B(x)) \to 1$ as $\alpha \to \infty$ and

$$\widetilde{c} < r + \delta. \tag{21}$$

In particular, we claim that (21) can be regarded as $\widetilde{c} < r$ by choosing sufficiently small $\delta > 0$. Indeed, let $\widetilde{E}(r_0)$ be a connected component of the level set $\{x \in D_r(w) : \widetilde{w}(x) < r_0,\ \sup_{\overline{B(x)}} w < r_0 < r\}$ containing $\overline{B(x)}$. Then,

$$\widetilde{c}_1 := \sup_{x\in D_r(w)\setminus\widetilde{E}(r_0),\ y\in\widetilde{E}(r_0)} c_{x,y}(\widetilde{w}) \quad \text{and} \quad \widetilde{c}_2 := \sup_{x,y\in D_r(w)\setminus\widetilde{E}(r_0)} c_{x,y}(\widetilde{w})$$

are strictly less than r by the definition of the critical depth. On the other hand, since $\widetilde{E}(r_0)$ is also a sub-level domain of $\widetilde{w}$,

$$\widetilde{c}_3 := \sup_{x,y\in\widetilde{E}(r_0)} c_{x,y}(\widetilde{w}) = c(\widetilde{E}(r_0), \widetilde{w}) < d(\widetilde{E}(r_0), \widetilde{w}) = r_0 + \delta$$

and thus, $\widetilde{c}_3$ is also strictly less than r by choosing the sufficiently small $\delta > 0$. Noting $\widetilde{c} = \max\{\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3\}$ we conclude that the claim is true. Therefore we see that $\widetilde{c} < r(\alpha)$ for sufficiently large $\alpha > 0$ and

$$\mathbf{P}_{f_\alpha}^{\alpha\widetilde{w},*}\left(\sigma_{B(x)}^{\alpha\widetilde{w}} < e^{\alpha r(\alpha)}\right) \ge \mathbf{P}_{f_\alpha}^{\alpha\widetilde{w},*}\left(X\left(e^{\alpha r(\alpha)}\right) \in B(x)\right) \longrightarrow 1 \quad \text{as } \alpha \to \infty$$

by virtue of Lemma 4.2. Here $\sigma_B^{\alpha\widetilde{w}}$ denotes the first hitting time of the diffusion $X(t)$ to B. Since the processes $\{X(t), \mathbf{P}_x^{\alpha w}\}$ and $\{X(t), \mathbf{P}_x^{\alpha\widetilde{w}}\}$ have the same law on $B(x)^c$,

$$\mathbf{P}_{f_\alpha}^{\alpha w}\left(\sigma_{B(x)}^{\alpha w} < e^{\alpha r(\alpha)}\right) = \mathbf{P}_{f_\alpha}^{\alpha\widetilde{w},*}\left(\sigma_{B(x)}^{\alpha\widetilde{w}} < e^{\alpha r(\alpha)}\right) \longrightarrow 1 \quad \text{as } \alpha \to \infty$$

which implies that $B(x) \subset U_\varepsilon(\mathcal{X}(e^{\alpha r(\alpha)}))$, $\mathbf{P}_{f_\alpha}^{\alpha w}$-a.s.. Hence we have

$$\overline{D_{r-\varepsilon''}(w)} \subset U_\varepsilon(\mathcal{X}(e^{\alpha r(\alpha)})) \quad \text{as } \alpha \to \infty$$

under $\mathcal{P}_{f_\alpha}^{\alpha}$. Letting $\varepsilon'' \to 0$, we can derive (20) from Lemma 4.1. □

Remark 4.3. (i) Excepting the one dimensional case, we do not know how to determine the compact set $K(w, r) \subset \mathbf{R}^n$ such that under $\mathcal{P}_f$, $\alpha^{-\lambda}\mathcal{X}(e^{\alpha r})$ converges in probability to $K(w, r)$ as $\alpha \to \infty$ without the second assumption of (**A.2**).

(ii) It is possible to apply our result to an arbitrary initial distribution (that is, a point). To deduce this, one may use a priori Gaussian bounds on the transition probabilities.

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