J. Korean Math. Soc. **48** (2011), No. 1, pp. 117–132
DOI 10.4134/JKMS.2011.48.1.117

A LINEAR APPROACH TO LIE TRIPLE AUTOMORPHISMS OF H^*-ALGEBRAS

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Abstract. By developing a linear algebra program involving many different structures associated to a three-graded H^*-algebra, it is shown that if L is a Lie triple automorphism of an infinite-dimensional topologically simple associative H^*-algebra A, then L is either an automorphism, an anti-automorphism, the negative of an automorphism or the negative of an anti-automorphism. If A is finite-dimensional, then there exists an automorphism, an anti-automorphism, the negative of an automorphism or the negative of an anti-automorphism $F : A \to A$ such that $\delta := F - L$ is a linear map from A onto its center sending commutators to zero. We also describe L in the case of having A zero annihilator.

1. Introduction

Let A be an associative algebra over the complex field $\mathbb{C}$. Then A is a Lie algebra under the Lie multiplication $[a, b] = ab - ba$, and a Lie triple system under the triple product $[a, b, c] = [[a, b], c]$. A linear isomorphism $L : A \to A'$ between two associative algebras is called a *Lie triple isomorphism* if $L([a, b, c]) = [L(a), L(b), L(c)]$ for all $a, b, c \in A$. Isomorphisms, anti-isomorphisms, negative of isomorphisms and negative of anti-isomorphisms $f : A \to A'$ are examples of Lie triple isomorphisms. We also recall that a linear isomorphism $l : A \to A'$ is said to be a *Lie isomorphism* if $l([a, b]) = [l(a), l(b)]$ for any $a, b \in A$.

The structure of Lie isomorphisms has attracted some attention over past years: in the case of rings [2, 3, 4, 7, 16, 17], in the case of Banach algebras [5, 26, 27] and in the case of H^*-algebras [11]. Obviously, every Lie isomorphism is a Lie triple isomorphism. The converse is, in general, not true. In the recent years, there also has been some interest in studying Lie triple isomorphisms

Received April 22, 2009.
2010 *Mathematics Subject Classification.* 17B40, 46L40, 47B47, 46K70.
Key words and phrases. H^*-algebra, graded algebra, Jordan pair, Lie triple automorphism.
Supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM-298 and FQM-336, by the project of the Spanish Ministerio de Educación y Ciencia MTM2004-06580-C02-02 and with fondos FEDER.

and derivations of different structures [24, 30]. In this paper we consider Lie triple automorphisms of H^*-algebras.

Using functional identities one can describe Lie triple isomorphisms at a high level of generality, [6, Corollary 1]. However, we use entirely different methods to characterize Lie triple isomorphisms of H^*-algebras. In fact, we note that the development of a linear algebra program relating many different structures associated to a three-graded H^*-algebra, see Proposition 2.2, to the treatment of this problem is perhaps the most interesting novelty in this paper.

We recall that an *H^*-algebra* A over $\mathbb{C}$ is a, non-necessarily associative, $\mathbb{C}$-algebra whose underlying vector space is a complex Hilbert space with inner product $(\cdot \mid \cdot)$, endowed with a conjugate-linear map $* : A \to A$ $(x \mapsto x^*)$, such that $(x^*)^* = x$, $(xy)^* = y^*x^*$ for any $x, y \in A$ and the following identities, (H^*-identities), hold

$$(xy|z) = (x|zy^*) = (y|x^*z)$$

for all $x, y, z \in A$. The map $*$ will be called the *involution* of the H^*-algebra. The continuity of the product of A is proved in [20]. We call the H^*-algebra A, *topologically simple* if $A^2 \neq 0$ and A has no nontrivial closed ideals. H^*-algebras were introduced and studied by Ambrose [1] in the associative case, and have been also considered in the case of the most familiar classes of nonassociative context [8, 18, 20, 29] and even in the general nonassociative context [19, 20]. In [20] it is prove that any H^*-algebra A with continuous involution splits into the orthogonal direct sum $A = \mathrm{Ann}(A) \perp \overline{\mathcal{L}(A^2)}$, where $\mathrm{Ann}(A) := \{x \in A : xA = Ax = 0\}$ is the *annihilator* of A, and $\overline{\mathcal{L}(A^2)}$ is the closure of the vector span of A^2, which turns out to be an H^*-algebra with zero annihilator. Moreover, each H^*-algebra A with zero annihilator satisfies $A = \overline{\perp I_\alpha}$, where $\{I_\alpha\}_\alpha$ denotes the family of minimal closed ideals of A, each of them being a topologically simple H^*-algebra. We also recall that any isomorphism f on arbitrary H^*-algebras with zero annihilator is continuous and that if f commutes with the involution, (we say that f is an *$*$-isomorphism*), and the H^*-algebras are topologically simple, then f is isometric up to a positive multiple of the inner product [19].

H^*-triple systems appears as the ternary version of H^*-algebras. It is well known that H^*-triple systems and their isomorphisms have similar properties to those of H^*-algebras described above (see [15, 14]).

Let H be a complex Hilbert space and let $\mathcal{HS}(H)$ be the algebra of all Hilbert-Schmidt operators on H. If $\{\phi_i\}_{i\in\mathcal{A}}$ is a complete orthonormal system of H and $f, g \in \mathcal{HS}(H)$, then the sum $\sum(f(\phi_i)|g(\phi_i))$ is independent of the choice of $\{\phi_i\}_{i\in\mathcal{A}}$. It can be proved that $\mathcal{HS}(H)$ becomes an associative H^*-algebra under the inner product

$$(f|g) = \sum(f(\phi_i)|g(\phi_i))$$

and the involution $f \mapsto f^*$, where f^* is the adjoint operator of f, that is, the unique element in $\mathcal{HS}(H)$ such that $(f(x)|y) = (x|f^*(y))$ for all $x, y \in H$. It is prove in [1] that any infinite dimensional topologically simple associative

H^*-algebra is $*$-isometrically isomorphic (up to a positive multiple of the inner product) to $\mathcal{HS}(H)$ with H a complex Hilbert space of infinite hilbertian dimension. We can give a matrix expression of topologically simple associative H^*-algebras as follows: In the infinite-dimensional case, if A is a non-empty set, we denote by $\mathcal{M}_A(\mathbb{C})$ the associative topologically simple complex H^*-algebra of all matrices $B = (b_{i,j})_{i,j \in A}$ with elements in $\mathbb{C}$ such that $\sum ||b_{i,j}||^2$ converges. The inner product is $((a_{i,j})|(b_{i,j})) = \sum a_{i,j}\overline{b_{i,j}}$ and the involution $(a_{i,j})^* = (\overline{a_{j,i}})$. Fixed $B = \{\phi_i\}_{i \in A}$ a complete orthonormal system of a complex Hilbert space H, if $\{E_{i,j}\}_{(i,j)\in A\times A}$ is the family of the operators of $\mathcal{HS}(H)$ defined by $E_{i,j}(\phi_k) = \delta_{j,k}\phi_i$ (where $\delta_{j,k}$ is the Kronecker delta), then the continuous linear mapping

$$\begin{aligned} \Phi : \mathcal{HS}(H) &\to \mathcal{M}_A(\mathbb{C}) \\ E_{i,j} &\mapsto e_{i,j} \end{aligned}$$

where $e_{i,j}$ denotes the usual elemental matrix, is an isometric $*$-isomorphism of associative H^*-algebras. We also recall that any simple associative H^*-algebra A with finite dimension is isometrically $*$-isomorphic (up to a positive multiple of the inner product) to an H^*-algebra of the type $\mathcal{M}_n(\mathbb{C})$.

The paper is organized as follows. In Section 2 we introduce the different H^*-structures associated to an associative three-graded H^*-algebra, and study the transference of the topological simplicity among them. These results will make us easier the development of the following section. Finally, we study in Section 3 the Lie triple automorphisms of topologically simple associative H^*-algebras and of associative H^*-algebras with zero annihilator.

2. Different structures on H^*-algebras

Let K be a unitary commutative ring. A *three-graded K-algebra A* is a K-algebra which splits into the direct sum $A = A_{-1} \oplus A_0 \oplus A_1$ of nonzero K-submodules satisfying $A_0A_i + A_iA_0 \subset A_i$ for all $i \in \{-1, 0, 1\}$,

$$A_{-1}A_1 + A_1A_{-1} \subset A_0$$

and $A_1A_1 = A_{-1}A_{-1} = 0$. A *three-graded H^*-algebra* is an H^*-algebra which is a three-graded algebra such that A_{-1}, A_0, A_1 are closed orthogonal subspaces satisfying $(A_i)^* = A_{-i}$ for $i \in \{-1, 0, 1\}$. We note that a topologically simple associative H^*-algebra can be three-graded in several different ways (see the proof of Theorem 3.1).

The following result, of easy verification, is proved in [11, Proposition 2.2].

Proposition 2.1. *Let A be an H^*-algebra with zero annihilator, and let D be a self-adjoint derivation on A with minimal polynomial $x^3 - x$ and satisfying $* \circ D = -D \circ *$. Then $A = A_{-1} \perp A_0 \perp A_1$ with $A_i = \mathrm{Ker}(D - iId)$, $i \in \{-1, 0, 1\}$, is a three-graded H^*-algebra.*

Lemma 2.1. *Let A and D be as in Proposition 2.1. Then*

$$\mathrm{Ker}(D + Id) + \mathrm{Ker}(D - Id) = \mathrm{Ker}(D^2 - Id).$$

Proof. The inclusion $\mathrm{Ker}(D+Id)+\mathrm{Ker}(D-Id)\subset\mathrm{Ker}(D^2-Id)$ is clear. Conversely, if $x\in\mathrm{Ker}(D^2-Id)$, by Proposition 2.1 we can write $x=x_{-1}+x_0+x_1$ with $x_i\in\mathrm{Ker}(D-iId)$, $i\in\{-1,0,1\}$. As $x=D^2(x)=DD(x_{-1}+x_0+x_1)=D(-x_{-1}+x_1)=x_{-1}+x_1$, we finally obtain $x\in\mathrm{Ker}(D+Id)+\mathrm{Ker}(D-Id)$. □

Let A be a complex vector space. We say that A is a *ternary algebra* if it is endowed with a trilinear map $\langle\cdot,\cdot,\cdot\rangle : A\times A\times A\to A$, called the triple product, satisfying

$$\langle\langle x,y,z\rangle,u,v\rangle=\langle x,\langle y,z,u\rangle,v\rangle=\langle x,y,\langle z,u,v\rangle\rangle$$

for any $x,y,z,u,v\in A$. If A is an associative algebra, then the underlying vector space of A with the triple product $\langle x,y,z\rangle := xyz$ is a ternary algebra. We denote it $\mathcal{TA}(A)$. A ternary H^*-algebra T is a ternary algebra with an involution $*$ (involutive conjugate-linear map $*: A\to A$ such that $\langle x,y,z\rangle^*=\langle z^*,y^*,x^*\rangle$), and a Hilbert space structure with inner product $(\cdot|\cdot)$, satisfying

$$(\langle x,y,z\rangle|u)=(x|\langle u,z^*,y^*\rangle)=(y|\langle x^*,u,z^*\rangle)=(z|\langle y^*x^*,u\rangle)$$

for any $x,y,z,u\in T$. If A is an associative H^*-algebra, then $\mathcal{TA}(A)$ with the same involution and inner product than A is a ternary H^*-algebra.

Let us construct some H^*-structures associated to an associative three-graded H^*-algebra. Let $A=A_{-1}\perp A_0\perp A_1$ be an associative three-graded H^*-algebra. Then $\mathcal{AP}(A):=(A_{-1},A_1)$ and $\mathcal{PTA}(A):=A_{-1}\perp A_1$ are, respectively, an associative H^*-pair with the triple products $\langle a^\sigma,b^{-\sigma},c^\sigma\rangle := a^\sigma b^{-\sigma}c^\sigma$ for any $a^\sigma,c^\sigma\in A_\sigma$, $b^{-\sigma}\in A_{-\sigma}$; and a polarized ternary H^*-algebra with the triple product $\langle(a^1,a^{-1}),(b^1,b^{-1}),(c^1,c^{-1})\rangle := (a^1b^{-1}c^1,a^{-1}b^1c^{-1})$ for any $a^\sigma,b^\sigma,c^\sigma\in A_\sigma$, and with the involutions and inner products induced by the ones in A. See [9, 12, 15] for the definitions and first results on associative H^*-pairs and polarized ternary H^*-algebras.

We have that A with the Lie bracket $[a,b]=ab-ba$ and with the triple product

$$[a,b,c]=[[a,b]c] \tag{1}$$

(and the same involution and inner product), becomes respectively a Lie H^*-algebra, (or L^*-algebra), and a Lie H^*-triple system, (or L^*-triple), that we denote by A^- and $\mathcal{LT}(A)$ resp. We also can check that

$$\mathcal{Z}_2\mathcal{L}(A):=A_0\perp(A_{-1}+A_1)$$

with the Lie bracket is a $\mathbb{Z}_2$-graded L^*-algebra, and that $\mathcal{PLT}(A):=A_{-1}\perp A_1$ is a polarized L^*-triple with the triple product (1) (and with the involutions and inner products induced by A). See [29, 18, 10] for the definitions and basic results on L^*-algebras, L^*-triples and $\mathbb{Z}_2$-graded L^*-algebras. See also [13, 28] for the definition of polarized L^*-triple.

Finally, we have that $\mathcal{JP}(A):=(A_{-1},A_1)$ and $\mathcal{PJT}(A):=A_{-1}\perp A_1$ are, respectively, a Jordan H^*-pair with the quadratic operators $Q^\sigma(a^\sigma)(b^{-\sigma})=a^\sigma b^{-\sigma}a^\sigma$ for any $a^\sigma\in A_\sigma$, $b^{-\sigma}\in A_{-\sigma}$; and a polarized Jordan H^*-triple system with the quadratic operator $Q^\sigma((a^1,a^{-1}))((b^1,b^{-1}))=(a^1b^{-1}a^1,a^{-1}b^1a^{-1})$ for all $a^\sigma, b^\sigma \in A_\sigma$, the involutions and inner products being the ones induced by A. See [9, 12, 13] for the definitions and preliminary results on Jordan H^*-pairs and polarized Jordan H^*-triple systems.

Proposition 2.2. *Let $A = A_{-1} \perp A_0 \perp A_1$ be an infinite dimensional three-graded associative H^*-algebra. Then the following assertions are equivalent:*

(1) *A is topologically simple.*
(2) *The ternary H^*-algebra $\mathcal{TA}(A)$ is topologically simple and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*
(3) *The associative H^*-pair $\mathcal{AP}(A) = (A_{-1}, A_1)$ is topologically simple and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*
(4) *The polarized ternary H^*-algebra $\mathcal{PTA}(A) = A_{-1} \perp A_1$ is topologically simple and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*
(5) *The L^*-algebra A^- is topologically simple.*
(6) *The L^*-triple $\mathcal{LT}(A)$ is topologically simple.*
(7) *The $\mathbb{Z}_2$-graded L^*-algebra $\mathcal{Z}_2\mathcal{L}(A) = A_0 \perp (A_{-1} + A_1)$ is topologically simple in graded sense and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*
(8) *The polarized L^*-triple $\mathcal{PLT}(A) = A_{-1} \perp A_1$ is topologically simple and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*
(9) *The Jordan H^*-pair $\mathcal{JP}(A) = (A_{-1}, A_1)$ is topologically simple and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*
(10) *The polarized Jordan H^*-triple system $\mathcal{PJT}(A) = A_{-1} \perp A_1$ is topologically simple and $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$.*

Proof. (1)⇒(2). By §1, we can write $A = \mathcal{HS}(H)$ and so, by the classification of topologically simple ternary H^*-algebras in [15, Main Theorem], we deduce $\mathcal{TA}(A)$ is also topologically simple. By the continuity of the product, $A_{-1} \perp \overline{A_{-1}A_1 + A_1A_{-1}} \perp A_1$ is a nonzero closed ideal of A and so $A_0 = \overline{A_{-1}A_1 + A_{-1}A_1}$.

(2)⇒(3). Let (I_{-1}, I_1) be a nonzero closed ideal of $\mathcal{AP}(A) = (A_{-1}, A_1)$. As $I := I_{-1} \perp \overline{(I_{-1}A_1 + A_{-1}I_1 + I_1A_{-1} + A_1I_{-1})} \perp I_1$ is a nonzero closed ideal of $\mathcal{TA}(A)$, then $I = \mathcal{TA}(A)$ and so $(I_{-1}, I_1) = (A_{-1}, A_1)$.

(3)⇒(4). Let I be a nonzero closed ideal of $\mathcal{PTA}(A)$. If we denote by $\pi_i(I)$, $i \in \pm 1$ the orthogonal projections of I over A_i, $i \in \pm 1$, it is easy to check that $(\pi_{-1}(I), \pi_1(I))$ is a closed ideal of $\mathcal{AP}(A)$. Since $I \neq 0$, $\pi_i(I) = A_i$. Therefore, for any $x_{-1} \in A_{-1}$, there exists $x_1 \in A_1$ such that $x_{-1} + x_1 \in I$. Given now any $a_{-1} \in A_{-1}$ and $a_1 \in A_1$ we have $(x_{-1} + x_1)a_1a_{-1} = x_{-1}a_1a_{-1} \in I \cap A_{-1}$. If $I \cap A_{-1} = 0$, then $A_{-1}A_1A_{-1} = 0$, contradicting the topological simplicity of $\mathcal{AP}(A)$. As $(I \cap A_{-1}, I \cap A_1)$ is an ideal of $\mathcal{AP}(A)$, we conclude $I \cap A_i = A_i$ and therefore $I = A_{-1} + A_1 = \mathcal{PTA}(A)$.

(4)⇒(1). By the classification of topologically simple ternary H^*-algebras [15], the $\mathbb{Z}_2$-graded envelope H^*-algebra of $\mathcal{PTA}(A)$ is topologically simple. As $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$, we have that A is the $\mathbb{Z}_2$-graded envelope H^*-algebra of $\mathcal{PTA}(A)$ and so it is topologically simple.

(1)⇔(5). It follows from the classifications of topologically simple associative and Lie H^*-algebras given in [20] and [18].

(5)⇒(6). $\mathcal{LT}(A)$ has $A \perp A$ as a $\mathbb{Z}_2$-graded envelope L^*-algebra. As $A \perp A$ is topologically simple in graded sense, then $\mathcal{LT}(A)$ is topologically simple.

(6)⇒(5). It is consequence of the fact that any ideal of A^- is also an ideal of $\mathcal{LT}(A)$.

(5)⇒(7). It is trivial.

(7)⇔(8). It is well known that an L^*-triple is topologically simple if and only if its $\mathbb{Z}_2$-graded L^*-algebra envelope is topologically simple in graded sense [10]. From $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$, we have $\mathcal{Z}_2\mathcal{L}(A)$ is a $\mathbb{Z}_2$-graded L^*-algebra envelope of $\mathcal{PLT}(A)$ and the equivalence is proved.

(8)⇔(9)⇔(10). See [13, 28].

(9)⇒(3). It is clear. See also [9, Theorem 2]. □

The following result can be proved as in [28] (see also [13, Proposition 3.2]).

Lemma 2.2. *Let V be a topologically simple Lie or Jordan H^*-triple system with two polarizations $V = V_{-1} \perp V_1$ and $V = W_{-1} \perp W_1$. Then either $V_i = W_i$ or $V_i = W_{-i}$, $i \in \pm 1$.*

Lemma 2.3. *Let $A = A_{-1} \perp A_0 \perp A_1$ and $A' = A'_{-1} \perp A'_0 \perp A'_1$ be two three-graded associative H^*-algebras. If $f : \mathcal{PLT}(A) \to \mathcal{PLT}(A')$ is a $*$-isomorphism of Lie triple systems, then $f : \mathcal{PJT}(A) \to \mathcal{PJT}(A')$ is a $*$-isomorphism of Jordan triple system.*

Proof. It is immediate, taking into account that the quadratic operator of the polarized Jordan triple system $\mathcal{PJT}(A)$ can be written as

$$Q((x_{-1}, x_1))((y_{-1}, y_1)) = ([x_{-1}, y_1, x_{-1}], [x_1, y_{-1}, x_1]),$$

where $[\cdot, \ \cdot, \ \cdot]$ denotes the triple product of $\mathcal{PLT}(A)$ (the same applies to $\mathcal{PJT}(A')$). □

3. Lie triple automorphisms

3.1. The infinite dimensional case

Theorem 3.1. *Let L be a Lie triple $*$-automorphism of an infinite dimensional topologically simple associative H^*-algebra A. Then, A can be three-graded as $A = A_{-1} \perp A_0 \perp A_1$ and as $A = A'_{-1} \perp A'_0 \perp A'_1$, in such a way that $A_0 = A_{00} \perp A_{01}$ and $A'_0 = A'_{00} \perp A'_{01}$ with A_{00}, A_{01}, A'_{00}, A'_{01} topologically simple associative H^*-algebras, A'_{00} and A'_{00} of arbitrary finite dimension, and there exists an automorphism or an anti-automorphism F of A such that $F(A_i) = L(A_i) = A'_i$, $i \in \pm 1$; $F(A_{0j}) = L(A_{0j}) = A'_{0j}$, $j \in \{0, 1\}$, and $F = L$ on $A_{-1} \perp A_1$.*

Proof. First, let us three-graduate A in two adequate ways. From §1, there is no loss of generality in writing $A = \mathcal{HS}(H)$, being H a complex Hilbert space with infinite hilbertian dimension. As L is a $*$-automorphism of the topologically simple L^*-triple $\mathcal{LT}(A)$, we have that L is isometric up to a positive multiple of the inner product (see §1). Let $\{\phi_i\}_{i\in\mathcal{A}}$ be a complete orthogonal system of H. We can express

$$\{\phi_i\}_{i\in\mathcal{A}} = \{\phi_i\}_{i\in\mathcal{B}} \cup \{\phi_i\}_{i\in\mathcal{C}},$$

being $\mathcal{B}\cup\mathcal{C} = \mathcal{A}$, $\mathcal{B}\cap\mathcal{C} = \emptyset$, $\mathcal{B},\mathcal{C} \neq \emptyset$ and $\mathcal{B}$ a finite set. Consider the map $ad(x) : \mathcal{HS}(H) \to \mathcal{HS}(H)$, given by $ad(x)(a) = [x, a] := xa - ax$, being $x : H \to H$ defined by $x(\phi_i) = -\phi_i$ if $i \in \mathcal{B}$ and $x(\phi_i) = 0$ if $i \in \mathcal{C}$. It is easy to prove that $ad(x)$ is a self-adjoint derivation on $\mathcal{HS}(H)$ with minimal polynomial $x^3 - x$ and satisfies

$$* \circ ad(x) = -ad(x) \circ *.$$

By Proposition 2.1,

$$A = A_{-1} \perp A_0 \perp A_1$$

is a three-graded associative H^*-algebra, (topologically simple), being $A_{-1} = \mathcal{HS}(H_\mathcal{C}, H_\mathcal{B})$, $A_0 = \mathcal{HS}(H_\mathcal{B}) \perp \mathcal{HS}(H_\mathcal{C})$, (we will write $A_{00} = \mathcal{HS}(H_\mathcal{B})$, $A_{01} = \mathcal{HS}(H_\mathcal{C})$); and $A_1 = \mathcal{HS}(H_\mathcal{B}, H_\mathcal{C})$, where $H_\mathcal{B}, H_\mathcal{C}$ denote the Hilbert subspaces of H generated by $\{\phi_i\}_{i\in\mathcal{B}}$ and $\{\phi_i\}_{i\in\mathcal{C}}$ respectively, and $\mathcal{HS}(H_\mathcal{C}, H_\mathcal{B})$ is the set of elements $g \in \mathcal{HS}(H)$ such that $g(H_\mathcal{B}) = 0$ and $g(H_\mathcal{C}) \subset H_\mathcal{B}$ (the same applies to

$$\mathcal{HS}(H_\mathcal{B}) = \mathcal{HS}(H_\mathcal{B}, H_\mathcal{B}), \quad \mathcal{HS}(H_\mathcal{C}) = \mathcal{HS}(H_\mathcal{C}, H_\mathcal{C})$$

and $\mathcal{HS}(H_\mathcal{B}, H_\mathcal{C})$).

Let us show that $ad(L(x))$ is a self-adjoint derivation of A with minimal polynomial $x^3 - x$ and such that $* \circ ad(L(x)) = -ad(L(x)) \circ *$: As $ad(x)$ has minimal polynomial $x^3 - x$, that is, $[x, [x, [x, y]]] = [x, y]$ for any $y \in A$, then $[[x, [x, [x, y]]], z] = [[x, y], z]$ for any $y, z \in A$. Therefore

$$[[L(x), [L(x), [L(x), L(y)]]], L(z)] = [[L(x), L(y)], L(z)],$$

that is,

$$[ad(L(x))^3(L(y)), L(z)] = [ad(L(x))(L(y)), L(z)]$$

for any $y, z \in A$ and so

$$[ad(L(x))^3(L(y)) - ad(L(x))(L(y)), L(z)] = 0.$$

But A^- is topologically simple (see Proposition 2.2), and so $\mathrm{Ann}(A^-) = 0$. Since L is bijective, then $ad(L(x))^3 = ad(L(x))$. A similar argument gives us $ad(L(x))^2 \neq Id$ and so $ad(L(x))$ has minimal polynomial $x^3 - x$. The fact that $*\circ ad(L(x)) = -ad(L(x))\circ *$ is a consequence of $x^* = x$, since for any $y \in A^-$ we have $[L(x), y]^* = [y^*, L(x)^*] = [y^*, L(x^*)] = [y^*, L(x)]$. Finally, the self-adjoint character of $ad(L(x))$ easily follows from the H^*-identities in A^- and $L(x)^* = L(x)$. By applying Proposition 2.1, we have that A becomes a three-graded associative H^*-algebra as $A = A'_{-1} \perp A'_0 \perp A'_1$ with $A'_i = \mathrm{Ker}(ad(L(x)) - iId)$, $i \in \{-1, 0, 1\}$. To distinguish this new three-graduation of A, we will denoted $A' := A'_{-1} \perp A'_0 \perp A'_1$.

Second, let us prove that $L(A_i) = A'_i$, $i \in \{-1, 0, 1\}$. We begin by observing that the mapping L is clearly a $*$-automorphism from

$$\mathcal{LT}(A) = \mathcal{LT}(A_{-1} \perp A_0 \perp A_1)$$

onto

$$\mathcal{LT}(A) = \mathcal{LT}(A'_{-1} \perp A'_0 \perp A'_1).$$

Now, we can deduce $L(A_{-1} \perp A_1) = A'_{-1} \perp A'_1$, taking into account that given $a \in A_i$, $i \in \pm 1$, $[x, [x, a]] = a$. From here, $[L(x), [L(x), L(a)]] = L(a)$ and so

$$L(a) \in \mathrm{Ker}(ad(L(x))^2 - Id).$$

Lemma 2.1 gives now $L(a) \in A'_{-1} \perp A'_1$. Conversely, given $y \in A'_i$, $i \in \pm 1$, a similar argument shows $L^{-1}(y) \in A_{-1} \perp A_1$, and then $L(A_{-1} \perp A_1) = A'_{-1} \perp A'_1$. We now have that $L(A_{-1}) \perp L(A_1)$ is another polarization of $\mathcal{PLT}(A') = A'_{-1} \perp A'_1$, and as $\mathcal{PLT}(A')$ is topologically simple by Proposition 2.2, Lemma 2.2 shows either $L(A_i) = A'_i$ or $L(A_i) = A'_{-i}$. In the second case we define $A_i := A_{-i}$. The isometric character (up to a positive multiple of the inner product) of L implies $L(A_0) = A'_0$.

Finally, let us prove the existence of F. We can apply Lemma 2.3 to conclude that $L : \mathcal{PJT}(A) \to \mathcal{PJT}(A')$ is a $*$-isomorphism of Jordan H^*-triple systems. We know that $\mathcal{PJT}(A)$ is a topologically simple polarized Jordan H^*-triple system coming from symmetrizing the topologically simple polarized ternary H^*-algebra $\mathcal{HS}(H_\mathcal{C}, H_\mathcal{B}) \perp \mathcal{HS}(H_\mathcal{B}, H_\mathcal{C})$. Moreover, Proposition 2.2 gives us that $\mathcal{PTA}(A') = A'_{-1} \perp A'_1$ is a topologically simple polarized ternary H^*-algebra, and that $A'_0 = \overline{A'_{-1}A'_1 + A'_1A'_{-1}}$. Since $\mathcal{PJT}(A')$ is then a topologically simple polarized Jordan H^*-triple system coming from symmetrizing a topologically simple polarized ternary H^*-algebra, the classifications of topologically simple polarized Jordan H^*-triple systems and ternary H^*-algebras in [12] and [15] give us that $\mathcal{PJT}(A') = \mathcal{HS}(H'_{\mathcal{C}'}, H'_{\mathcal{B}'}) \perp \mathcal{HS}(H'_{\mathcal{B}'}, H'_{\mathcal{C}'})$ and that $A'_0 = A'_{00} \perp A'_{01}$ where $A'_{00} = \mathcal{HS}(H'_{\mathcal{B}'})$ and $A'_{01} = \mathcal{HS}(H'_{\mathcal{C}'})$; with $H'_{\mathcal{B}'}$ and $H'_{\mathcal{C}'}$ as above.

By applying now D'Amour's result ([21, Theorem B] or [9, Theorem 1]) as in the proof of [9, Theorem 2], $L : \mathcal{PJT}(A) \to \mathcal{PJT}(A)'$ extends to an isomorphism of two-graded associative algebras

$$F : A \oplus A^{op} \to A' \oplus (A')^{op},$$

being $A = \mathcal{HS}(H) \simeq \begin{pmatrix} \mathcal{HS}(H_\mathcal{B}) & \mathcal{HS}(H_\mathcal{C}, H_\mathcal{B}) \\ \mathcal{HS}(H_\mathcal{B}, H_\mathcal{C}) & \mathcal{HS}(H_\mathcal{C}) \end{pmatrix}$ and

$$A' = \mathcal{HS}(H') \simeq \begin{pmatrix} \mathcal{HS}(H'_{\mathcal{B}'}) & \mathcal{HS}(H'_{\mathcal{C}'}, H'_{\mathcal{B}'}) \\ \mathcal{HS}(H'_{\mathcal{B}'}, H'_{\mathcal{C}'}) & \mathcal{HS}(H'_{\mathcal{C}'}) \end{pmatrix}.$$

We note with respect to the above conclusion that, following [21, Theorem B] and [9, Theorem 2], $A \oplus A^{op}$ and $A' \oplus (A')^{op}$ are, respectively, $\mathbb{Z}_2$-graded δ-tight algebra envelopes of $\mathcal{PJT}(A)$ and $\mathcal{PJT}(A')$ with $\delta(x, y) := (y, x)$, and that the infinite dimensional nature of the above Jordan triple systems forces its hermitian character (following McCrimmons terminology), and so the Zel'manov polynomials in [21, Theorem B] do not vanish on them (see [25, p. 143]).

By an easy argument, we have two possibilities for F, either $F(A \oplus \{0\}) = A \oplus \{0\}$ or

$$F(A \oplus \{0\}) = \{0\} \oplus A^{op}.$$

We have in the first case that $F(xy) = F(x)F(y)$ for any $x, y \in A$, being $L(a_i) = F(a_i)$ for $a_i \in A_{-1} \cup A_1$ because F extends L, and in the second case that $F(xy) = F(y)F(x)$ for any $x, y \in A$, also being $L(a_i) = F(a_i)$ for $a_i \in A_{-1} \cup A_1$. We note that F is continuous (see §1). This fact, the continuity of $*$ and the equality $A_0 = \overline{A_{-1}A_1 + A_1A_{-1}}$ give us that F commutes with $*$ and so F is isometric (up to a positive multiple of the inner product) and then $F(A_0) = A'_0$. Finally, let us show that $F(A_{0j}) = L(A_{0j}) = A'_{0j}$, $j \in \{0, 1\}$. As $F, L : \mathcal{LT}(A_0) \to \mathcal{LT}(A'_0)$ are isometric (up to a positive multiple of the inner product), $*$-isomorphisms of Lie triple systems, being A_{01}, A'_{01} the only infinite dimensional closed ideals of $\mathcal{LT}(A_0)$ and $\mathcal{LT}(A'_0)$ respectively, then $F(A_{01}) = L(A_{01}) = A'_{01}$, and therefore $F(A_{00}) = L(A_{00}) = A'_{00}$. □

Taking into account the matrix representation of a topologically simple associative H^*-algebra, see §1, we can formulate Theorem 3.1 as follows:

Theorem 3.2. *Let $A = \mathcal{M}_{\mathcal{A}}(\mathbb{C})$ be an infinite dimensional topologically simple associative H^*-algebra and $L : A \to A$ a Lie triple $*$-automorphism. Then $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ with $\mathcal{B} \cap \mathcal{C} = \emptyset$, $\mathcal{B}, \mathcal{C} \neq \emptyset$, $\mathcal{B}$ of arbitrary finite cardinality, and there exists an automorphism or an anti-automorphism F of A such that $L = F$ on $\mathcal{M}_{\mathcal{B},\mathcal{C}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C},\mathcal{B}}(\mathbb{C})$ and $L(\mathcal{M}_{\mathcal{B}}(\mathbb{C})) = F(\mathcal{M}_{\mathcal{B}}(\mathbb{C}))$, $L(\mathcal{M}_{\mathcal{C}}(\mathbb{C})) = F(\mathcal{M}_{\mathcal{C}}(\mathbb{C}))$.*

Let us observe that the mapping $f := LF^{-1} : A \to A$ is an automorphism of the Lie triple system $\mathcal{LTS}(A)$ acting as the identity on $\mathcal{M}_{\mathcal{B},\mathcal{C}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C},\mathcal{B}}(\mathbb{C})$ and $f(\mathcal{M}_{\mathcal{B}}(\mathbb{C})) = \mathcal{M}_{\mathcal{B}}(\mathbb{C})$, $f(\mathcal{M}_{\mathcal{C}}(\mathbb{C})) = \mathcal{M}_{\mathcal{C}}(\mathbb{C})$. Let us study such a mapping:

Theorem 3.3. *Let $A = \mathcal{M}_{\mathcal{A}}(\mathbb{C})$ be an infinite dimensional topologically simple associative H^*-algebra. Suppose $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ with $\mathcal{B} \cap \mathcal{C} = \emptyset$, $\mathcal{B}, \mathcal{C} \neq \emptyset$, and let $f : A \to A$ be a Lie triple automorphism of A such that $f(x) = x$ for any $x \in \mathcal{M}_{\mathcal{B},\mathcal{C}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C},\mathcal{B}}(\mathbb{C})$ and $f(\mathcal{M}_{\mathcal{B}}(\mathbb{C})) = \mathcal{M}_{\mathcal{B}}(\mathbb{C})$, $f(\mathcal{M}_{\mathcal{C}}(\mathbb{C})) = \mathcal{M}_{\mathcal{C}}(\mathbb{C})$. Then f is either the identity map or $f(x) = -x$ for $x \in \mathcal{M}_{\mathcal{B}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C}}(\mathbb{C})$.*

Proof. If we denote the elementary matrices in $A = \mathcal{M}_{\mathcal{A}}(\mathbb{C})$ by e_{ij} as usual, then we know that $f(e_{ij}) = e_{ij}$ for any $(i, j) \in (\mathcal{B} \times \mathcal{C}) \cup (\mathcal{C} \times \mathcal{B})$. Let us compute $f(e_{ii})$ and $f(e_{kk})$ for $(i, k) \in \mathcal{B} \times \mathcal{C}$: We can write

$$f(e_{ii}) = \sum \lambda_{pq} e_{pq},$$

where $p, q \in \mathcal{B}$ and

$$f(e_{kk}) = \sum \mu_{rs} e_{rs}$$

with $r, s \in \mathcal{C}$, $\lambda_{pq}, \mu_{rs} \in \mathbb{C}$. Applying f to the equality $[e_{ii}, e_{ik}, e_{ki}] = e_{ii} - e_{kk}$ we obtain

$$[\sum \lambda_{pq} e_{pq}, e_{ik}, e_{ki}] = \sum \lambda_{pq} e_{pq} - \sum \mu_{rs} e_{rs}. \tag{2}$$

As we also have

$$[\sum \lambda_{pq} e_{pq}, e_{ik}, e_{ki}] = \sum_p \lambda_{pi} e_{pi} - \lambda_{ii} e_{kk}, \tag{3}$$

(2) and (3) give us

$$\sum \lambda_{pq} e_{pq} - \sum \mu_{rs} e_{rs} = \sum_p \lambda_{pi} e_{pi} - \lambda_{ii} e_{kk}$$

and so $\lambda_{pq} = 0$ if $q \neq i$, $\mu_{rs} = 0$ if $(r, s) \neq (k, k)$ and $\mu_{kk} = \lambda_{ii}$. As we also have $[e_{ii}, e_{ki}, e_{ik}] = e_{ii} - e_{kk}$, by applying f as above we obtain

$$[\sum \lambda_{pq} e_{pq}, e_{ki}, e_{ik}] = \sum \lambda_{pq} e_{pq} - \sum \mu_{rs} e_{rs},$$

and as $[\sum \lambda_{pq} e_{pq}, e_{ki}, e_{ik}] = \sum_q \lambda_{iq} e_{iq} - \lambda_{ii} e_{kk}$, we conclude $\sum \lambda_{pq} e_{pq} - \sum \mu_{rs} e_{rs} = \sum_q \lambda_{iq} e_{iq} - \lambda_{ii} e_{kk}$ and so $\lambda_{pq} = 0$ if $p \neq i$. We have proved $f(e_{ii}) = \lambda_{ii} e_{ii}$ and $f(e_{kk}) = \lambda_{ii} e_{kk}$. Moreover, since $[e_{ii}, e_{ik}, e_{ii}] = -e_{ik}$, we have $\lambda_{ii}^2 [e_{ii}, e_{ik}, e_{ii}] = -e_{ik}$ and so $\epsilon := \lambda_{ii} = \pm 1$. From here, $f(e_{ii}) = \epsilon e_{ii}$ and $f(e_{kk}) = \epsilon e_{kk}$ for any $i \in \mathcal{B}$ and $k \in \mathcal{C}$.

Let us compute $f(e_{ij})$ with $i, j \in \mathcal{B}$, $i \neq j$: We can write

$$f(e_{ij}) = \sum \gamma_{pq} e_{pq}$$

with any $p, q \in \mathcal{B}$. Let us fix $k \in \mathcal{C}$. As $e_{ij} = [e_{ij}, e_{jk}, e_{kj}]$, we have $\sum \gamma_{pq} e_{pq} = [\sum \gamma_{pq} e_{pq}, e_{jk}, e_{kj}] = \sum_p \gamma_{pj} e_{pj} - \gamma_{jj} e_{kk}$. From here, $\gamma_{pq} = 0$ if $q \neq j$ and $\gamma_{jj} = 0$. In a similar way, the equality $e_{ij} = [e_{ij}, e_{ki}, e_{ik}]$ gives us $\sum \gamma_{pq} e_{pq} = \sum_q \gamma_{iq} e_{iq} - \gamma_{ii} e_{kk}$ and so $\gamma_{pq} = 0$ if $p \neq i$ and $\gamma_{ii} = 0$. We conclude $f(e_{ij}) = \gamma_{ij} e_{ij}$. From $[e_{ij}, e_{jk}, e_{kk}] = e_{ik}$, we deduce $\gamma_{ij} \epsilon [e_{ij}, e_{jk}, e_{kk}] = e_{ik}$ and then $\gamma_{ij} = \epsilon$. We obtain a similar result for any $f(e_{ij})$ with $i, j \in \mathcal{C}$, $i \neq j$, and the theorem is proved. □

Taking into account the comment after Theorem 3.2, we have the following corollary:

Corollary 3.1. *Let $A = \mathcal{M}_{\mathcal{A}}(\mathbb{C})$ be an infinite dimensional topologically simple associative H^*-algebra, $L : A \to A$ a Lie triple automorphism of A and $F : A \to A$ an automorphism or an anti-automorphism of A, such that $L = F$ on $\mathcal{M}_{\mathcal{B},\mathcal{C}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C},\mathcal{B}}(\mathbb{C})$ and $L(\mathcal{M}_{\mathcal{B}}(\mathbb{C})) = F(\mathcal{M}_{\mathcal{B}}(\mathbb{C}))$, $L(\mathcal{M}_{\mathcal{C}}(\mathbb{C})) = F(\mathcal{M}_{\mathcal{C}}(\mathbb{C}))$. Then either $L = F$ or $L = fF$ where f is the identity map on $\mathcal{M}_{\mathcal{B},\mathcal{C}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C},\mathcal{B}}(\mathbb{C})$ and the negative of the identity on $\mathcal{M}_{\mathcal{B}}(\mathbb{C}) \oplus \mathcal{M}_{\mathcal{C}}(\mathbb{C})$.*

We observe that in the second case, as f is the negative of an automorphism, then L is the negative of an automorphism when F is an automorphism, and L is the negative of an anti-automorphism in the case that F is an anti-automorphism. This observation together with Theorem 3.2 allow us to claim the following theorem.

Theorem 3.4. *Let L be a Lie triple $*$-automorphism of an infinite dimensional topologically simple associative H^*-algebra A. Then L is either a $*$-automorphism, a $*$-anti-automorphism, the negative of a $*$-automorphism or the negative of a $*$-anti-automorphism of A.*

Corollary 3.2. *Let $L : A \to A'$ be a Lie triple $*$-isomorphism between two infinite dimensional topologically simple associative H^*-algebras A, A'. Then L is either a $*$-isomorphism, a $*$-anti-isomorphism, the negative of a $*$-isomorphism or the negative of a $*$-anti-isomorphism.*

Proof. Since $L : \mathcal{LT}(A) \to \mathcal{LT}(A')$ is a $*$-isomorphism and by [14] isometric (up to a positive multiple of the inner product), the hilbertian dimensions of A and A' agree. So, A and A' are $*$-isomorphic to $\mathcal{HS}(H)$, via ϕ, ϕ' respectively. If we write by L' the only mapping making commutative the below diagram, the commutativity of the diagram joint with Theorem 3.4 complete the proof.

$$\begin{array}{ccccc} & & L & & \\ & A & \to & A' & \\ \phi & \downarrow & & \downarrow & \phi' \\ & \mathcal{HS}(H) & \to & \mathcal{HS}(H) & \\ & & L' & & \end{array}$$

□

Corollary 3.3. *Let $L : A \to A'$ be a Lie triple isomorphism between two infinite dimensional topologically simple associative H^*-algebras A, A'. Then L is either an isomorphism, an anti-isomorphism, the negative of an isomorphism or the negative of an anti-isomorphism.*

Proof. Consider in A' the unique L^*-triple structure making $L : \mathcal{LT}(A) \to \mathcal{LT}(A')$ an isometric $*$-isomorphism (of topologically simple L^*-triples). As $A \perp A$ is a $\mathbb{Z}_2$-graded L^*-algebra envelope of $\mathcal{LT}(A)$, then the $\mathbb{Z}_2$-graded L^*-algebra envelope of A' is of the type $B \perp B$ with B a topologically simple associative H^*-algebra, and being $B \perp B$ $*$-isometrically isomorphic to $A \perp A$ (see [10, Proposition 2.1]). From here, $L : A \to B$, where B is A' considered with its new involution and inner product, is a Lie triple $*$-isomorphism between two infinite dimensional topologically simple associative H^*-algebras. By Corollary 3.2, L is either a $*$-isomorphism, a $*$-anti-isomorphism, the negative of a $*$-isomorphism or the negative of a $*$-anti-isomorphism. The result follows from here. □

3.2. The finite dimensional case

The infinite dimension of the polarized Jordan triple systems $\mathcal{PJT}(A)$ and $\mathcal{PJT}(A)'$ in Theorem 3.1 is needed to apply the key D'Amour's extension results [21, Theorem B]. Therefore, we will develop new arguments to study the finite dimensional case.

As it is well-known, any topologically simple associative H^*-algebra A with finite dimension $n > 1$ is isomorphic to an algebra of the type $\mathcal{M}_n(\mathbb{C})$.

Lemma 3.1. *If we write* $A = \mathcal{M}_n(\mathbb{C}) = \mathbb{C}Id \oplus [\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]$ *and denote by* L *a Lie triple automorphism of* A*. Then* $L(\mathbb{C}Id) = \mathbb{C}Id$ *and*

$$L([\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]) = [\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})].$$

Proof. Let us write by T the Lie triple system $\mathcal{LTS}(A)$. It is easy to verify that $\mathrm{Ann}(T) = \mathbb{C}Id$. Indeed, clearly $\mathbb{C}Id \subset \mathrm{Ann}(T)$ and conversely, given $x \in T$ such that $[x, a, b] = [[x, a], b] = 0$ for any $a, b \in T$, we have $[x, a] = \lambda Id$, with $\lambda \in \mathbb{C}$. From here, $[x, a] \in \mathbb{C}Id \cap \mathrm{sl}(n, \mathbb{C}) = 0$, then $[x, a] = 0$ for all $a \in T$ and so $x \in \mathbb{C}Id$. As L is a Lie triple automorphism of T, we conclude $L(\mathbb{C}Id) = \mathbb{C}Id$. Since $A_{n-1} = [\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]$ is the only nonzero minimal ideal of T, then $L([\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]) = [\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]$. □

So let $L|_{A_l} : A_l \to A_l$ be a Lie triple automorphism and consider the well defined map $G : A_l \to A_l$ such that $G(\sum_i [x_i, y_i]) = \sum_i [L(x_i), L(y_i)]$ for $x_i, y_i \in A_l$. For convenience we will write $L := L|_{A_l}$. Next we prove the identity $G([[a, b], [c, d]]) = [G([a, b]), G([c, d])]$ for $a, b, c, d \in A_l$, by using the fact $[[a, b], [c, d]] = -[[b, [c, d]], a] - [[[c, d], a], b]$. Thus

$$\begin{aligned} G([[a, b], [c, d]]) &= -[[L(b), [L(c), L(d)]], L(a)] - [[[L(c), L(d)], L(a)], L(b)] \\ &= [[L(a), L(b)], [L(c), L(d)]] \\ &= [G([a, b]), G([c, d])], \end{aligned} \tag{4}$$

which proves that G is a Lie algebra automorphism of A_l. On the other hand we also have

$$L([x, y]) = [G(x), L(y)] = [L(x), G(y)]$$

for $x, y \in A_l$, since $x = \sum_i [a_i, b_i]$ and

$$\begin{aligned} L([x, y]) &= \sum_i L([[a_i, b_i], y]) = \sum_i [[L(a_i), L(b_i)], L(y)] \\ &= [G(\sum_i [a_i, b_i]), L(y)] = [G(x), L(y)]. \end{aligned}$$

In a similar way it is proved that $L([x, y]) = [L(x), G(y)]$. Thus we have proved the set of equalities:

$$\begin{aligned} G([x, y]) &= [G(x), G(y)] = [L(x), L(y)], \\ L([x, y]) &= [L(x), G(y)] = [G(x), L(y)]. \end{aligned} \tag{5}$$

Lemma 3.2. *We have* $(L^{-1}G)^2 = (GL^{-1})^2 = 1$.

Proof. As

$$[L(x), G(y)] = [L(x), L(L^{-1}G(y))] = [G(x), GL^{-1}G(y)]$$

$$= [G(x), LL^{-1}GL^{-1}G(y)] = [L(x), GL^{-1}GL^{-1}G(y)],$$

we deduce $G(y) = GL^{-1}GL^{-1}G(y)$ and so $L^{-1}GL^{-1}G = 1$, $GL^{-1}GL^{-1} = 1$. The lemma is proved. □

Since $(L^{-1}G)^2 = 1$ we can decompose A_l as the direct sum of two subspaces so that $L^{-1}G$ acts as the identity on one of the direct summands while it acts as the negative of the identity on the other summand. Thus L and G agree on one of the summands and $L = -G$ on the other one. Next we show that the subspace on which L and G agree is a Lie triple ideal of A_l so that given the simplicity of A_l as a Lie triple, this subspace is 0 or the whole A_l.

Corollary 3.4. $L = \pm G$.

Proof. For $\epsilon = \pm 1$ write $T_\epsilon = \{x \in A_l : L(x) = \epsilon G(x)\}$ so that $A_l = T_1 \oplus T_{-1}$. It is immediate that $[T_\alpha, T_\beta, T_\gamma] \subset T_{\alpha\beta\gamma}$ for $\alpha, \beta, \gamma \in \{-1, 1\}$. But taking $x \in T_1$ and $y \in T_{-1}$ we have $[L(x), G(y)] = [G(x), -L(y)]$ which compared with (5) gives $[L(x), G(y)] = 0$, that is, $[L(T_1), G(T_{-1})] = [L(T_1), L(T_{-1})] = 0$. Hence $[L(T_1), L(T_{-1}), L(T)] = 0$ implying $[T_1, T_{-1}, T] = 0$, that is, $[T_1, T_{-1}] = 0$. Also $[T_1, T_1, T_{-1}] = 0$ applying Jacobi identity. Thus $[T_1, T, T] = [T_1, T_1, T] = [T_1, T_1, T_1] \subset T_1$ which proves that T_1 is an ideal of the triple system A_l. The simplicity of this Lie triple implies that $T_1 = 0$ (implying $L = -G$) or $T_1 = A_l$ (hence $L = G$). □

Theorem 3.5. *Let $L : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, $n > 1$, be a Lie triple automorphism. Then there exists an automorphism, an anti-automorphism, the negative of an automorphism or the negative of an anti-automorphism $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ such that $\delta := L - F$ is a linear map from $\mathcal{M}_n(\mathbb{C})$ onto its center sending commutators to zero.*

Proof. By Corollary 3.4 we have two possibilities for $L|_{A_l}$. In the first one, $L|_{A_l}$ is an automorphism of the Lie algebra A_l. By applying [22, Theorem 5, p. 283], $L|_{A_l} : A_l \to A_l$ extends to an automorphism or the negative of an anti-automorphism $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$. In the second one, $-L|_{A_l}$ is an automorphism of the Lie algebra A_l. [22, Theorem 5, p. 283] gives us now that $-L|_{A_l}$ extends to an automorphism or the negative of an anti-automorphism F' of $\mathcal{M}_n(\mathbb{C})$. In this case $L|_{A_l}$ extends to $F := -F' : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, F being the negative of an automorphism or an anti-automorphism. If we call $\delta := L - F$, we can prove that $\delta(\mathcal{M}_n(\mathbb{C})) \subset Z(\mathcal{M}_n(\mathbb{C})) = \mathbb{C}Id$ and $\delta([\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]) = 0$: Given any $x \in \mathcal{M}_n(\mathbb{C})$, $x = c + a$ with $c \in \mathbb{C}Id$ and $a \in [\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]$. The character of automorphism, anti-automorphism, negative of an automorphism or negative of an anti-automorphism of F gives us $F(c) \in \mathbb{C}Id$. By Lemma 3.1, we also have $L(c) \in \mathbb{C}Id$. Finally, as $F(a) = L(a)$ for any $a \in [\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]$, we conclude $\delta(\mathcal{M}_n(\mathbb{C})) \subset \mathbb{C}Id$ and

$$\delta([\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{C})]) = 0.$$

□

3.3. The annihilator zero case

Theorem 3.6. *Let L be a continuous Lie triple automorphism of an associative H^*-algebra with zero annihilator A. Then there exist closed ideals P, Q, R, S of A and a $\mathbb{C}$-linear bijective mapping $F : A \to A$ such that $A = P \perp Q \perp R \perp S$, and if we denote by $\{I_\alpha\}$ the family of the minimal closed ideals of A, then:*

1. *F restricted to P is an isomorphism.*
2. *F restricted to Q is an anti-isomorphism.*
3. *F restricted to R is the negative of an isomorphism.*
4. *F restricted to S is the negative of an anti-isomorphism.*
5. *If I_α is infinite dimensional, then $F|_{I_\alpha} = L|_{I_\alpha}$.*
6. *If I_α is finite dimensional, then $\delta_\alpha := L|_{I_\alpha} - F|_{I_\alpha}$ is a linear mapping from I_α onto the center of A sending commutators to zero.*

Proof. Denote by $\{I_\alpha\}_{\alpha\in\Lambda}$ the family of minimal closed ideals of A. Let us consider $I_{\alpha_0} \in \{I_\alpha\}_{\alpha\in\Lambda}$.

If I_{α_0} is infinite dimensional, as I_{α_0} is a topologically simple L^*-triple with $\mathbb{Z}_2$-graded L^*-algebra envelope $I_{\alpha_0} \perp I_{\alpha_0}$, then $L^{-1}(I_{\alpha_0})$ is also a topologically simple L^*-triple with a $\mathbb{Z}_2$-graded L^*-algebra envelope of the type $L^{-1}(I_{\alpha_0}) \perp L^{-1}(I_{\alpha_0})$, $L^{-1}(I_{\alpha_0})$ being an infinite dimensional minimal closed ideal of A. If we denote by F_{α_0} the restriction of L to $L^{-1}(I_{\alpha_0})$, Corollary 3.3 shows that F_{α_0} is either an isomorphism, an anti-isomorphism, a negative of an isomorphism or a negative of an anti-isomorphism.

If I_{α_0} is finite dimensional with $\dim I_{\alpha_0} > 1$, as I_{α_0} is isomorphic to an associative algebra of the type $\mathcal{M}_n(\mathbb{C})$, $n > 1$, then we have as in Lemma 3.1 that $L^{-1}(I_{\alpha_0})$ is also a minimal ideal of A isomorphic to $\mathcal{M}_n(\mathbb{C})$. If we denote by L_{α_0} the restriction of L to $L^{-1}(I_{\alpha_0})$, Theorem 3.5 gives us that there exists an automorphism, an anti-automorphism, a negative of an automorphism or a negative of an anti-automorphism $F_{\alpha_0} : L^{-1}(I_{\alpha_0}) \to I_{\alpha_0}$ such that $\delta_{\alpha_0} := L_{\alpha_0} - F_{\alpha_0}$ is a linear map from $L^{-1}(I_{\alpha_0})$ onto the center of I_{α_0} sending commutators to zero.

Let I_{α_0} be such that $\dim I_{\alpha_0} = 1$. Since we have the family of linear isomorphisms $\{F_\beta\}$, $F_\beta : L_{\beta_0}^{-1}(I_\beta) \to I_\beta$ among the minimal closed ideals of dimension not 1, we define the unique linear isomorphism $F_{\alpha_0} : I_{\alpha_0} \to I_{\alpha_0}$ given by $F_{\alpha_0}(1) = 1$ which turns out to be an automorphism (of associative algebras).

Let consider any $I_\alpha \in \{I_\alpha\}_{\alpha\in\Lambda}$ with the unique H^*-structure that makes F_α either a $*$-isometric isomorphism, a $*$-isometric anti-isomorphism, a $*$-isometric negative of an isomorphism or a $*$-isometric negative of an anti-isomorphism. As $A = \overline{\perp_{\alpha\in\Lambda} I_\alpha}$, the isometric character of any F_α, $\alpha \in \Lambda$, enable us to extend $\{F_\alpha\}_{\alpha\in\Lambda}$ to an isometric linear mapping $F : A \to A$ such that

$$A = (\overline{\perp_{\alpha\in\Lambda_1} I_\alpha}) \perp (\overline{\perp_{\alpha\in\Lambda_2} I_\alpha}) \perp (\overline{\perp_{\alpha\in\Lambda_3} I_\alpha}) \perp (\overline{\perp_{\alpha\in\Lambda_4} I_\alpha}),$$

with $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4 = \Lambda$, $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$, and being F restricted to $P := \overline{\perp_{\alpha\in\Lambda_1} I_\alpha}$ an isomorphism, F restricted to $Q := \overline{\perp_{\alpha\in\Lambda_2} I_\alpha}$ an anti-isomorphism, F restricted to $R := \overline{\perp_{\alpha\in\Lambda_3} I_\alpha}$ a negative of an automorphism, and F restricted to $S := \overline{\perp_{\alpha\in\Lambda_4} I_\alpha}$ a negative of an anti-isomorphism. It is clear that P, Q, R, S and F satisfy the conditions of Theorem 3.6. □

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