J. Korean Math. Soc. **48** (2011), No. 1, pp. 105–115
DOI 10.4134/JKMS.2011.48.1.105

$[r,\ s,\ t;\ f]$-COLORING OF GRAPHS

Yong Yu and Guizhen Liu

Abstract. Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$, let r, s and t be non-negative integers. An *f-coloring* of G is an edge-coloring of G such that each vertex $v \in V(G)$ has at most $f(v)$ incident edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G and denoted by $\chi'_f(G)$. An $[r,\ s,\ t;\ f]$-*coloring* of a graph G is a mapping c from $V(G) \bigcup E(G)$ to the color set $C = \{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i and v_j, $|c(e_i) - c(e_j)| \geq s$ and $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G)$, $\alpha \in C$ where $\alpha(v_i)$ denotes the number of α-edges incident with the vertex v_i and e_i, e_j are edges which are incident with v_i but colored with different colors, $|c(e_i) - c(v_j)| \geq t$ for all pairs of incident vertices and edges. The minimum k such that G has an $[r,\ s,\ t;\ f]$-*coloring* with k colors is defined as the $[r,\ s,\ t;\ f]$-*chromatic number* and denoted by $\chi_{r,s,t;f}(G)$. In this paper, we present some general bounds for $[r,\ s,\ t;\ f]$-coloring firstly. After that, we obtain some important properties under the restriction $\min\{r, s, t\} = 0$ or $\min\{r, s, t\} = 1$. Finally, we present some problems for further research.

1. Introduction

In this paper, the term *graph* is used to denote a simple connected graph G with a finite vertex set $V(G)$ and a finite edge set $E(G)$. If multiple edges are allowed, G is called a multigraph. The degree of a vertex v in G is the number of edges incident with v and denoted by $d(v)$. We write $\delta(G) = \min\{d(v) : v \in V(G)\}$ and $\Delta(G) = \max\{d(v) : v \in V(G)\}$ to denote the minimum degree and maximum degree of G, respectively. Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. We define $\Delta_f(G) = \max_{v \in V(G)}\{\lceil d(v)/f(v) \rceil\}$. Let C denote the set of colors $\{0, 1, \ldots, k-1\}$. A *vertex* (*res. edge*) *coloring* of a graph G is a *mapping* c from $V(G)$ (res. $E(G)$) to the color set C. A *proper vertex* (*res. edge*) *coloring* of

Received April 18, 2009; Revised October 13, 2009.
2010 *Mathematics Subject Classification.* 05C15.
Key words and phrases. f-coloring, $[r,s,t]$-coloring, $[r,s,t;f]$-coloring, f-total coloring, $[r,s,t;f]$-chromatic number.
This work is supported by NSFC(10901097), NNSF(10871119) and RSDP(200804220001) of China.

a graph G is a vertex (res. edge) coloring such that any two adjacent vertices (res. edges) get different colors. The minimum k such that G has a proper vertex (res. edge) coloring with color set $C = \{0, 1, \ldots, k-1\}$ is called the chromatic number (res. edge chromatic number) of graph G, and denoted by $\chi(G)$ (res. $\chi'(G)$). We use $c(v)$ to denote the color assigned to the vertex v and $c(e)$ for the edge e of graph G. An edge colored with color $\alpha \in C$ is called an *α-edge*. The number of α-edges of G incident with the vertex v is denoted by $\alpha(v)$. Our terminology and notation will be standard except where indicated. Readers are referred to [1] for undefined terms.

Hakimi and Kariv [4] generalized the *proper edge-coloring* to *f-coloring* and obtained many interesting results. Let G be a multigraph and f be a function defined as above. An *f-coloring* of G is an edge-coloring of G such that each vertex $v \in V(G)$ has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G and denoted by $\chi'_f(G)$. Zhang and Liu [7, 8, 9] studied the f-coloring of graphs and got many interesting results.

Kemnitz and Marangio [6] studied the *$[r, s, t]$-coloring* of a graph G. Given non-negative integers r, s and t, an *$[r, s, t]$-coloring* of a graph G is a mapping c from $V(G) \bigcup E(G)$ to the color set $C = \{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i and v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(e_i) - c(v_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The *$[r, s, t]$-chromatic number* $\chi_{r,s,t}(G)$ of G is the minimum k such that G has an *$[r, s, t]$-coloring.* Dekar, et al. [3] gave exact values of $\chi_{r,s,t}(G)$ of stars except one case.

Here we present a new coloring which is defined as *$[r, s, t; f]$-coloring.* Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$, let r, s and t be non-negative integers. An *$[r, s, t; f]$-coloring* of a graph G is a mapping c from $V(G) \bigcup E(G)$ to the color set $C = \{0, 1, \ldots, k-1\}$ such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i and v_j, $|c(e_i) - c(e_j)| \geq s$ and $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G), \alpha \in C$ where $\alpha(v_i)$ denotes the number of α-edges incident with the vertex v_i and e_i, e_j are edges which are incident with v_i but colored with different colors, $|c(e_i) - c(v_j)| \geq t$ for all pairs of incident vertices and edges. The minimum k such that G has an *$[r, s, t; f]$-coloring* is defined as the *$[r, s, t; f]$-chromatic number* and denoted by $\chi_{r,s,t;f}(G)$. Clearly, if $s = 1$, $r = t = 0$, then c is an f-coloring; if $f(v) = 1$ for all $v \in V(G)$ (we will write $f \equiv 1$ for short in the following), then c is an *$[r, s, t]$-coloring*; if $f \equiv 1$ and $r = 1, s = t = 0$, then c is a *proper vertex coloring*; if $f \equiv 1$ and $s = 1, r = t = 0$, then c is a *proper edge coloring*; if $f \equiv 1$ and $r = s = t = 1$, then c is a *total coloring.* Similarly, let $r = s = t = 1$, we get another new coloring which we define as *f-total coloring.*

In this paper, we at first discuss some interesting results for this new coloring. Then we focus on the case $r = s = 1$ which are not considered in the *$[r, s, t]$-coloring.*

2. Basic results

At first, we give two obvious lemmas.

Lemma 2.1. *If $H \subseteq G$, then $\chi_{r,s,t;f}(H) \leq \chi_{r,s,t;f}(G)$.*

Proof. It is obvious that the restriction of an [r, s, t; f]-coloring of G to the element of $H \subseteq G$ is still an [r, s, t; f]-coloring of H. □

Lemma 2.2. *Let f and f' be two functions defined as in the definition of [r, s, t; f]-coloring. If $f'(v) \geq f(v)$ for all $v \in V(G)$, and $r' \leq r$, $s' \leq s$, $t' \leq t$, then $\chi_{r',s',t';f'}(G) \leq \chi_{r,s,t;f}(G)$.*

Proof. The proof is trivial. We leave it to the readers. □

These two lemmas are obvious but useful to determine bounds and exact values of the [r, s, t; f]-chromatic number of graphs.

Theorem 2.3. *If $a \geq 0$ is an integer, then $\chi_{ar,as,at;f}(G) = a(\chi_{r,s,t;f}(G) - 1) + 1$.*

Proof. If $a = 0$ or 1, then the assertion is obvious. Suppose $a \geq 2$ and c is an [r, s, t; f]-coloring of G with $\chi_{r,s,t;f}(G)$ colors. Then $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i and v_j, $|c(e_i) - c(e_j)| \geq s$, $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G), \alpha \in C$ where $\alpha(v_i)$ denotes the number of α-edges incident with the vertex v_i and e_i, e_j are edges which are incident with v_i but colored with different colors, $|c(e_i) - c(v_j)| \geq t$ for all pairs of incident vertices and edges. Let $c'(x) = a \cdot c(x)$ for all $x \in V(G) \bigcup E(G)$, and we use α', C' denote the new color and the new color set, respectively. Then we have

$$|c'(v_i) - c'(v_j)| = a \cdot |c(v_i) - c(v_j)| \geq ar,$$
$$|c'(e_i) - c'(e_j)| = a \cdot |c(e_i) - c(e_j)| \geq as,$$
$$|c'(e_i) - c'(v_j)| = a \cdot |c(e_i) - c(v_j)| \geq at.$$

For $\alpha' \in C'$, if $\alpha'(v_i) \neq 0$, then there is color $\alpha \in C$ such that $\alpha' = a\alpha$ and $\alpha'(v_i) = \alpha(v_i) \leq f(v_i)$; if $\alpha'(v_i) = 0$, obviously we have $\alpha'(v_i) \leq f(v_i)$. Anyway,

$$\alpha'(v_i) \leq f(v_i) \text{ for all } v_i \in V(G),\ \alpha' \in C'.$$

Therefore, c' is an [ar, as, at; f]-coloring of G with colors $\{0, 1, \ldots, a(\chi_{r,s,t;f}(G) -1)\}$.

On the other hand, assume that G has an [ar, as, at; f]-coloring c with color set $\{0, 1, \ldots, a(\chi_{r,s,t;f}(G) - 1) - 1\}$, $a \geq 2$. Then we have $|c(v_i) - c(v_j)| \geq ar$ for every two adjacent vertices v_i and v_j, $|c(e_i) - c(e_j)| \geq as$, $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G), \alpha \in C$ where $\alpha(v_i)$ denotes the number of α-edges incident with the vertex v_i and e_i, e_j are edges which are incident with v_i but colored with different colors, $|c(e_i) - c(v_j)| \geq at$ for all pairs of incident vertices and edges. We define a coloring c' by $c'(x) = \lfloor c(x)/a \rfloor$ for all $x \in V(G) \bigcup E(G)$, in which

$\lfloor c(x)/a \rfloor$ is the largest integer not larger than $c(x)/a$. Let $\alpha' = \lfloor \alpha/a \rfloor \in C'$, C' denote the color set of c'. Clearly, $|\lfloor x \rfloor| \geq \lfloor |x| \rfloor$ for any real number x. So we have

$$|c'(v_i) - c'(v_j)| \geq \lfloor | \frac{c(v_i) - c(v_j)}{a} | \rfloor \geq r,$$

$$|c'(e_i) - c'(e_j)| \geq \lfloor | \frac{c(e_i) - c(e_j)}{a} | \rfloor \geq s,$$

$$|c'(e_i) - c'(v_j)| \geq \lfloor | \frac{c(e_i) - c(v_j)}{a} | \rfloor \geq t.$$

Let e_i, e_j are two edges incident with v_i, if they are both α-edges, then $c'(e_i) = c'(e_j) = \alpha'$; if $c(e_i) = \alpha$ and $c(e_j) \neq \alpha$, then $|c(e_i) - c(e_j)| \geq as$ for c is an $[ar$, as, at; $f]$-coloring of G. This implies $|c'(e_i) - c'(e_j)| \geq \lfloor | \frac{c(e_i)-c(e_j)}{a} | \rfloor \geq s \geq 1$. Therefore, $\alpha'(v_i) = \alpha(v_i) \leq f(v_i)$. So

$$\alpha'(v_i) \leq f(v_i) \text{ for all } v_i \in V(G), \alpha' \in C'.$$

That is, c' is an $[r$, s, t; $f]$-coloring of G with colors

$$\{0, 1, \ldots, \lfloor \frac{a(\chi_{r,s,t;f}(G) - 1) - 1}{a} \rfloor\},$$

where $\lfloor \frac{a(\chi_{r,s,t;f}(G)-1)-1}{a} \rfloor \leq \chi_{r,s,t;f}(G) - 2$. We get an $[r,s,t]$-coloring of G with no more than $\chi_{r,s,t;f}(G) - 1$ colors, a contradiction. □

Corollary 2.4. *If $r = s = t$ and $f(v) \equiv 1$, then*

$$\chi_{r,s,t;f}(G) = r(\chi^{''}(G) - 1) + 1,$$

where $\chi^{''}(G)$ is the total chromatic number of graph G.

Corollary 2.5. *Let G be a graph and let r, s, t, f be defined as in the definition of $[r, s, t; f]$-coloring. Then*

$$\chi_{r,0,0;f}(G) = r(\chi(G) - 1) + 1,$$
$$\chi_{0,s,0;f}(G) = s(\chi^{'}_f(G) - 1) + 1,$$
$$\chi_{0,0,t;f}(G) = t + 1.$$

Lemma 2.6 ([4]). *Let G be a graph. Then*

$$\Delta_f(G) \leq \chi^{'}_f(G) \leq \max_{v \in V(G)} \{\lceil (1 + d(v))/f(v) \rceil\} \leq \Delta_f(G) + 1.$$

Theorem 2.7. *Let G be a graph and let r, s, t, f be defined as in the definition of $[r, s, t; f]$-coloring. Then*

$$\max\{r(\chi(G) - 1) + 1, s(\chi^{'}_f(G) - 1) + 1, t + 1\}$$
$$\leq \chi_{r,s,t;f}(G) \leq r(\chi(G) - 1) + s(\chi^{'}_f(G) - 1) + t + 1.$$

Proof. (a) If $f(v) = d(v)$ for all $v \in V(G)$, then

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil d(v)/f(v) \rceil\} = 1,$$

and we can use one color to f-color G. Therefore, $\chi'_f(G) = \Delta_f(G) = 1$. Let c be an $[r, 0, 0; f]$-coloring of G with $r(\chi(G) - 1) + 1$ colors. Then we assign color $r(\chi(G) - 1) + t$ to all the edges of G, we get an $[r,\ s,\ t;\ f]$-coloring with $r(\chi(G) - 1) + t + 1$ colors. This is the upper bound, and the lower bound is obvious by Lemma 2.2 and Corollary 2.5.

(b) If there is a vertex $u \in V(G)$ such that $f(u) < d(u)$, then $\chi'_f(G) \geq 2$. In this case, consider c mentioned in part (a). We use colors $r(\chi(G) - 1) + t, r(\chi(G) - 1) + t + s, \ldots, r(\chi(G) - 1) + t + s(\chi'_f(G) - 1)$ to color the edges. Then we get an $[r,\ s,\ t;\ f]$-coloring with $r(\chi(G) - 1) + s(\chi'_f(G) - 1) + t + 1$ colors. The lower bound can be got by Lemma 2.2 and Corollary 2.5. □

Lemma 2.8. *Let G be a graph and let r, s, t, f be defined as in the definition of $[r, s, t; f]$-coloring. If $t > r(\chi(G) - 1) + s(\chi'_f(G) - 1)$, then*

$$\chi_{r,s,t;f}(G) \geq r(\chi(G) - 1) + s(\delta_f(G) - 1) + t + 1,$$

where $\delta_f(G) = \min_{v \in V(G)}\{\lceil d(v)/f(v) \rceil\}$.

Proof. Let c be an $[r,\ s,\ t;\ f]$-coloring of G with $\chi_{r,s,t;f}(G)$ colors. By Theorem 2.7 and the assumption on t we obtain $2t + 1 > r(\chi(G) - 1) + s(\chi'_f(G) - 1) + t + 1 \geq \chi_{r,s,t;f}(G)$. So $\chi_{r,s,t;f}(G) \leq 2t$. If there is a vertex v and incident edges e_1, e_2 such that $c(e_1) < c(v) < c(e_2)$ or an edge $e = v_1v_2$ such that $c(v_1) < c(e) < c(v_2)$, then at least $2t + 1$ colors are needed which contradicts with the conclusion $\chi_{r,s,t;f}(G) \leq 2t$. Therefore, if x is an arbitrary element of G, then $c(x) < c(y)$ for all elements y that are incident to x or $c(x) > c(y)$ for all y. By induction, we obtain either $c(v) < c(e)$ for all vertices v and all edges e incident to v or always $c(v) > c(e)$. Without loss of generality, we assume $c(v) < c(e)$.

Consider the vertex u which obtains the greatest color $c(u)$. In order to proper coloring the vertex set of graph G, at least $\chi_{r,0,0;f}(G)$ colors are needed. By Corollary 2.5 we have $\chi_{r,0,0;f}(G) = r(\chi(G) - 1) + 1$. Therefore, $c(u) \geq r(\chi(G) - 1)$. In the f-coloring, denote by $r(u)$ the color numbers appeared on the edges which are incident with u. Obviously, we have $r(u)f(u) \geq d(u)$, which implies $r(u) \geq \lceil d(u)/f(u) \rceil \geq \min_{v \in V(G)}\{\lceil d(v)/f(v) \rceil\} = \delta_f(G)$. That is to say, there are at least $\delta_f(G)$ different colors which are greater than $c(u)$ by our assumption appeared on u. Then we get $\chi_{r,s,t;f}(G) \geq c(u) + t + s(\delta_f(G) - 1) + 1 \geq r(\chi(G) - 1) + s(\delta_f(G) - 1) + t + 1$. □

By Lemma 2.6, all graphs are partitioned into two classes. One is graphs with $\chi'_f(G) = \Delta_f(G)$, called C_f 1, or f-class 1, and the other with $\chi'_f(G) = \Delta_f(G) + 1$, called C_f 2, or f-class 2.

Just as the case we discussed in Theorem 2.7, $\chi_f^{'}(G) = \Delta_f(G) = 1$ when $f(v) = d(v)$ for all $v \in V(G)$. This also implies that $\delta_f(G) = 1$. So by Theorem 2.7 and Lemma 2.8 we have the following result.

Corollary 2.9. *Suppose that* $t > r(\chi(G) - 1) + s(\chi_f^{'}(G) - 1)$.

(1) *If* $f(v) = d(v)$ *for all* $v \in V(G)$*, then*

$$\chi_{r,s,t;f}(G) = r(\chi(G) - 1) + t + 1;$$

(2) *If* (1) *is not satisfied, but* G *is a* C_f 1 *graph with* $\Delta_f(G) = \delta_f(G)$*, then*

$$\chi_{r,s,t;f}(G) = r(\chi(G) - 1) + s(\chi_f^{'}(G) - 1) + t + 1.$$

Corollary 2.9 provides a subclass of graphs that can reach the upper bound of Theorem 2.7.

In Section 3 and Section 4, we will give some restriction to the parameters r, s, t, f in order to obtain some new results.

3. $\min\{r, s, t\} = 0$

We consider the case only one of r, s, t equals 0. The case where two of r, s, t equal 0 is discussed in Corollary 2.5.

Theorem 3.1. *Let* G *be a graph. Then*

$$\chi_{r,s,0;f}(G) = \max\{r(\chi(G) - 1) + 1, s(\chi_f^{'}(G) - 1) + 1\}.$$

Proof. This equation can be obtained by Theorem 2.7 and the fact that vertices and edges can be colored independently. □

Lemma 3.2 ([6])**.** *Let* G *be a graph. Then*

(1) *If* $\chi(G) = 2$*, then*

$$\chi_{r,0,t}(G) = \begin{cases} r+1 & \text{if } r \geq 2t; \\ 2t+1 & \text{if } t \leq r < 2t; \\ r+t+1 & \text{if } r < t. \end{cases}$$

(2) *If* $\chi(G) \geq 3$ *and* $r \geq t$*, then*

$$\chi_{r,0,t}(G) = r(\chi(G) - 1) + 1;$$

(3) *If* $\chi(G) \geq 3$ *and* $r < t$*, then*

$$\max\{r(\chi(G) - 1) + 1, t + 1\} \leq \chi_{r,0,t}(G) \leq r(\chi(G) - 3) + t + 1 + \min\{t, 2r\}.$$

Theorem 3.3. *Let* G *be a graph. If* $f(v) = d(v)$ *for all* $v \in V(G)$*, then* $\chi_{r,0,t;f}(G) = \chi_{r,0,t}(G)$*, where* $\chi_{r,0,t}(G)$ *is the same as that in Lemma 3.2.*

Proof. If $f(v) = d(v)$ for all $v \in V(G)$, then $\chi_f^{'}(G) = \Delta_f(G) = 1$. That is, we can color all the edges of G with one color and the condition $\alpha(v_i) \leq f(v_i)$ for all $v_i \in V(G), \alpha \in C$ in the definition of $[r,\ s,\ t;\ f]$-coloring has no influence. Therefore, we have $\chi_{r,0,t;f}(G) = \chi_{r,0,t}(G)$. □

Note that if there is a vertex $u \in V(G)$ such that $f(u) < d(u)$, then at least 2 colors are needed for the edges of G. Therefore, $s = 0$ is impossible in this case.

Lemma 3.4 ([6]). *Let G be a graph. Then*

(1) *If $\Delta(G) \geq 2$ and G is of class 1, then*

$$\chi_{0,s,t}(G) = \begin{cases} s(\Delta(G)-1)+1 & \text{if } s \geq 2t; \\ s(\Delta(G)-1)+2t-s+1 & \text{if } t \leq s < 2t; \\ s(\Delta(G)-1)+t+1 & \text{if } s < t. \end{cases}$$

(2) *If $\Delta(G) \geq 2$, G is of class 2 and $s \geq t$, then*

$$\chi_{0,s,t}(G) = s(\chi'(G)-1)+1;$$

(3) *If $\Delta(G) \geq 2$, G is of class 2 and $s < t$, then*

$$s(\Delta(G)-1)+t+1 \leq \chi_{0,s,t}(G) \leq \min\{s\Delta(G)+t+1, t\Delta(G)+1\}.$$

Theorem 3.5. *Let G be a graph. Then*

(a) *if $f(v) = d(v)$ for all $v \in V(G)$, then $\chi_{0,s,t;f}(G) = t+1$;*

(b) *otherwise,*

(1) *If $\Delta_f(G) \geq 2$ and G is of C_f 1, then*

$$\chi_{0,s,t;f}(G) = \begin{cases} s(\Delta_f(G)-1)+1 & \text{if } s \geq 2t; \\ s(\Delta_f(G)-1)+2t-s+1 & \text{if } t \leq s < 2t; \\ s(\Delta_f(G)-1)+t+1 & \text{if } s < t. \end{cases}$$

(2) *If $\Delta_f(G) \geq 2$, G is of C_f 2 and $s \geq t$, then*

$$\chi_{0,s,t;f}(G) = s(\chi'_f(G)-1)+1;$$

(3) *If $\Delta_f(G) \geq 2$, G is of C_f 2 and $s < t$, then*

$$s(\Delta_f(G)-1)+t+1 \leq \chi_{0,s,t;f}(G) \leq \min\{s\Delta_f(G)+t+1, t\Delta_f(G)+1\}.$$

Proof. (a) If $f(v) = d(v)$ for all $v \in V(G)$, then we can color all the vertices with color 0 and all the edges with color t. Then we obtain an $[0, s, t; f]$-coloring of G with $t+1$ colors. On the other hand, by Theorem 2.7 we get $\chi_{0,s,t;f}(G) \geq t+1$. Therefore, $\chi_{0,s,t;f}(G) = t+1$.

(b) If there is a vertex $u \in V(G)$ such that $f(u) < d(u)$, then the proof is similar to the proof in [4] (see A. Kemnitz, M. Marangio [4] Lemmas 7, 8, 9) just using $\Delta_f(G)$ instead of $\Delta(G)$. We don't mention it here. □

4. $\min\{r, s, t\} = 1$

In this section we will consider the three parameters $\chi_{r,1,1;f}(G)$, $\chi_{1,s,1;f}(G)$, $\chi_{1,1,t;f}(G)$, especially the last one.

Theorem 4.1. *If $r \geq \frac{\chi'_f(G)}{\chi(G)-1}+1$, then $\chi_{r,1,1;f}(G) = r(\chi(G)-1)+1$.*

Proof. Let c be an $[r,0,0]$-coloring of G with colors $0, r, \dots, r(\chi(G)-1)$. The assumption implies $\chi'_f(G) \leq (r-1)(\chi(G)-1)$. So we can use the colors which are not used by c to f-color the edges of G. Then we get an $[r,1,1;f]$-coloring of G. The lower bound can be got by Theorem 2.7. Therefore, $\chi_{r,1,1;f}(G) = r(\chi(G)-1)+1$. □

Theorem 4.2. *If $f(v) = d(v)$ for all $v \in V(G)$, then $\chi_{1,s,1;f}(G) = \chi(G)+1$. If there is a vertex $u \in V(G)$ such that $f(u) < d(u)$ but $s \geq \frac{\chi(G)}{\chi'_f(G)-1}+1$, then $\chi_{1,s,1;f}(G) = s(\chi'_f(G)-1)+1$.*

Proof. (1) If $f(v) = d(v)$ for all $v \in V(G)$, then just one color can f-coloring the edges of G. Suppose c is a proper vertex coloring of G with $\chi(G)$ colors. Then we assign color $\chi(G)$ to all the edges of G and obtain an $[1,s,1;f]$-coloring of G. $\chi_{1,s,1;f} \geq \chi(G)+1$ is obvious by Theorem 2.7.

(2) If there is a vertex $u \in V(G)$ such that $f(u) < d(u)$, then $\chi'_f(G) \geq 2$. By $s \geq \frac{\chi(G)}{\chi'_f(G)-1}+1$ we have $\chi(G) \leq (s-1)(\chi'_f(G)-1)$. Therefore, we can use the $(s-1)(\chi'_f(G)-1)$ colors which are not used in the f-coloring of G to obtain a proper vertex coloring of G. Then we get an $[1,s,1;f]$-coloring of G with $s(\chi'_f(G)-1)+1$ colors. The lower bound can be got by Theorem 2.7. □

Lemma 4.3. *Let G be a graph and let t and f be defined as in the definition of $[r,s,t;f]$-coloring. Then we have*

$$\Delta_f(G) + t \leq \chi_{1,1,t;f}(G) \leq \chi(G) + \chi'_f(G) + t - 1.$$

Proof. The upper bound can be obtained by Theorem 2.7. On the other hand, by Lemma 2.2 we get $\chi_{1,1,t;f}(G) \geq \chi_{0,1,t;f}(G)$. Then by Theorem 3.5 we obtain the lower bound. □

When we investigate the $[r,\ s,\ t;\ f]$-chromatic number under the special case $r = s = 1$, we can improve the result in Lemma 4.3 as Theorem 4.6.

Lemma 4.4 ([7])**.** *Let G be a complete graph K_n. If k and n are odd integers, $f(v) = k$ and $k|d(v)$ for all $v \in V(G)$, then G is of C_f 2. Otherwise, G is of C_f 1.*

Lemma 4.5 ([2], Brook's Theorem)**.** *$\chi(G) \leq \Delta(G)+1$ holds for every graph G. Moreover, $\chi(G) = \Delta(G)+1$ if and only if either $\Delta(G) \neq 2$ and G has a complete graph $K_{\Delta(G)+1}$ as a connected component, or $\Delta(G) = 2$ and G has an odd cycle as a connected component.*

Theorem 4.6. *Let G be a graph and let t, f be defined as in the definition of $[r,s,t;f]$-coloring. Then we have*

$$\chi_{1,1,t;f}(G) \leq \Delta(G) + \Delta_f(G) + t.$$

Proof. We now consider three cases depending on G.

Case 1. If G is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$ by Lemma 4.5 and $\chi_f^{'}(G) \leq \Delta_f(G) + 1$ by Lemma 2.6. Hence, the inequality is true.

Case 2. G is the complete graph K_n on n vertices. By Lemma 4.4 we know that K_n is of C_f 1 except one case. Then we have $\chi_f^{'}(G) = \Delta_f(G)$. By Lemma 4.3, we have $\chi_{1,1,t;f}(G) \leq (\Delta(G)+1) + \Delta_f(G) + t - 1 = \Delta(G) + \Delta_f(G) + t$. Now we assume that k and n are odd integers, $f(v) = k$ and $k|d(v)$ for all $v \in V(G)$. Then Lemma 4.4 implies that G is of C_f 2.

Case 2.1. If $f(v) = d(v)$, then we have $\chi_f^{'}(G) = \Delta_f(G) = 1$. We can assign all the edges with one color $n + t - 1$ and assign the vertices differently with colors $0, 1, \ldots, n-1$. Therefore, we obtain an $[1, 1, t; f]$-coloring of K_n with $n + t = \Delta(K_n) + \Delta_f(K_n) + t$ colors.

Case 2.2. If $f(v) \equiv 1$, then it becomes an $[1, 1, t]$-coloring of K_n. Let c be a proper edge coloring of K_n with n colors and M_i $(1 \leq i \leq n)$ be the matchings corresponding to the color classes. Further more, each M_i contains all vertices but one v_i (We know that it is true for K_n when n is odd, because $\chi'(K_n) = n = \Delta + 1, |M_i| \leq \frac{n-1}{2},\ 1 \leq i \leq n$, and if there is an integer j such that $|M_j| < \frac{n-1}{2}$, then $\chi'(K_n)\frac{n-1}{2} > \varepsilon(K_n) = \frac{n(n-1)}{2}$, a contradiction). For $1 \leq i \leq n$, color the vertex v_i with color $n - i$ and the edges in M_i with $n+t-3+i$. Since v_1 is not incident to M_1, then we obtain an $[1, 1, t; 1]$-coloring of K_n with $2n + t - 3 = \Delta(K_n) + \Delta_1(K_n) + t$ colors.

Case 2.3. If $1 < f(v) < d(v)$, then $f(v) = k \geq 3$ and

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil d(v)/f(v) \rceil\} = \frac{n-1}{k} \overset{\text{def}}{=} 2\alpha.$$

Let M_i be defined as in Case 2.2 and let $M_1' = M_1$, $M_i' = \bigcup_{j=2}^{k+1} M_{(i-2)k+j}$, $2 \leq i \leq 2\alpha + 1$. Color the vertex v_i with color $n - i$ and the edges in M_i' with color $n + t - 3 + i$, $2 \leq i \leq 2\alpha + 1$. We obtain an $[1, 1, t; f]$-coloring of K_n with $n + t - 3 + (2\alpha + 1) + 1 = \Delta(K_n) + \Delta_f(K_n) + t$ colors.

Case 3. G is an odd cycle. Then $\Delta = 2, \Delta_f(G) = \max_{v \in V(G)}\{\lceil \frac{d(v)}{f(v)} \rceil\} \leq 2$.

Case 3.1. If $f(v) = d(v)$ for all $v \in V(G)$, then $\chi_f^{'}(G) = \Delta_f(G) = 1$. We assign colors 0 and 1 to the vertices along the odd cycle alternately and assign color 2 to the final vertex. Then we color all the edges of G with color $\Delta(G) + \Delta_f(G) + t - 1 = t + 2$. We obtain an $[1, 1, t; f]$-coloring of G with $\Delta(G) + \Delta_f(G) + t$ colors.

Case 3.2. If there is a vertex $u \in V(G)$ such that $f(u) < d(u) = 2$, which implies $f(u) = 1,\ \Delta_f(G) = 2$. We color u with color 2 and the other vertices with 0 and 1 alternately. Denoted by e_1, e_2 the edges incident with u. Next, we color edge e_1 with color $t + 2$, color the edge adjacent with e_1 but not e_2

with color $t+1$. In this order, we color the edges along the cycle with colors $t+2, t+1$ alternately except for coloring e_2 with color $t+3$. Then we obtain an $[1,1,t;f]$-coloring of G with $t+4 = \Delta(G) + \Delta_f(G) + t$ colors.

In any case, we all prove that $\chi_{1,1,t;f}(G) \le \Delta(G) + \Delta_f(G) + t$. □

Lemma 4.7. *Let $t \ge 2$ be an integer. Then*

(1) *If $\delta_f(G) = \Delta_f(G)$, then $\chi_{1,1,t;f}(G) \ge \Delta_f(G) + t + 1$;*
(2) *If $t \ge \Delta_f(G)$, then $\chi_{1,1,t;f}(G) \ge \Delta_f(G) + t + 1$.*

Proof. Assume that we have an $[1,1,t;f]$-coloring of G with colors $\{0,1,\ldots,\Delta_f(G)+t-1\}$. We at first prove that the vertex u with $\lceil d(u)/f(u) \rceil = \Delta_f(G)$ must be assigned color 0 or $\Delta_f(G)+t-1$. Consider u and all the edges which are incident to it. We denote the subgraph by H. Then at least $\Delta_f(G)$ colors are needed for $[1,1,t;f]$-coloring the edges of H. Without loss of generality, we denote the colors by $C_1 < C_2 < \cdots < C_{\Delta_f}$. If there is an integer i, such that $C_i < c(u) < C_{i+1}$, then $C_{\Delta_f} \ge 2t + \Delta_f(G) - 2 > \Delta_f(G) + t - 1$, a contradiction. If $c(u) < C_1$, then $C_1 \ge t$ which implies that $c(u) = 0$ and $C_1 = t+1, C_2 = t+2, \ldots, C_{\Delta_f} = \Delta_f(G) + t - 1$; If $c(u) > C_{\Delta_f}$, then we can get $c(u) = \Delta_f(G) + t - 1$ and $C_1 = t+1, C_2 = t+2, \ldots, C_{\Delta_f} = \Delta_f(G) + t - 1$ by the same way. Without loss of generality, we assume that $c(u) = 0$.

(1) If $\delta_f(G) = \Delta_f(G)$, then every vertex must be assigned color 0 or $\Delta_f(G) + t - 1$. Let uv be an edge colored with color $\Delta_f(G) + t - 1$. We see that v can be labeled by neither 0 nor $\Delta_f(G) + t - 1$, a contradiction.

(2) If $t \ge \Delta_f(G)$, let uv be an edge colored with color t, then $c(v) \ge 2t \ge \Delta_f(G) + t$ by the assumption $t \ge \Delta_f(G)$, a contradiction. □

Lemma 4.8 ([7])**.** *Let $G(V,E)$ be a bipartite graph and*

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil d(v)/f(v) \rceil\}.$$

Then $\chi'_f(G) = \Delta_f(G)$.

Theorem 4.9. *Let $G(V,E)$ be a bipartite graph. Then*

(1) *$\Delta_f(G) + t \le \chi_{1,1,t;f}(G) \le \Delta_f(G) + t + 1$;*
(2) *If $t \ge \Delta_f(G)$ or $\delta_f(G) = \Delta_f(G)$, then $\chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1$.*

Proof. If G is a bipartite graph, then $\chi(G) = 2$ and $\chi'_f(G) = \Delta_f(G)$ by Lemma 4.8. Together with Lemma 4.3 we obtain (1).

(2) can be obtained by Lemma 4.7 and (1) of Theorem 4.9. □

Note that for a bipartite graph G, $\chi(G) = 2$ and $\chi'_f(G) = \Delta_f(G)$. If $t \ge \Delta_f(G)$, by (2) of Theorem 4.9 we get $\chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1$; If $t = 0$, by Theorem 3.1 we get $\chi_{1,1,0;f}(G) = \max\{2, \Delta_f(G)\}$; If $1 \le t < \Delta_f(G)$, by (1) of Theorem 4.9 we have $\chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1$ or $\Delta_f(G) + t$. We may ask what conditions are needed for a bipartite graph G with $1 \le t < \Delta_f(G)$ to satisfy $\chi_{1,1,t;f}(G) = \Delta_f(G) + t + 1$?

5. Problems for further research

In this paper, we present a new coloring of a graph G for the first time. We named it an $[r,\ s,\ t;\ f]$-*coloring* of G and investigate some interesting properties on the $[r,\ s,\ t;\ f]$-chromatic number. Some are the generalization of the results about the $[r,\ s,\ t]$-coloring and the other are new. However, all the results in our paper are correct for $[r,\ s,\ t]$-coloring just let $f(v) = 1$ for all $v \in V(G)$.

Finally, we present the following problems for further research.

Problem 1. Find the properties of the f-*total coloring* as we defined in Section 1. Is there a conjecture like the TCC for it?

Problem 2. Find the other results on the chromatic number $\chi_{1,1,t;f}(G)$.

Problem 3. Find the exact values of $\chi_{r,s,t;f}(G)$ for some special graphs.

Acknowledgements. We would like to thank the referees for various comments whose suggestions greatly improved the present paper.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[2] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941), 194–197.

[3] L. Dekar, et al., $[r, s, t]$-*coloring of trees and bipartite graphs*, Discrete Math. (2008) doi:10.1016/j.disc.2008.09.021.

[4] S. L. Hakimi and O. Kariv, *A generalization of edge-coloring in graphs*, J. Graph Theory **10** (1986), no. 2, 139–154.

[5] F. Havet and M. L. Yu, $(p, 1)$-*total labelling of graphs*, Discrete Math. **308** (2008), no. 4, 496–513.

[6] A. Kemnitz and M. Marangio, $[r, s, t]$-*colorings of graphs*, Discrete Math. **307** (2007), no. 2, 199–207.

[7] X. Zhang and G. Liu, *The classification of complete graphs* K_n *on* f-*coloring*, J. Appl. Math. Comput. **19** (2005), no. 1-2, 127–133.

[8] ———, *Some sufficient conditions for a graph to be of* C_f *1*, Appl. Math. Lett. **19** (2006), no. 1, 38–44.

[9] ———, *Some graphs of class 1 for* f-*colorings*, Appl. Math. Lett. **21** (2008), no. 1, 23–29.

Yong Yu
School OF Mathematics
Shandong University Jinan
Shandong, 250100, P. R. China
E-mail address: yuyong6834@yahoo.com.cn

Guizhen Liu
School OF Mathematics
Shandong University Jinan
Shandong, 250100, P. R. China
E-mail address: gzliu@sdu.edu.cn