

COMPOUND-COMMUTING ADDITIVE MAPS ON MATRIX SPACES

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ABSTRACT. In this note, compound-commuting additive maps on matrix spaces are studied. We show that compound-commuting additive maps send rank one matrices to matrices of rank less than or equal to one. By using the structural results of rank-one nonincreasing additive maps, we characterize compound-commuting additive maps on four types of matrices: triangular matrices, square matrices, symmetric matrices and Hermitian matrices.

1. Introduction

Let \mathcal{M}_1 and \mathcal{M}_2 be matrix spaces over the same field. Let Φ be a matrix function such that $\Phi(A) \in \mathcal{M}_i$ whenever $A \in \mathcal{M}_i$ for $i = 1, 2$. We say that a map $\psi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is Φ -commuting if $(\psi \circ \Phi)(A) = (\Phi \circ \psi)(A)$ for all $A \in \mathcal{M}_1$. In 1982, Sinkhorn [13] first studied Φ -commuting linear maps on n -square complex matrices with $\Phi(A) = \text{adj}A$, where $\text{adj}A$, the adjugate of A , is the matrix whose (i, j) -th entry is the cofactor of the row j and column i of A . By using the classical theorem of Frobenius [6] concerning determinant linear preservers, he gave a general form of adjugate-commuting linear maps on n -square complex matrices and showed that if $n \geq 3$, then the map is of the form

$$A \mapsto \lambda PAP^{-1} \quad \text{or} \quad A \mapsto \lambda PA^t P^{-1},$$

where $\lambda \in \mathbb{C}$ with $\lambda^{n-2} = 1$, P is an invertible n -square complex matrix, and A^t denotes the transpose of A . This result was generalized by Chan et al. in [2] to arbitrary infinite fields based on the structural results of linear maps preserving invertibility. In the same paper, linear maps on the spaces of square matrices and symmetric matrices that commute with the exponential function, i.e., $\Phi(A) = e^A$, were also considered. Later on, Chan and Lim [1] characterized linear maps on square matrices that commute with the k -th power function,

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i.e., $\Phi(A) = A^k$ for some fixed positive integer k . More recently, by relaxing the linearity assumption, some researchers considered Φ -commuting additive maps. For instance, adjugate-commuting additive maps were studied on the space of complex Hermitian matrices in [14], square matrices and symmetric matrices in [15], and triangular matrices in [15, 3], respectively. We refer the reader to [12, Chapter 9] and [17, Chapter 10] for more information.

Let \mathbb{F} be a field and let $\mathcal{M}_{m,n}(\mathbb{F})$ denote the linear space of $m \times n$ rectangular matrices over \mathbb{F} . We write $\mathcal{M}_{n,n}(\mathbb{F})$ as $\mathcal{M}_n(\mathbb{F})$. If $A \in \mathcal{M}_n(\mathbb{F})$, then the $(n-1)$ -th *compound matrix*, or simply *compound matrix* of A , denoted by $C_{n-1}(A)$, is the n -square matrix whose entries are

$$C_{n-1}(A)_{ij} = \det(A[n+1-i | n+1-j])$$

with $1 \leq i, j \leq n$, where $A[i|j]$ is the $(n-1)$ -square submatrix of A obtained by excluding its i -th row and j -th column. In this note, we study Φ -commuting additive maps in the context of $\Phi(A) = C_{n-1}(A)$, and call such a map *compound-commuting*. Nevertheless, to our knowledge, compound-commuting additive maps on matrix spaces have not been studied yet. We show, in Lemma 2.5, that every nonzero compound-commuting additive map is rank-one nonincreasing, i.e., the map sends rank one matrices to matrices of rank less than or equal to one. By using the structural results of rank-one nonincreasing additive maps on the space of block triangular matrices in [3], symmetric matrices in [8, 7], and Hermitian matrices in [9, 10, 11], respectively, we characterize compound-commuting additive maps on the spaces of square matrices, triangular matrices and symmetric matrices over arbitrary fields, and on the space of Hermitian matrices over a field with proper involution. We will see in Theorem 2.8 that the classification of compound-commuting additive maps on triangular matrices is quite different and essentially more complicated than the corresponding theorems on spaces of square matrices, symmetric matrices, and Hermitian matrices.

We introduce some notation that will be needed in our discussion. Throughout this note, unless otherwise specified, we assume \mathbb{F} is an arbitrary field, and m and n are positive integers with $n \leq m$. Let k, n_1, \dots, n_k be positive integers satisfying $n_1 + \dots + n_k = n$. By $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$, we designate the subalgebra of $\mathcal{M}_n(\mathbb{F})$ consisting of block matrices (A_{ij}) of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}$$

with $A_{ij} \in \mathcal{M}_{n_i, n_j}(\mathbb{F})$ for every $1 \leq i \leq j \leq k$. We shall call such an algebra $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ a *block triangular matrix algebra*. In particular, when $n_i = 1$ for all i , then it constitutes the algebra of n -square triangular matrices and is abbreviated to $\mathcal{T}_n(\mathbb{F})$, and $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) = \mathcal{M}_n(\mathbb{F})$ when $k = 1$. For each block triangular matrix algebra $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$, we associate a chain of nonnegative

integers $0 = \delta_0 < \delta_1 < \dots < \delta_{k-1} < \delta_k = n$ with $\delta_i = n_1 + \dots + n_i$ for each $i = 1, \dots, k$. Let $\bar{} : \mathbb{F} \rightarrow \mathbb{F}$ be a field involution (i.e., $\overline{a+b} = \bar{a} + \bar{b}$, $\overline{ab} = \bar{b}\bar{a}$, and $\bar{\bar{a}} = a$ for every $a, b \in \mathbb{F}$). A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is *symmetric* if $A^t = A$, and is called a *Hermitian matrix on the involution $\bar{}$ of \mathbb{F}* , or simply *Hermitian*, if $A^* := \overline{A^t} = A$, where $\bar{}$ is applied on A entrywise. Let $\mathcal{S}_n(\mathbb{F})$ and $\mathcal{H}_n(\mathbb{F})$ denote, respectively, the set of all n -square symmetric matrices and n -square Hermitian matrices over \mathbb{F} . Note that $\mathcal{H}_n(\mathbb{F}) = \mathcal{S}_n(\mathbb{F})$ if the involution $\bar{}$ is identity. We will use E_{ij} and F_{ij} to designate the matrix units of $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{M}_m(\mathbb{F})$, respectively, whose (i, j) -th entry is one and the others are zero. We denote by I_n the n -square identity matrix, and Z_n the diagonal matrix $\sum_{i=1}^n (-1)^{i+1} E_{ii}$. Given a matrix $A = (a_{ij}) \in \mathcal{M}_{m,n}(\mathbb{F})$, by $\text{rank} A$, A^σ and A^\sim , we denote, respectively, the rank of A , the $m \times n$ matrix whose (i, j) -th entry is $\sigma(a_{ij})$ with σ a field homomorphism on \mathbb{F} , and the $n \times m$ matrix whose (i, j) -th entry is $a_{m+1-j, n+1-i}$. It is easily verified that $A^\sim = J_n A^t J_m$ where $J_n := E_{n1} + E_{n-1,2} + \dots + E_{1n}$.

2. Results

In this section we characterize compound-commuting additive maps on the space of square matrices, triangular matrices, symmetric matrices, and Hermitian matrices, respectively. For the construction of our main results, we establish the following lemmas.

We start with the following interesting result proved by Fošner and Šemrl in [5].

Lemma 2.1 ([5, Lemma 2.1]). *Let \mathbb{F} be a field, and let m and n be positive integers with $m > n$. If A_1, \dots, A_m are matrices in $\mathcal{M}_n(\mathbb{F})$ such that $\det(\sum_{i \in I} A_i) = 0$ for every nonempty proper subset I of $\{1, \dots, m\}$, then $\det(A_1 + \dots + A_m) = 0$.*

We recall from the introduction that

$$J_n = \sum_{i=1}^n E_{n+1-i,i} \quad \text{and} \quad Z_n = \sum_{i=1}^n (-1)^{i+1} E_{ii}.$$

The proofs of the following Lemmas 2.2 and 2.3 are standard and will be omitted.

Lemma 2.2. *Let \mathbb{F} be a field, and let n be an integer with $n \geq 2$. Then the following hold true.*

- (i) $C_{n-1}(I_n) = I_n$ and $C_{n-1}(0) = 0$.
- (ii) If $A = J_n$ or $A = Z_n$, then

$$C_{n-1}(A) = \begin{cases} -A & \text{if } n = 4k - 1, 4k, \text{ for } k = 1, 2, \dots, \\ A & \text{otherwise.} \end{cases}$$

- (iii) Let $a \in \mathbb{F}$. Then

- (a) $C_{n-1}(I_n - E_{n+1-i, n+1-i} + (a-1)E_{jj}) = aE_{ii}$ for every $1 \leq i \leq n$, where j is an integer with $1 \leq j \neq n+1-i \leq n$.
- (b) $C_{n-1}(I_n - E_{n+1-i, n+1-i} - E_{n+1-j, n+1-j} + (-1)^{i+j+1}aE_{n+1-j, n+1-i}) = aE_{ij}$ for every $1 \leq i \neq j \leq n$.

Lemma 2.3. *Let \mathbb{F} be a field, and let n be an integer with $n \geq 2$. Let $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ be a field homomorphism and let $- : \mathbb{F} \rightarrow \mathbb{F}$ be a field involution. Let $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$. Then the following statements hold true.*

- (i) $C_{n-1}(\alpha A) = \alpha^{n-1}C_{n-1}(A)$.
- (ii) $C_{n-1}(AB) = C_{n-1}(A)C_{n-1}(B)$.
- (iii) $C_{n-1}(A^{-1}) = C_{n-1}(A)^{-1}$ when A is invertible.
- (iv) $C_{n-1}(A^\sigma) = C_{n-1}(A)^\sigma$.
- (v) $C_{n-1}(A^t) = C_{n-1}(A)^t$.
- (vi) $C_{n-1}(A^\sim) = \overline{C_{n-1}(A)}^\sim$.
- (vii) $C_{n-1}(\overline{A}) = \overline{C_{n-1}(A)}$.
- (viii) $\text{rank}C_{n-1}(A) = n$ whenever $\text{rank}A = n$.
- (ix) $\text{rank}C_{n-1}(A) = 1$ whenever $\text{rank}A = n-1$.
- (x) $\text{rank}C_{n-1}(A) = 0$ whenever $\text{rank}A \leq n-2$.
- (xi) $C_{n-1}(A) = Z_n((\text{adj}A)^\sim)Z_n$.

Lemma 2.4. *Let \mathbb{F} be a field and \mathbb{K} be a field with an involution $-$. Let n be an integer with $n \geq 2$. Let $\mathcal{S} = \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ or $\mathcal{H}_n(\mathbb{K})$. If $A \in \mathcal{S}$ is of rank one, then there is a rank $n-1$ matrix $B \in \mathcal{S}$ such that $A = C_{n-1}(B)$.*

Proof. Let $A \in \mathcal{S}$ be a rank one matrix. We first claim that there exists a rank $n-1$ matrix $D \in \mathcal{S}$ such that $A = \text{adj}D$. We distinguish our proof into the following two cases.

Case I: $\mathcal{S} = \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$. Then there exist $E_{st} \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ and invertible matrices $P_1, P_2 \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ such that $A = P_1 E_{st} P_2$ (see [4, Lemma 2.1]). Let $\alpha_i = (\det P_i)^{n-2}$ and let $Q_i = \text{adj}P_i$ for $i = 1, 2$. Clearly, $\alpha_1, \alpha_2 \in \mathbb{F}$ are nonzero and $Q_1, Q_2 \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ are invertible. We let

$$D = \begin{cases} Q_2 \left(I_n - E_{ss} + \left(\frac{1}{\alpha_1 \alpha_2} - 1 \right) E_{jj} \right) Q_1 & \text{if } s = t, \\ Q_2 \left(I_n - E_{ss} - E_{tt} - \frac{1}{\alpha_1 \alpha_2} E_{st} \right) Q_1 & \text{if } s \neq t. \end{cases}$$

Here, j is an integer with $1 \leq j \neq s \leq n$. It is clear that $D \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ is of rank $n-1$. Since $\text{adj}Q_i = (\det P_i)^{n-2}P_i = \alpha_i P_i$ for $i = 1, 2$, it follows that $\text{adj}D = (\alpha_1 P_1) \left(\frac{1}{\alpha_1 \alpha_2} E_{st} \right) (\alpha_2 P_2) = P_1 E_{st} P_2 = A$.

Case II: $\mathcal{S} = \mathcal{H}_n(\mathbb{K})$. If $A \in \mathcal{S}$ is of rank one, then there exist an invertible matrix $Q \in \mathcal{M}_n(\mathbb{K})$ and a scalar $\lambda \in \mathbb{K}$ with $\overline{\lambda} = \lambda$ such that $A = \lambda Q E_{11} Q^*$ (see [16, Proposition 1.32]). Let $\beta = (\det Q)^{n-2} (\overline{\det Q})^{n-2}$. It is clear that $\overline{\beta} = \beta \neq 0$. Let $U = \text{adj}Q$, and let

$$D = U^* \left(I_n - E_{11} - E_{22} + \frac{\lambda}{\beta} E_{22} \right) U.$$

It is clear that $D \in \mathcal{H}_n(\mathbb{K})$ because $\overline{\lambda/\beta} = \lambda/\beta$. Since $\text{adj}(U^*) = (\text{adj}U)^*$, we have

$$\begin{aligned} \text{adj}D &= (\text{adj}U) \text{adj} \left(I_n - E_{11} - E_{22} + \frac{\lambda}{\beta} E_{22} \right) (\text{adj}U)^* \\ &= (\det Q)^{n-2} Q \left(\frac{\lambda}{\beta} E_{11} \right) Q (\overline{\det Q})^{n-2} Q^* = (\lambda E_{11}) Q^* = A. \end{aligned}$$

Our claim is proved.

Let $B = Z_n D \sim Z_n$. Evidently, B is a rank $n-1$ matrix in \mathcal{S} , and

$$(1) \quad C_{n-1}(B) = C_{n-1}(Z_n D \sim Z_n) = C_{n-1}(Z_n) C_{n-1}(D \sim) C_{n-1}(Z_n).$$

By the fact that $\text{adj}(D \sim) = (\text{adj}D) \sim$ and Lemma 2.3(xi), we have $C_{n-1}(D \sim) = Z_n (\text{adj}D) Z_n$. By (1) and Lemma 2.2(ii), $C_{n-1}(B) = Z_n C_{n-1}(D \sim) Z_n = Z_n (Z_n (\text{adj}D) Z_n) Z_n = \text{adj}D = A$, as desired. \square

Lemma 2.5. *Let \mathbb{F} be a field and let \mathbb{K} be a field with an involution $\bar{\cdot}$. Let m and n be integers with $m, n \geq 3$. Let ψ be a compound-commuting additive map from $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ into $\mathcal{M}_m(\mathbb{F})$ or from $\mathcal{H}_n(\mathbb{K})$ into $\mathcal{H}_m(\mathbb{K})$. Then ψ is rank-one nonincreasing. Moreover, if $\psi \neq 0$, then $m = n$ and $\psi(I_n)$ is of rank n .*

Proof. Let $\mathcal{S} = \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ or $\mathcal{H}_n(\mathbb{K})$. Let $A \in \mathcal{S}$ be a rank one matrix. We see that $\text{rank} \psi(A) \neq m$ because $n \geq 3$ and $C_{n-1}(\psi(A)) = \psi(C_{n-1}(A)) = \psi(0) = 0$. By Lemma 2.4, there exists a rank $n-1$ matrix $B \in \mathcal{S}$ such that $A = C_{n-1}(B)$, and so, $\psi(A) = \psi(C_{n-1}(B)) = C_{n-1}(\psi(B))$. Since $\text{rank} \psi(A) \neq m$, it follows that $\text{rank} \psi(B) \leq m-1$, and hence, $\text{rank} \psi(A) \leq 1$. Consequently, ψ is a rank-one nonincreasing additive map.

Let ψ be a nonzero map. Since $C_{n-1}(\psi(I_n)) = \psi(C_{n-1}(I_n)) = \psi(I_n)$, it follows from Lemma 2.3(viii)-(x) that either $\psi(I_n) = 0$ or $\psi(I_n)$ is of rank m . Suppose $\psi(I_n) = 0$. Then $\psi(E_{ii}) = 0$ for all $i = 1, \dots, n$. This is because, for each $1 \leq i \leq n$, $\psi(E_{ii}) = \psi(C_{n-1}(I_n - E_{n+1-i, n+1-i})) = C_{n-1}(\psi(I_n - E_{n+1-i, n+1-i})) = C_{n-1}(-\psi(E_{n+1-i, n+1-i})) = 0$. For each $a \in \mathbb{F}$ (respectively, $a \in \mathbb{K}$ with $\bar{a} = a$ if $\mathcal{S} = \mathcal{H}_n(\mathbb{K})$) and $E_{ii} \in \mathcal{S}$, in view of Lemma 2.2(iii), we see that $aE_{ii} = C_{n-1}(I_n - E_{n+1-i, n+1-i} + (a-1)E_{jj})$, where j is an integer with $1 \leq j \neq n+1-i \leq n$. Thus, $\psi(aE_{ii}) = C_{n-1}(\psi(I_n - E_{n+1-i, n+1-i} + (a-1)E_{jj})) = C_{n-1}(\psi((a-1)E_{jj})) = 0$. Therefore, we have

$$(2) \quad \psi(aE_{ii}) = 0$$

for every $E_{ii} \in \mathcal{S}$ and $a \in \mathbb{F}$ (respectively, $a \in \mathbb{K}$ with $\bar{a} = a$ if $\mathcal{S} = \mathcal{H}_n(\mathbb{K})$).

When $\mathcal{S} = \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$, for each $a \in \mathbb{F}$ and $E_{ij} \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ with $1 \leq i \neq j \leq n$, we have

$$aE_{ij} = C_{n-1}(I_n - E_{n+1-i, n+1-i} - E_{n+1-j, n+1-j} + (-1)^{i+j+1} a E_{n+1-j, n+1-i})$$

by Lemma 2.2(iii). So, by (2), we see that

$$\psi(aE_{ij}) = C_{n-1}(\psi((-1)^{i+j+1} a E_{n+1-j, n+1-i})) = 0$$

since ψ is rank-one nonincreasing. Therefore, $\psi(aE_{ij}) = 0$ for every $a \in \mathbb{F}$ and $E_{ij} \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ with $1 \leq i \neq j \leq n$.

When $\mathcal{S} = \mathcal{H}_n(\mathbb{K})$, for each $a \in \mathbb{K}$ and $1 \leq i \neq j \leq n$, for the sake of simplicity, we denote

$$G = I_n - E_{n+1-i, n+1-i} - E_{n+1-j, n+1-j} + aE_{n+1-i, n+1-j} + \bar{a}E_{n+1-j, n+1-i}.$$

We see that $G \in \mathcal{H}_n(\mathbb{K})$ and $C_{n-1}(G) = -a\bar{a}(I_n - E_{ii} - E_{jj}) + (-1)^{i+j+1}\bar{a}E_{ij} + aE_{ji}$. By (2), we obtain $\psi(C_{n-1}(G)) = \psi(-a\bar{a}(I_n - E_{ii} - E_{jj}) + (-1)^{i+j+1}\bar{a}E_{ij} + aE_{ji}) = (-1)^{i+j+1}\psi(\bar{a}E_{ij} + aE_{ji})$. On the other hand, by (2), we see that

$$\begin{aligned} \psi(C_{n-1}(G)) &= C_{n-1}(\psi(G)) = C_{n-1}(\psi(aE_{n+1-i, n+1-j} + \bar{a}E_{n+1-j, n+1-i})) \\ &= \psi(C_{n-1}(aE_{n+1-i, n+1-j} + \bar{a}E_{n+1-j, n+1-i})) = 0. \end{aligned}$$

Hence, $\psi(\bar{a}E_{ij} + aE_{ji}) = 0$ for all $a \in \mathbb{K}$ and $1 \leq i \neq j \leq n$.

In both cases, together with (2), we conclude that $\psi = 0$, which contradicts to our hypothesis. Then $\psi(I_n) \neq 0$. Hence, ψ is a rank-one nonincreasing additive mapping with $\psi(I_n)$ being of rank m . By the additivity of ψ , we see that

$$m = \text{rank}\psi(I_n) \leq \sum_{i=1}^n \text{rank}\psi(E_{ii}) \leq n.$$

Suppose $n > m$. Since $\psi(I_n)$ is of rank m , it follows from Lemma 2.1 that there exist integers $1 \leq s_1 < \dots < s_k \leq n$ with $m \leq k \leq n-1$ such that $\text{rank}\psi(E_{s_1 s_1} + \dots + E_{s_k s_k}) = m$. Thus,

$$m = \text{rank}C_{n-1}(\psi(E_{s_1 s_1} + \dots + E_{s_k s_k})) = \text{rank}\psi(C_{n-1}(E_{s_1 s_1} + \dots + E_{s_k s_k})) \leq 1,$$

a contradiction. Hence, $m = n$. This completes our proof. \square

Let $n \geq 1$ and let \mathbb{F} be a field. Let $r, \epsilon_1, \dots, \epsilon_r$ be positive integers such that $\epsilon_1 + \dots + \epsilon_r = n$. For each integer $1 \leq i \leq r$, we denote $d_i = \epsilon_1 + \dots + \epsilon_i$ and $d_0 = 0$. Given a matrix $A \in \mathcal{M}_n(\mathbb{F})$, we associate a unique n -square block diagonal matrix

$$\bigoplus_{i=1}^r A_i := \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_r \end{pmatrix}$$

where, for each $1 \leq i \leq r$, A_i is the ϵ_i -square principal submatrix of A lying in rows $(d_{i-1} + 1), \dots, d_i$ and columns $(d_{i-1} + 1), \dots, d_i$. Such a block matrix $\bigoplus_{i=1}^r A_i$ is called the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by A .

Given an n -square block triangular matrix algebra $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$, we denote

$$\Omega = \{\delta_i : n_i = 1, 1 \leq i \leq k\} \cup \{1, n\}.$$

Let h be a positive integer with $h \leq |\Omega|$. By H_{h, n_1, \dots, n_k} , we designate the totality of strictly increasing sequences of h integers $\alpha = (\alpha_1, \dots, \alpha_h)$ chosen from Ω satisfying $\alpha_1 = 1$ and $\alpha_h = n$. In particular, when $n_i = 1$ for all i , we write H_{h, n_1, \dots, n_k} as $H_{h, n}$. It is clear that when $n = 1$ (i.e., $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) =$

$\mathcal{T}_1(\mathbb{F})$, $\Omega = \{1\}$ and $\alpha = 1$. Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_h) \in H_{h,n}$, we denote $\alpha^\sim = (n+1-\alpha_h, \dots, n+1-\alpha_2, n+1-\alpha_1) \in H_{h,n}$, and clearly, $(\alpha^\sim)^\sim = \alpha$. For example, if $\alpha = (1, 2, 6, 9) \in H_{4,9}$, then $\alpha^\sim = (1, 4, 8, 9) \in H_{4,9}$.

For each matrix $A = (a_{ij}) \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$ and $\alpha = (\alpha_1, \dots, \alpha_h) \in H_{h, n_1, \dots, n_k}$, we associate with a unique n -square matrix, denoted $A(\alpha)$, to be $A(\alpha) := A$ when $n = 1$, and

$$(3) \quad A(\alpha) := \sum_{\substack{1 \leq s < h \\ s \text{ odd}}} A_{\alpha_s} + \sum_{\substack{1 \leq s < h \\ s \text{ even}}} A_{\alpha_s}^t - \sum_{1 < s < h} a_{\alpha_s \alpha_s} E_{\alpha_s \alpha_s} \quad \text{when } n \geq 2$$

where, for each $1 \leq s < h$, $A_{\alpha_s} = (b_{ij})$ is the n -square matrix whose entries are given by

$$b_{ij} = \begin{cases} a_{ij} & \text{if } \alpha_s \leq i, j \leq \alpha_s + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $\sigma_1, \dots, \sigma_{h-1}$ are field homomorphisms on \mathbb{F} , we use the notation $A(\alpha)^{\sigma_1 \dots \sigma_{h-1}}$ to designate the n -square matrix defined by

$$(4) \quad A(\alpha)^{\sigma_1 \dots \sigma_{h-1}} := \sum_{\substack{1 \leq s < h \\ s \text{ odd}}} A_{\alpha_s}^{\sigma_s} + \sum_{\substack{1 \leq s < h \\ s \text{ even}}} (A_{\alpha_s}^{\sigma_s})^t - \sum_{1 < s < h} \sigma_s(a_{\alpha_s \alpha_s}) E_{\alpha_s \alpha_s}$$

when $n \geq 2$, and $A(\alpha)^{\sigma_1 \dots \sigma_{h-1}} := A^{\sigma_1}$ when $n = 1$. If $\sigma_1 = \dots = \sigma_{h-1} = \sigma$, then $A(\alpha)^{\sigma_1 \dots \sigma_{h-1}} = A(\alpha)^\sigma$. For example, if $A = (a_{ij}) \in \mathcal{T}_6(\mathbb{F})$, $\alpha = (1, 2, 3, 6) \in H_{4,6}$, and $\sigma_1, \sigma_2, \sigma_3$ are field homomorphisms on \mathbb{F} , then

$$A(\alpha)^{\sigma_1, \sigma_2, \sigma_3} = \begin{pmatrix} \sigma_1(a_{11}) & \sigma_1(a_{12}) & 0 & 0 & 0 & 0 \\ 0 & \sigma_1(a_{22}) & 0 & 0 & 0 & 0 \\ 0 & \sigma_2(a_{23}) & \sigma_2(a_{33}) & \sigma_3(a_{34}) & \sigma_3(a_{35}) & \sigma_3(a_{36}) \\ 0 & 0 & 0 & \sigma_3(a_{44}) & \sigma_3(a_{45}) & \sigma_3(a_{46}) \\ 0 & 0 & 0 & 0 & \sigma_3(a_{55}) & \sigma_3(a_{56}) \\ 0 & 0 & 0 & 0 & 0 & \sigma_3(a_{66}) \end{pmatrix}.$$

Let m, n and p be integers with $1 \leq n \leq m$ and $0 \leq p \leq m - n$. We recall that E_{ij} and F_{ij} denote the matrix units in $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{M}_m(\mathbb{F})$, respectively. Let $\Phi : \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ be the additive map defined by

- $\Phi(A) = \begin{pmatrix} 0_p & 0 & 0 \\ 0 & A(\alpha)^{\sigma_1 \dots \sigma_{h-1}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all $A = (a_{ij}) - \sum_{s=1}^h a_{\alpha_s \alpha_s} E_{\alpha_s \alpha_s} \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$,
- (I) • $\Phi(aE_{\alpha_s \alpha_s}) = \sum_{j=1}^m g_{sj}(a) F_{p+\alpha_s, j}$ for all $a \in \mathbb{F}$, whenever $1 \leq s \leq h$ is odd, and
- $\Phi(aE_{\alpha_s \alpha_s}) = \sum_{j=1}^m g_{sj}(a) F_{j, p+\alpha_s}$ for all $a \in \mathbb{F}$, whenever $1 \leq s \leq h$ is even,

where $\alpha = (\alpha_1, \dots, \alpha_h) \in H_{h, n_1, \dots, n_k}$, $\sigma_1, \dots, \sigma_{h-1}$ are nonzero field homomorphisms on \mathbb{F} , and g_{11}, \dots, g_{hm} are additive maps on \mathbb{F} such that

- (i) for each $1 \leq s \leq h$, $g_{s,p+\alpha_s}(1) = 1$ and $g_{sj}(1) = 0$ for $j \neq p + \alpha_s$,
- (ii) when $n_1 \geq 2$, $g_{1,p+1} = \sigma_1$ and $g_{1j} = 0$ for every $j \neq p + 1$, and
- (iii) when $n_k \geq 2$, $g_{h,p+n} = \sigma_{h-1}$ and $g_{hj} = 0$ for every $j \neq p + n$.

In view of Definitions (3) and (4), it is easy to check that $\text{rank} A(\alpha)^{\sigma_1 \cdots \sigma_{h-1}} \leq 1$ whenever A is of rank one. Consequently, the map Φ defined in (I) is a rank-one nonincreasing additive map with $\Phi(I_n) = 0_p \oplus I_n \oplus 0_{m-p-n}$.

An additive map $\psi : \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ is said to be (n, p) -block rank-one nonincreasing if ψ or ψ^t is of the form Φ defined in (I). Here, ψ^t is the map defined by $\psi^t(A) = \psi(A)^t$ for all $A \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F})$.

We state the following result proved in [3, Theorem 2.10].

Lemma 2.6. *Let \mathbb{F} be a field, and let m and n be positive integers with $m \geq n \geq 1$. Then $\psi : \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ is a rank-one nonincreasing additive map with $\psi(I_n)$ being of rank n if and only if there exist invertible matrices $P, Q \in \mathcal{M}_m(\mathbb{F})$ and integers $0 = s_0 < s_1 < \cdots < s_r = k$ such that*

$$\psi(A) = P \left(\sum_{i=1}^r \Phi_i(A_i) \right) Q \quad \text{for all } A \in \mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}).$$

Here, $\bigoplus_{i=1}^r A_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by A with $\epsilon_i = \delta_{s_i} - \delta_{s_{i-1}}$, $\delta_{s_i} = n_1 + \cdots + n_{s_i}$ and $\delta_0 = 0$, and for each $1 \leq i \leq r$, $\Phi_i : \mathcal{T}_{n_{s_{i-1}+1}, \dots, n_{s_i}}(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ is a $(\epsilon_i, \delta_{s_{i-1}})$ -block rank-one nonincreasing additive map as described in (I).

Lemma 2.7. *Let \mathbb{F} be a field. Let $r, \epsilon_1, \dots, \epsilon_r$ be positive integers such that $\epsilon_1 + \cdots + \epsilon_r = n$. Let $A \in \mathcal{T}_n(\mathbb{F})$, and let $\alpha_i \in H_{k_i, \epsilon_i}$ with positive integer $k_i \leq \epsilon_i$ for $i = 1, \dots, r$. Then*

$$C_{n-1} \left(\bigoplus_{i=1}^r A_i(\alpha_i) \right) = \left(\bigoplus_{i=1}^r C_{n-1}(A^\sim)_i(\alpha_i) \right)^\sim,$$

where $\bigoplus_{i=1}^r A_i$ and $\bigoplus_{i=1}^r C_{n-1}(A^\sim)_i$ are the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrices induced by A and $C_{n-1}(A^\sim)$, respectively.

Proof. Let $N = \{1, \dots, n\}$ and let

$$\mathcal{S} = \left\{ (s, t) \in N \times N \mid E_{st} \in \mathcal{T}_n(\mathbb{F}) \text{ and } \bigoplus_{i=1}^r (E_{st})_i \neq 0 \right\},$$

where $\bigoplus_{i=1}^r (E_{st})_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by E_{st} . Let $A \in \mathcal{T}_n(\mathbb{F})$. We denote

$$X = (x_{st}) = \text{adj} \left(\bigoplus_{i=1}^r A_i(\alpha_i) \right) \quad \text{and} \quad Y = (y_{st}) = \bigoplus_{i=1}^r (\text{adj} A)_i(\alpha_i),$$

where $\bigoplus_{i=1}^r A_i$ and $\bigoplus_{i=1}^r (\text{adj} A)_i$ are the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrices induced by A and $\text{adj} A$, respectively. We first claim that $X = Y$. Since the classical adjoint of a block triangular matrix is also a block triangular matrix

of the same type, it follows that $x_{st} = 0 = y_{st}$ for every $(s, t) \notin \mathcal{S}$. We now consider x_{pq} with $(p, q) \in \mathcal{S}$. Then there exists $1 \leq i_0 \leq r$ such that $(E_{pq})_{i_0} \neq 0$. So,

$$\begin{aligned} x_{pq} &= (-1)^{p+q} \det \left(\left(\bigoplus_{i=1}^r A_i(\alpha_i) \right) [q|p] \right) \\ &= (-1)^{p+q} \left(\prod_{i=1, i \neq i_0}^r \det A_i(\alpha_i) \right) \times \det(A_{i_0}(\alpha_{i_0})[q|p]). \end{aligned}$$

Since each A_i is upper triangular, by Definition (3), we have $\det A_i(\alpha_i) = \det A_i$, and so

$$(5) \quad x_{pq} = (-1)^{p+q} \left(\prod_{i=1, i \neq i_0}^r \det A_i \right) \times \det(A_{i_0}(\alpha_{i_0})[q|p]).$$

We now consider the following two cases. If $(E_{pq})_{i_0}(\alpha_{i_0}) = 0$, then $y_{pq} = 0$ since $\text{adj}A \in \mathcal{T}_n(\mathbb{F})$. On the other hand, it can be verified that $\det(A_{i_0}(\alpha_{i_0})[q|p]) = 0$, and so, $x_{pq} = 0$ by (5). Hence, $x_{pq} = 0 = y_{pq}$. Next, we consider $(E_{st})_{i_0}(\alpha_{i_0}) \neq 0$. Since

$$\det(A_{i_0}(\alpha_{i_0})[q|p]) = \begin{cases} \det(A_{i_0}[q|p]) & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{pq})_{i_0}, \\ \det(A_{i_0}[p|q]) & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{pq})_{i_0}^t = (E_{qp})_{i_0}. \end{cases}$$

By (5),

$$\begin{aligned} x_{pq} &= \begin{cases} (-1)^{p+q} \left(\prod_{i=1, i \neq i_0}^r \det A_i \right) \times \det(A_{i_0}[q|p]) & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{pq})_{i_0} \\ (-1)^{p+q} \left(\prod_{i=1, i \neq i_0}^r \det A_i \right) \times \det(A_{i_0}[p|q]) & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{qp})_{i_0}, \end{cases} \\ &= \begin{cases} (-1)^{p+q} \det(A[q|p]) & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{pq})_{i_0} \\ (-1)^{p+q} \det(A[p|q]) & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{qp})_{i_0}, \end{cases} \\ &= \begin{cases} (\text{adj}A)_{pq} & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{pq})_{i_0} \\ (\text{adj}A)_{qp} & \text{if } (E_{pq})_{i_0}(\alpha_{i_0}) = (E_{qp})_{i_0}, \end{cases} \\ &= y_{pq} \end{aligned}$$

since $\text{adj}A \in \mathcal{T}_n(\mathbb{F})$. Hence, $x_{st} = y_{st}$ for every $1 \leq s, t \leq n$, as claimed. So, we have

$$(6) \quad \text{adj} \left(\bigoplus_{i=1}^r A_i(\alpha_i) \right) = \bigoplus_{i=1}^r (\text{adj}A)_i(\alpha_i).$$

Since $C_{n-1}(A) \sim Z_n(\text{adj}A)Z_n$, and $Z_n = \bigoplus_{i=1}^r (Z_n)_i$ and $(Z_n)_i A_i(\alpha_i) (Z_n)_i = (Z_n A Z_n)_i(\alpha_i)$, where $\bigoplus_{i=1}^r (Z_n)_i$ and $\bigoplus_{i=1}^r (Z_n A Z_n)_i$ are the $(\epsilon_1, \dots, \epsilon_r)$ -block

diagonal matrices induced by Z_n and $Z_n A Z_n$, respectively, and together with (6) and Lemma 2.3(xi), we obtain

$$\begin{aligned} \left(C_{n-1} \left(\bigoplus_{i=1}^r A_i(\alpha_i) \right) \right)^\sim &= Z_n \left(\text{adj} \left(\bigoplus_{i=1}^r A_i(\alpha_i) \right) \right) Z_n \\ &= Z_n \left(\bigoplus_{i=1}^r (\text{adj} A)_i(\alpha_i) \right) Z_n \\ &= \bigoplus_{i=1}^r (Z_n)_i (\text{adj} A)_i(\alpha_i) (Z_n)_i \\ &= \bigoplus_{i=1}^r C_{n-1}(A^\sim)_i(\alpha_i). \end{aligned}$$

Here, $\bigoplus_{i=1}^r C_{n-1}(A^\sim)_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by $C_{n-1}(A^\sim)$. This completes our proof. \square

By Lemmas 2.5, 2.6 and 2.7, we give a general form of compound-commuting additive maps from triangular matrices to square matrices over an arbitrary field. More precisely, we have:

Theorem 2.8. *Let \mathbb{F} be a field, and let m and n be integers with $m, n \geq 3$. Then $\psi : \mathcal{T}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ is a compound-commuting additive map if and only if either $\psi = 0$, or $m = n$ and there exist positive integers $0 = s_0 < s_1 < \dots < s_r = n$ with $s_i - s_{i-1} = s_{r+1-i} - s_{r-i}$ for $i = 1, \dots, r$, and a nonzero field homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ such that*

$$\psi(A) = P \left(\bigoplus_{i=1}^r [\lambda_i A_i(\alpha_i) + (1 - \lambda_i) A_i(\alpha_i)^t]^\sigma \right) Q \quad \text{for all } A \in \mathcal{T}_n(\mathbb{F}).$$

Here, $\bigoplus_{i=1}^r A_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by A with $\epsilon_i = s_i - s_{i-1}$ and $\epsilon_i = \epsilon_{r+1-i}$ for $i = 1, \dots, r$; $\alpha_i \in H_{k_i, \epsilon_i}$ for some positive integer $k_i \leq \epsilon_i$ with $k_i = k_{r+1-i}$ and $\alpha_i^\sim = \alpha_{r+1-i}$ for $i = 1, \dots, r$; $P, Q \in \mathcal{M}_n(\mathbb{F})$ are invertible with $C_{n-1}(P) = P(\bigoplus_{i=1}^r \mu_i I_{\epsilon_i})$ and $C_{n-1}(Q) = (\bigoplus_{i=1}^r \mu_i^{-1} I_{\epsilon_i})Q$ for some nonzero scalars $\mu_1, \dots, \mu_r \in \mathbb{F}$; and $\lambda_1, \dots, \lambda_r \in \{0, 1\}$ with $\lambda_{r+1-i} = \lambda_i$ when k_i even, or $\lambda_{r+1-i} = 1 - \lambda_i$ when k_i odd for $i = 1, \dots, r$.

Proof. Sufficiency part. It is clear that $\psi = 0$ is a compound-commuting additive map. Suppose $\psi \neq 0$ and it is of the form stated in Theorem 2.8. Let $A \in \mathcal{T}_n(\mathbb{F})$. By Lemma 2.7, it is easily checked that

$$\begin{aligned} &C_{n-1} \left(\bigoplus_{i=1}^r \lambda_i A_i(\alpha_i) + (1 - \lambda_i) A_i(\alpha_i)^t \right) \\ &= \left(\bigoplus_{i=1}^r \lambda_i C_{n-1}(A^\sim)_i(\alpha_i) + (1 - \lambda_i) C_{n-1}(A^\sim)_i(\alpha_i)^t \right)^\sim, \end{aligned}$$

where $\bigoplus_{i=1}^r A_i$ and $\bigoplus_{i=1}^r C_{n-1}(A^\sim)_i$ are the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrices induced by A and $C_{n-1}(A^\sim)$, respectively.

Since $(\bigoplus_{i=1}^r A_i)^\sim = \bigoplus_{i=1}^r (A_{r+1-i})^\sim$ and

$$(7) \quad A_i(\alpha_i)^\sim = \begin{cases} (A_i)^\sim(\alpha_i^\sim) & \text{when } \alpha_i \in H_{k_i, \epsilon_i} \text{ with } k_i \text{ even,} \\ (A_i)^\sim(\alpha_i^\sim)^t & \text{when } \alpha_i \in H_{k_i, \epsilon_i} \text{ with } k_i \text{ odd,} \end{cases}$$

it follows that

$$\begin{aligned} & \left(\bigoplus_{i=1}^r \lambda_i C_{n-1}(A^\sim)_i(\alpha_i) + (1 - \lambda_i) C_{n-1}(A^\sim)_i(\alpha_i)^t \right)^\sim \\ &= \bigoplus_{i=1}^r [\lambda_{r+1-i} C_{n-1}(A^\sim)_{r+1-i}(\alpha_{r+1-i}) + (1 - \lambda_{r+1-i}) C_{n-1}(A^\sim)_{r+1-i}(\alpha_{r+1-i})^t]^\sim \\ &= \bigoplus_{i=1}^r \beta_{r+1-i} (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_{r+1-i}^\sim) + (1 - \beta_{r+1-i}) (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_{r+1-i}^\sim)^t \end{aligned}$$

where $\beta_{r+1-i} = \lambda_{r+1-i}$ when k_{r+1-i} is even, or $\beta_{r+1-i} = 1 - \lambda_{r+1-i}$ when k_{r+1-i} is odd for $i = 1, \dots, r$. Since, for each $1 \leq i \leq r$, we have $\lambda_{r+1-i} = \lambda_i$ when k_i is even, or $\lambda_{r+1-i} = 1 - \lambda_i$ when k_i is odd, it follows that $\beta_{r+1-i} = \lambda_i$ for $i = 1, \dots, r$. Together with the fact that $k_{r+1-i} = k_i$ and $\alpha_{r+1-i}^\sim = \alpha_i$ for $i = 1, \dots, r$, we obtain

$$\begin{aligned} & \bigoplus_{i=1}^r \beta_{r+1-i} (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_{r+1-i}^\sim) + (1 - \beta_{r+1-i}) (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_{r+1-i}^\sim)^t \\ &= \bigoplus_{i=1}^r \lambda_i (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_i) + (1 - \lambda_i) (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_i)^t. \end{aligned}$$

Further, since $\epsilon_{r+1-i} = \epsilon_i$ for $i = 1, \dots, r$, we have

$$\bigoplus_{i=1}^r (A^\sim)_i = \bigoplus_{i=1}^r (A_{r+1-i})^\sim,$$

and so,

$$\bigoplus_{i=1}^r (A_{r+1-i})^\sim(\alpha_i) = \bigoplus_{i=1}^r (A^\sim)_i(\alpha_i),$$

where $\bigoplus_{i=1}^r (A^\sim)_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by A^\sim . Together with the fact that $C_{n-1}(A)^\sim = C_{n-1}(A^\sim)$, we see that

$$\begin{aligned} & \bigoplus_{i=1}^r \lambda_i (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_i) + (1 - \lambda_i) (C_{n-1}(A^\sim)_{r+1-i})^\sim(\alpha_i)^t \\ &= \bigoplus_{i=1}^r \lambda_i (C_{n-1}(A^\sim)^\sim)_i(\alpha_i) + (1 - \lambda_i) (C_{n-1}(A^\sim)^\sim)_i(\alpha_i)^t \end{aligned}$$

$$= \bigoplus_{i=1}^r \lambda_i C_{n-1}(A)_i(\alpha_i) + (1 - \lambda_i) C_{n-1}(A)_i(\alpha_i)^t.$$

Therefore,

$$\begin{aligned} C_{n-1}(\psi(A)) &= C_{n-1} \left(P \left(\bigoplus_{i=1}^r [\lambda_i A_i(\alpha_i) + (1 - \lambda_i) A_i(\alpha_i)^t]^\sigma \right) Q \right) \\ &= C_{n-1}(P) C_{n-1} \left(\bigoplus_{i=1}^r [\lambda_i A_i(\alpha_i) + (1 - \lambda_i) A_i(\alpha_i)^t]^\sigma \right) C_{n-1}(Q) \\ &= P \left(\bigoplus_{i=1}^r \lambda_i C_{n-1}(A)_i(\alpha_i) + (1 - \lambda_i) C_{n-1}(A)_i(\alpha_i)^t \right)^\sigma Q \\ &= \psi(C_{n-1}(A)). \end{aligned}$$

Hence, ψ is a compound-commuting additive map.

Necessity part. If $\psi \neq 0$, then, by Lemma 2.5, we have $m = n$, and ψ is a rank-one nonincreasing additive map with $\psi(I_n)$ being of rank n . In view of Lemma 2.6, there exist integers $0 = s_0 < s_1 < \dots < s_r = n$, and invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$ such that

$$(8) \quad \psi(A) = P \left(\sum_{i=1}^r \Phi_i(A_i) \right) Q \quad \text{for all } A \in \mathcal{T}_n(\mathbb{F}).$$

Here, $\bigoplus_{i=1}^r A_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by A with $\epsilon_i = s_i - s_{i-1}$ for $i = 1, \dots, r$, and each $\Phi_i : \mathcal{T}_{\epsilon_i}(\mathbb{F}) \rightarrow \mathcal{M}_n(\mathbb{F})$ is a (ϵ_i, s_{i-1}) -block rank-one nonincreasing additive map as described in **(I)**, i.e., either Φ_i or Φ_i^t is of the form:

- (a) $A \mapsto 0_{s_{i-1}} \oplus A(\alpha_i)^{\sigma_1 \dots \sigma_{k_i-1}} \oplus 0$ for all $A = (a_{pq}) - \sum_{t=1}^{k_i} a_{\alpha_{it}, \alpha_{it}} E_{\alpha_{it}, \alpha_{it}} \in \mathcal{T}_{\epsilon_i}(\mathbb{F})$,
- (II)**(b) $a E_{\alpha_{it}, \alpha_{it}} \mapsto \sum_{j=1}^n g_{tj}(a) E_{s_{i-1} + \alpha_{it}, j}$ for all $a \in \mathbb{F}$, whenever $1 \leq t \leq k_i$ is odd, and
- (c) $a E_{\alpha_{it}, \alpha_{it}} \mapsto \sum_{j=1}^n g_{tj}(a) E_{j, s_{i-1} + \alpha_{it}}$ for all $a \in \mathbb{F}$, whenever $1 \leq t \leq k_i$ is even.

Here, $\sigma_1, \dots, \sigma_{k_i-1}$ are nonzero field homomorphisms on \mathbb{F} , $\alpha_i = (\alpha_{i1}, \dots, \alpha_{ik_i}) \in H_{k_i, \epsilon_i}$ for $i = 1, \dots, n$, and $g_{11}, \dots, g_{k_i, n}$ are additive maps on \mathbb{F} such that for each $1 \leq t \leq k_i$, $g_{t, s_{i-1} + \alpha_{it}}(1) = 1$ and $g_{tj}(1) = 0$ for every $j \neq s_{i-1} + \alpha_{it}$. Let $\varphi : \mathcal{T}_n(\mathbb{F}) \rightarrow \mathcal{M}_n(\mathbb{F})$ be the additive map defined by

$$(9) \quad \varphi(A) = P^{-1} \psi(A) Q^{-1} \quad \text{for all } A \in \mathcal{T}_n(\mathbb{F}).$$

By (8) and (9), we get

$$(10) \quad \varphi(A) = \sum_{i=1}^r \Phi_i(A_i) \quad \text{for all } A \in \mathcal{T}_n(\mathbb{F}).$$

By (8) and the structure of Φ_i 's as described in **(II)**, we have $\psi(I_n) = PQ$. Since $C_{n-1}(\psi(I_n)) = \psi(C_{n-1}(I_n)) = \psi(I_n)$, we conclude that $C_{n-1}(PQ) = PQ$, and so, $P^{-1}C_{n-1}(P) = QC_{n-1}(Q^{-1})$. Let $1 \leq j \leq n$. Since $\psi(C_{n-1}(I_n - E_{n+1-j, n+1-j})) = C_{n-1}(\psi(I_n - E_{n+1-j, n+1-j}))$, it follows from (8) and **(II)** that $PE_{jj}Q = C_{n-1}(P)E_{jj}C_{n-1}(Q)$. Thus

$$P^{-1}C_{n-1}(P)E_{jj} = E_{jj}QC_{n-1}(Q^{-1}) = E_{jj}P^{-1}C_{n-1}(P) \quad \text{for all } j = 1, \dots, n.$$

Therefore, $P^{-1}C_{n-1}(P) = D$ for some invertible diagonal matrix $D = \text{diag}(d_1, \dots, d_n) \in \mathcal{M}_n(\mathbb{F})$. Hence

$$(11) \quad C_{n-1}(P) = PD \quad \text{and} \quad C_{n-1}(Q) = D^{-1}Q.$$

Since $\psi(C_{n-1}(A)) = C_{n-1}(\psi(A))$ for all $A \in \mathcal{T}_n(\mathbb{F})$, it follows from (9) and (11) that

$$(12) \quad \varphi(C_{n-1}(A)) = DC_{n-1}(\varphi(A))D^{-1} \quad \text{for all } A \in \mathcal{T}_n(\mathbb{F}).$$

In view of (10), and by the structure of Φ_i 's as described in **(II)**, we see that $\varphi(E_{ss}) = E_{ss}$ for all s , and when $s < t$, we have $\varphi(E_{st}) = E_{st}$ or $\varphi(E_{st}) = E_{ts}$ for every $E_{st} \notin \ker \varphi$. Here, $\ker \varphi$ stands for the kernel of φ . We now show that if $E_{st} \notin \ker \varphi$, then

$$(13) \quad \varphi(E_{st}) = E_{st} \quad \Rightarrow \quad \varphi(E_{n+1-t, n+1-s}) = E_{n+1-t, n+1-s},$$

$$(14) \quad \varphi(E_{st}) = E_{ts} \quad \Rightarrow \quad \varphi(E_{n+1-t, n+1-s}) = E_{n+1-s, n+1-t}.$$

We show only (13) as (14) can be proved similarly. By Lemma 2.2(iii)(b) and (12), we have

$$\begin{aligned} \varphi(E_{n+1-t, n+1-s}) &= \varphi(C_{n-1}(I_n - E_{ss} - E_{tt} + (-1)^{s+t+1}E_{st})) \\ &= DC_{n-1}(\varphi(I_n - E_{ss} - E_{tt} + (-1)^{s+t+1}E_{st}))D^{-1} \\ &= DC_{n-1}(I_n - E_{ss} - E_{tt} + (-1)^{s+t+1}E_{st})D^{-1} \\ &= D(E_{n+1-t, n+1-s})D^{-1} \\ &= (d_{n+1-t}d_{n+1-s}^{-1})E_{n+1-t, n+1-s}. \end{aligned}$$

Thus, $\varphi(E_{n+1-t, n+1-s}) \neq 0$. On the other hand, in view of the structure of Φ_i 's as described in **(II)**, we know that $\varphi(E_{n+1-t, n+1-s}) = E_{n+1-t, n+1-s}$ or $\varphi(E_{n+1-t, n+1-s}) = E_{n+1-s, n+1-t}$. So, we conclude that $\varphi(E_{n+1-t, n+1-s}) = E_{n+1-t, n+1-s}$, and thus, $(d_{n+1-t}d_{n+1-s}^{-1}) = 1$, that is, $d_{n+1-t} = d_{n+1-s}$. Consequently, (13) is proved. Moreover, we note that if $E_{n+1-t, n+1-s} \notin \ker \varphi$, then, by the structure of Φ_i 's as described in **(II)**, there exists an integer $1 \leq i_0 \leq r$ such that $s_{i_0-1} + 1 \leq n + 1 - s$, $n + t - t \leq s_{i_0}$. Therefore, the result $d_{n+1-t} = d_{n+1-s}$ whenever $E_{n+1-t, n+1-s} \notin \ker \varphi$ leads us to the conclusion $d_{s_{i-1}+1} = \dots = d_{s_i}$ for every $i = 1, \dots, r$. Hence, we obtain $D = \bigoplus_{i=1}^r \mu_i I_{\epsilon_i}$ for some nonzero scalars $\mu_1, \dots, \mu_r \in \mathbb{F}$, and by (11), we get $C_{n-1}(P) = P(\bigoplus_{i=1}^r \mu_i I_{\epsilon_i})$ and $C_{n-1}(Q) = (\bigoplus_{i=1}^r \mu_i^{-1} I_{\epsilon_i})Q$.

We claim that there exists a nonzero homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ such that either

$$(15) \quad \varphi(aE_{jj}) = \sigma(a)E_{jj} \quad \text{for every } a \in \mathbb{F} \text{ and } 1 \leq j \leq n.$$

By the structure of Φ_i 's as described in **(II)**, we see that, for each $1 \leq j \leq n$, $\varphi(aE_{jj})$ is of the form as described in (a), (b) or (c). We distinguish our proof into the following two cases:

Case I: Suppose there is an integer $1 \leq j_0 \leq n$ such that $\varphi(aE_{j_0, j_0})$ is of the form as described in (a), i.e., there exists a nonzero homomorphism $\sigma_{j_0} : \mathbb{F} \rightarrow \mathbb{F}$ such that $\varphi(aE_{j_0, j_0}) = \sigma_{j_0}(a)E_{j_0, j_0}$ for every $a \in \mathbb{F}$. Let $1 \leq j \neq j_0 \leq n$ and $a \in \mathbb{F}$. If $j \neq n+1-j_0$, then, by Lemma 2.2(iii)(a) and (12), we have

$$\begin{aligned} \varphi(aE_{jj}) &= \varphi(C_{n-1}(I_n - E_{n+1-j, n+1-j} + (a-1)E_{j_0, j_0})) \\ &= DC_{n-1}(\varphi(I_n - E_{n+1-j, n+1-j} + (a-1)E_{j_0, j_0}))D^{-1} \\ &= DC_{n-1}(I_n - E_{n+1-j, n+1-j} + (\sigma_{j_0}(a) - 1)E_{j_0, j_0})D^{-1} \\ &= D(\sigma_{j_0}(a)E_{jj})D^{-1} = \sigma_{j_0}(a)E_{jj}. \end{aligned}$$

If $j = n+1-j_0$, then, since $n \geq 3$, we select $1 \leq j_1 \leq n$ with $j_1 \neq j_0, n+1-j_0$, and so

$$\begin{aligned} \varphi(aE_{n+1-j_0, n+1-j_0}) &= \varphi(C_{n-1}(I_n - E_{j_0, j_0} + (a-1)E_{j_1, j_1})) \\ &= DC_{n-1}(I_n - E_{j_0, j_0} + (\sigma_{j_0}(a) - 1)E_{j_1, j_1})D^{-1} \\ &= \sigma_{j_0}(a)E_{n+1-j_0, n+1-j_0}. \end{aligned}$$

Consequently, we conclude that there exists a nonzero homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ such that $\varphi(aE_{jj}) = \sigma(a)E_{jj}$ for all $1 \leq j \leq n$ and $a \in \mathbb{F}$. We are done.

Case II: Suppose, for each $1 \leq j \leq n$, $\varphi(aE_{jj})$ is of the form as described in (b) or (c) only. We divide our argument into two cases:

Subcase 1: Suppose there exist integers $1 \leq s_1 \neq s_2 \leq n$ such that $\psi(aE_{s_1, s_1})$ and $\varphi(aE_{s_2, s_2})$ are of distinct forms. We may assume, without loss of generality, that $\psi(aE_{s_1, s_1})$ and $\varphi(aE_{s_2, s_2})$ are of the forms as described in (c) and (b), respectively, and $s_2 \neq n+1-s_1$. Let $\varphi(aE_{s_1, s_1}) = \sum_{j=1}^n g_{s_1, j}(a)E_{j, s_1}$ and $\varphi(aE_{s_2, s_2}) = \sum_{j=1}^n g_{s_2, j}(a)E_{s_2, j}$, where $g_{s_1, 1} \dots, g_{s_1, n}, g_{s_2, 1} \dots, g_{s_2, n}$, are additive maps on \mathbb{F} such that $g_{s_i, s_i}(1) = 1$ for $i = 1, 2$, and for each $i = 1, 2$, $g_{s_i, j}(1) = 0$ for all $j \neq s_i$. By (12), we see that

$$\begin{aligned} \varphi(aE_{s_1, s_1}) &= \varphi(C_{n-1}(I_n - E_{n+1-s_1, n+1-s_1} + (a-1)E_{s_2, s_2})) \\ &= DC_{n-1}(\varphi(I_n - E_{n+1-s_1, n+1-s_1} - E_{s_2, s_2} + aE_{s_2, s_2}))D^{-1} \\ &= DC_{n-1}\left(I_n - E_{n+1-s_1, n+1-s_1} - E_{s_2, s_2} + \sum_{j=1}^n g_{s_2, j}(a)E_{s_2, j}\right)D^{-1} \\ &= D(g_{s_2, s_2}(a)E_{s_1, s_1} + (-1)^{s_1+s_2+1}g_{s_2, n+1-s_1}(a)E_{s_1, n+1-s_2})D^{-1} \\ &= g_{s_2, s_2}(a)E_{s_1, s_1} + (-1)^{s_1+s_2+1}g_{s_2, n+1-s_1}(a)\beta_1 E_{s_1, n+1-s_2}, \end{aligned}$$

where $DE_{s_1, n+1-s_2}D^{-1} = \beta_1 E_{s_1, n+1-s_2}$ for some nonzero scalar $\beta_1 \in \mathbb{F}$. Then

$$\begin{aligned} \sum_{j=1}^n g_{s_1, j}(a)E_{j, s_1} &= \varphi(aE_{s_1, s_1}) \\ &= g_{s_2, s_2}(a)E_{s_1, s_1} + (-1)^{s_1+s_2+1}g_{s_2, n+1-s_1}(a)\beta_1 E_{s_1, n+1-s_2}, \end{aligned}$$

and so, $g_{s_1, j} = 0$ for all $j \neq s_1$. Hence, we obtain $\varphi(aE_{s_1, s_1}) = g_{s_1, s_1}(a)E_{s_1, s_1}$ for all $a \in \mathbb{F}$.

Subcase 2: Suppose all $\varphi(aE_{jj})$'s are of the same form. Without loss of generality, we assume $\varphi(aE_{jj})$ is of form (b) for all $j = 1, \dots, n$. Since $n \geq 3$, there exist integers $1 \leq s_1 < s_2 < s_3 \leq n$ such that $s_2 \neq n+1-s_1$ and $s_3 \neq n+1-s_2$. Let $\varphi(aE_{s_i, s_i}) = \sum_{j=1}^n g_{s_i, j}(a)E_{s_i, j}$ for $i = 1, 2, 3$, where $g_{s_1, 1}, \dots, g_{s_1, n}, g_{s_2, 1}, \dots, g_{s_3, n}$ are additive maps on \mathbb{F} with $g_{s_i, s_i}(1) = 1$ for $i = 1, 2, 3$, and for each $1 \leq i \leq 3$, $g_{s_i, j}(1) = 0$ for all $j \neq s_i$. By a similar argument as in Subcase 1, we see that $\varphi(aE_{s_2, s_2}) = \varphi(C_{n-1}(I_n - E_{n+1-s_2, n+1-s_2} + (a-1)E_{s_3, s_3}))$ leads to

$$\begin{aligned} \sum_{j=1}^n g_{s_2, j}(a)E_{s_2, j} &= \varphi(aE_{s_2, s_2}) \\ &= g_{s_3, s_3}(a)E_{s_2, s_2} + (-1)^{s_2+s_3+1}g_{s_3, n+1-s_2}(a)\beta_2 E_{s_2, n+1-s_3} \end{aligned}$$

for some nonzero scalar $\beta_2 \in \mathbb{F}$. So, $g_{s_2, j} = 0$ for all $j \neq s_2, n+1-s_3$, and hence,

$$(16) \quad \varphi(aE_{s_2, s_2}) = g_{s_2, s_2}(a)E_{s_2, s_2} + g_{s_2, n+1-s_3}(a)E_{s_2, n+1-s_3}.$$

Next, we consider $\varphi(aE_{s_1, s_1}) = \varphi(C_{n-1}(I_n - E_{n+1-s_1, n+1-s_1} + (a-1)E_{s_2, s_2}))$. By (16), since $s_1 \neq s_3$, we get $g_{s_1, j} = 0$ for all $j \neq s_1$, and hence, $\varphi(aE_{s_1, s_1}) = g_{s_1, s_1}(a)E_{s_1, s_1}$ for all $a \in \mathbb{F}$.

We denote $\sigma = g_{s_1, s_1}$ and $s_1 = s$ for the sake of simplicity. In view of the proofs of Subcase 1 and Subcase 2, we have shown that there exist an integer $1 \leq s \leq n$ and an additive map σ on \mathbb{F} with $\sigma(1) = 1$ such that $\varphi(aE_{ss}) = \sigma(a)E_{ss}$ for all $a \in \mathbb{F}$. Therefore, by repeating a similar argument as in the proof of **Case I**, we can show that $\varphi(aE_{jj}) = \sigma(a)E_{jj}$ for all $1 \leq j \leq n$ and $a \in \mathbb{F}$. To complete the proof of Claim (15), it suffices to show that σ is a field homomorphism on \mathbb{F} . Let $a, b \in \mathbb{F}$ and let $1 \leq t_1 \neq t_2 \leq n$ with $t_i \neq n+1-s$ for $i = 1, 2$. We see that $abE_{ss} = C_{n-1}(I_n - E_{n+1-s, n+1-s} + (a-1)E_{t_1, t_1} + (b-1)E_{t_2, t_2})$. Thus

$$\begin{aligned} \sigma(ab)E_{ss} &= \varphi(abE_{ss}) \\ &= C_{n-1}(\varphi(I_n - E_{n+1-s, n+1-s} + (a-1)E_{t_1, t_1} + (b-1)E_{t_2, t_2})) \\ &= C_{n-1}(I_n - E_{n+1-s, n+1-s} + (\sigma(a)-1)E_{t_1, t_1} + (\sigma(b)-1)E_{t_2, t_2}) \\ &= \sigma(a)\sigma(b)E_{ss}. \end{aligned}$$

Hence $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in \mathbb{F}$, so σ is a homomorphism on \mathbb{F} . Thus (15) as claimed.

We next claim that $\varphi(aE_{st}) = \sigma(a)E_{st}$ or $\varphi(aE_{st}) = \sigma(a)E_{ts}$ for all $E_{st} \in \mathcal{T}_n(\mathbb{F}) \setminus \ker \varphi$ and $a \in \mathbb{F}$. Suppose there exists an integer $1 \leq j_1 \leq n-1$ such that $E_{j_1, j_1+1} \notin \ker \varphi$. By the structure of Φ_i 's as described in **(II)**-(a), there exist an integer $1 \leq i_1 \leq r$ with $s_{i_1-1}+1 \leq j_1 \leq s_{i_1}$, and a nonzero homomorphism $\sigma_{j_1} : \mathbb{F} \rightarrow \mathbb{F}$ such that $\varphi(aE_{j_1, j_1+1}) = \sigma_{j_1}(a)E_{j_1, j_1+1}$ for all $a \in \mathbb{F}$, or $\varphi(aE_{j_1, j_1+1}) = \sigma_{j_1}(a)E_{j_1+1, j_1}$ for all $a \in \mathbb{F}$. We consider only the first case as the second case can be treated similarly. We first note, by (13), that $\varphi(E_{n-j_1, n+1-j_1}) = E_{n-j_1, n+1-j_1}$. Since $n \geq 3$, by choosing $1 \leq j \leq n$ with $j \neq j_1, j_1+1$, and together with (12) and (15), we have

$$\begin{aligned} & \varphi(aE_{j_1, j_1+1}) \\ &= \varphi(C_{n-1}(I_n - E_{n+1-j_1, n+1-j_1} - E_{n-j_1, n-j_1} + E_{n-j_1, n+1-j_1} + (a-1)E_{n+1-j, n+1-j})) \\ &= DC_{n-1}(I_n - E_{n+1-j_1, n+1-j_1} - E_{n-j_1, n-j_1} + E_{n-j_1, n+1-j_1} + (\sigma(a)-1)E_{n+1-j, n+1-j})D^{-1} \\ &= D(\sigma(a)E_{j_1, j_1+1})D^{-1}. \end{aligned}$$

Since $s_{i_1-1}+1 \leq j_1 \leq s_{i_1}$ and $D = \bigoplus_{i=1}^r \mu_i I_{\epsilon_i}$, we get $D(\sigma(a)E_{j_1, j_1+1})D^{-1} = \sigma(a)E_{j_1, j_1+1}$, and so, $\sigma_{j_1}(a)E_{j_1, j_1+1} = \varphi(aE_{j_1, j_1+1}) = \sigma(a)E_{j_1, j_1+1}$ for all $a \in \mathbb{F}$. Hence, $\sigma_{j_1} = \sigma$. Consequently, by (15) and together with the structures of Φ_i 's as described in **(II)**, the claim is proved.

In view of (9), (10), (15) and by the structures of Φ_i 's as described in **(II)**, we obtain

$$\psi(A) = P \left(\bigoplus_{i=1}^r [\lambda_i A_i(\alpha_i) + (1-\lambda_i)A_i(\alpha_i)^t]^\sigma \right) Q \quad \text{for all } A \in \mathcal{T}_n(\mathbb{F}).$$

where $\alpha_i \in H_{k_i, \epsilon_i}$ and $\lambda_i \in \{0, 1\}$ for $i = 1, \dots, r$, and $P, Q \in \mathcal{M}_n(\mathbb{F})$ are invertible matrices with $C_{n-1}(P) = P(\bigoplus_{i=1}^r \mu_i I_{\epsilon_i})$ and $C_{n-1}(Q) = (\bigoplus_{i=1}^r \mu_i^{-1} I_{\epsilon_i})Q$ for some nonzero scalars $\mu_1, \dots, \mu_r \in \mathbb{F}$. By (9), (13) and (14), we conclude that

$$E_{j, j+1} \notin \ker \psi \quad \Leftrightarrow \quad E_{n-j, n+1-j} \notin \ker \psi.$$

Thus, $\epsilon_i = \epsilon_{r+1-i}$ for every $i = 1, \dots, r$. Let $A \in \mathcal{T}_n(\mathbb{F})$. For each $1 \leq i \leq r$, we denote

$$X_i = [\lambda_i A_i(\alpha_i) + (1-\lambda_i)A_i(\alpha_i)^t]^\sigma$$

and

$$Y_i = [\lambda_i C_{n-1}(A)_i(\alpha_i) + (1-\lambda_i)C_{n-1}(A)_i(\alpha_i)^t]^\sigma,$$

where $\bigoplus_{i=1}^r C_{n-1}(A)_i$ is the $(\epsilon_1, \dots, \epsilon_r)$ -block diagonal matrix induced by $C_{n-1}(A)$. Since $C_{n-1}(\psi(A)) = \psi(C_{n-1}(A))$, it follows that

$$\begin{aligned} C_{n-1} \left(\bigoplus_{i=1}^r X_i \right) &= C_{n-1}(P)^{-1} \psi(C_{n-1}(A)) C_{n-1}(Q)^{-1} \\ &= \left(\bigoplus_{i=1}^r \mu_i^{-1} I_{\epsilon_i} \right) \left(\bigoplus_{i=1}^r Y_i \right) \left(\bigoplus_{i=1}^r \mu_i I_{\epsilon_i} \right) \end{aligned}$$

$$= \bigoplus_{i=1}^r Y_i.$$

Further since $\epsilon_i = \epsilon_{r+1-i}$ for all $i = 1, \dots, r$, we have

$$\left(\bigoplus_{i=1}^r A_i \right)^\sim = \bigoplus_{i=1}^r (A_{r+1-i})^\sim,$$

and together with (7) and Lemma 2.7, it is easy to see that

$$\begin{aligned} \bigoplus_{i=1}^r Y_i &= C_{n-1} \left(\bigoplus_{i=1}^r X_i \right) \\ &= C_{n-1} \left(\bigoplus_{i=1}^r [\lambda_i A_i(\alpha_i) + (1 - \lambda_i) A_i(\alpha_i)^t]^\sigma \right) \\ &= \left(\left(\bigoplus_{i=1}^r \lambda_i C_{n-1}(A^\sim)_i(\alpha_i) + (1 - \lambda_i) C_{n-1}(A^\sim)_i(\alpha_i)^t \right)^\sim \right)^\sigma \\ &= \bigoplus_{i=1}^r [\beta_{r+1-i} C_{n-1}(A)_i(\alpha_{r+1-i}^\sim) + (1 - \beta_{r+1-i}) C_{n-1}(A)_i(\alpha_{r+1-i}^\sim)^t]^\sigma, \end{aligned}$$

where $\beta_{r+1-i} = \lambda_{r+1-i}$ when k_{r+1-i} is even, or $\beta_{r+1-i} = 1 - \lambda_{r+1-i}$ when k_{r+1-i} is odd for $i = 1, \dots, r$. Thus, $Y_i = [\beta_{r+1-i} C_{n-1}(A)_i(\alpha_{r+1-i}^\sim) + (1 - \beta_{r+1-i}) C_{n-1}(A)_i(\alpha_{r+1-i}^\sim)^t]^\sigma$ for all $i = 1, \dots, r$. Hence, for each $1 \leq i \leq r$, we have $\alpha_i^\sim = \alpha_{r+1-i}$, $k_i = k_{r+1-i}$, and $\lambda_{r+1-i} = \lambda_i$ when k_i is even, or $\lambda_{r+1-i} = 1 - \lambda_i$ when k_i is odd. This completes our proof. \square

We give two examples to illustrate the result obtained in Theorem 2.8.

Example 2.9. Let \mathbb{F} be a field, and let σ be a nonzero field homomorphism on \mathbb{F} . Let $\phi : \mathcal{T}_5(\mathbb{F}) \rightarrow \mathcal{M}_5(\mathbb{F})$ be the map defined by

$$\phi(A) = \begin{pmatrix} \sigma(a_{11}) & \sigma(a_{12}) & 0 & 0 & 0 \\ 0 & \sigma(a_{22}) & 0 & 0 & 0 \\ 0 & \sigma(a_{23}) & \sigma(a_{33}) & 0 & 0 \\ 0 & \sigma(a_{24}) & \sigma(a_{34}) & \sigma(a_{44}) & \sigma(a_{45}) \\ 0 & 0 & 0 & 0 & \sigma(a_{55}) \end{pmatrix} \text{ for all } A = (a_{ij}) \in \mathcal{T}_5(\mathbb{F}).$$

We see that ϕ is a compound-commuting additive map and $\phi(A)^\sim = \phi(A^\sim)$ for all $A \in \mathcal{T}_5(\mathbb{F})$.

Example 2.10. Let \mathbb{F} be a field, and let σ be a nonzero homomorphism on \mathbb{F} . Let $\psi : \mathcal{T}_9(\mathbb{F}) \rightarrow \mathcal{M}_9(\mathbb{F})$ be the map defined by

$$\psi(A) = \begin{pmatrix} \sigma(a_{11}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma(a_{12}) & \sigma(a_{22}) & \sigma(a_{23}) & \sigma(a_{24}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma(a_{33}) & \sigma(a_{34}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma(a_{44}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(a_{55}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma(a_{66}) & \sigma(a_{67}) & \sigma(a_{68}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma(a_{77}) & \sigma(a_{78}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma(a_{88}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma(a_{89}) & \sigma(a_{99}) \end{pmatrix}$$

for all $A = (a_{ij}) \in \mathcal{T}_9(\mathbb{F})$. It is easy to check that ψ is a compound-commuting additive map and $\psi(A)^\sim = \psi(A^\sim)$ for all $A \in \mathcal{T}_9(\mathbb{F})$.

As an immediate consequence of Theorem 2.8, we have the following corollary.

Corollary 2.11. *Let \mathbb{F} be a field, and let m and n be integers with $m, n \geq 3$. Let $\psi : \mathcal{T}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ be a compound-commuting additive map. Then the following statements are equivalent.*

- (a) ψ is injective.
- (b) $\psi(E_{1n}) \neq 0$.
- (c) $m = n$ and there exist a nonzero field homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ and invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$, with $C_{n-1}(P) = \mu P$ and $C_{n-1}(Q) = \mu^{-1}Q$ for some nonzero element $\mu \in \mathbb{F}$, such that

$$\psi(A) = PA^\sigma Q \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}), \text{ or}$$

$$\psi(A) = P(A^\sigma)^t Q \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}).$$

Proof. (a) \Rightarrow (b) Trivial. (b) \Rightarrow (c) Since ψ is nonzero, it follows from Theorem 2.8 that $m = n$, and together with the fact that $\psi(E_{1n}) \neq 0$, we conclude that there exist a scalar $\lambda \in \{0, 1\}$, a nonzero field homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, and invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$, with $C_{n-1}(P) = \mu P$ and $C_{n-1}(Q) = \mu^{-1}Q$ for some nonzero element $\mu \in \mathbb{F}$, such that

$$\psi(A) = P(\lambda A^\sigma + (1 - \lambda)(A^\sigma)^t)Q \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}).$$

(c) \Rightarrow (a) Trivial. We are done. \square

By Lemmas 2.5 and 2.6, we obtain a characterization of compound-commuting additive maps between square matrix algebras over the same field.

Theorem 2.12. *Let \mathbb{F} be a field, and let m and n be integers with $m, n \geq 3$. Then $\psi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ is a compound-commuting additive map if and only if either $\psi = 0$, or $m = n$ and there exist a field homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, and invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$, with $C_{n-1}(P) = \mu P$ and $C_{n-1}(Q) = \mu^{-1}Q$ for some nonzero element $\mu \in \mathbb{F}$, such that*

$$\psi(A) = PA^\sigma Q \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}), \text{ or}$$

$$\psi(A) = P(A^\sigma)^t Q \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}).$$

Proof. The sufficiency is clear. We now consider the necessity. Suppose $\psi \neq 0$. It can be deduced from Lemma 2.5, applied on $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) = \mathcal{M}_n(\mathbb{F})$ (i.e., $k = 1$), we get $m = n$ and ψ is rank-one nonincreasing with $\psi(I_n)$ being of rank n . By Lemma 2.6, again applied on $\mathcal{T}_{n_1, \dots, n_k}(\mathbb{F}) = \mathcal{M}_n(\mathbb{F})$, there exist a scalar $\lambda \in \{0, 1\}$, a nonzero field homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, and invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$ such that

$$\psi(A) = P(\lambda A^\sigma + (1 - \lambda)(A^\sigma)^t)Q \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}).$$

Since $\psi(C_{n-1}(A)) = C_{n-1}(\psi(A))$ for all $A \in \mathcal{M}_n(\mathbb{F})$, we have

$$P^{-1}C_{n-1}(P)E_{ij} = E_{ij}QC_{n-1}(Q)^{-1}$$

for all $E_{ij} \in \mathcal{M}_n(\mathbb{F})$, and so, $P^{-1}C_{n-1}(P) = \mu I_n = QC_{n-1}(Q)^{-1}$ for some nonzero element $\mu \in \mathbb{F}$. Hence, $C_{n-1}(P) = \mu P$ and $C_{n-1}(Q) = \mu^{-1}Q$. We are done. \square

By Lemma 2.5, Corollary 2.6 in [8] and Theorem 2.1 in [7], we have:

Theorem 2.13. *Let \mathbb{F} be a field, and let m and n be integers with $m, n \geq 3$. Then $\psi : \mathcal{S}_n(\mathbb{F}) \rightarrow \mathcal{S}_m(\mathbb{F})$ is a compound-commuting additive map if and only if $\psi = 0$, or $m = n$ and there exist a nonzero field homomorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ and an invertible matrix $Q \in \mathcal{M}_n(\mathbb{F})$ with $C_{n-1}(Q) = \mu Q$ such that*

$$\psi(A) = \lambda QA^\sigma Q^t \quad \text{for all } A \in \mathcal{S}_n(\mathbb{F}),$$

where $\lambda, \mu \in \mathbb{F}$ are scalars such that $\lambda^{n-2}\mu^2 = 1$.

Proof. The sufficiency part can be easily checked. For the converse, we suppose $\psi \neq 0$. By Lemma 2.5, applied on $\mathcal{H}_n(\mathbb{F}) = \mathcal{S}_n(\mathbb{F})$ (i.e., the field involution $-$ on \mathbb{F} is identity), we get $m = n$ and ψ is a rank-one nonincreasing additive map with $\psi(I_n)$ being of rank n . By Corollary 2.6 in [8] and Theorem 2.1 in [7], we see that ψ is of the following forms:

- (a) $\psi(A) = \lambda QA^\sigma Q^t$ for every $A \in \mathcal{S}_n(\mathbb{F})$, or
- (b) $\psi(A) = P\zeta(A)P^t$ for every $A \in \mathcal{S}_n(\mathbb{F})$, only when $n = 3$ and $\mathbb{F} = \mathbb{Z}_2 := \{0, 1\}$.

Here, $Q \in \mathcal{M}_n(\mathbb{F})$ and $P \in \mathcal{M}_3(\mathbb{Z}_2)$ are invertible matrices, $\lambda \in \mathbb{F}$ is a nonzero scalar, σ is a nonzero field homomorphism on \mathbb{F} , and ζ is a rank-one nonincreasing linear map on $\mathcal{S}_3(\mathbb{Z}_2)$ with $\zeta(I_3)$ being of rank 3.

We first consider ψ is of form (a). By the fact that $C_{n-1}(\psi(A)) = \psi(C_{n-1}(A))$ for $A \in \mathcal{S}_n(\mathbb{F})$, we have $(\lambda Q)^{-1}C_{n-1}(\lambda Q)D_{ij} = D_{ij}Q^t C_{n-1}(Q^t)^{-1}$ for all $1 \leq i, j \leq n$. Here, $D_{ii} = E_{ii}$, and $D_{ij} = -I_n + E_{ii} + E_{jj} + (-1)^{i+j+1}(E_{ij} + E_{ji})$ when $i \neq j$. Thus, $(\lambda Q)^{-1}C_{n-1}(\lambda Q) = \mu^{-1}I_n = Q^t C_{n-1}(Q^t)^{-1}$ for some nonzero scalar $\mu \in \mathbb{F}$, and hence, $C_{n-1}(Q) = \mu Q$ and $\lambda^{n-2}\mu^2 = 1$.

Next, we suppose ψ is of form (b). The compound-commuting of ψ yields

$$(17) \quad \zeta(C_2(A)) = DC_2(\zeta(A))D^t \quad \text{for all } A \in \mathcal{S}_3(\mathbb{Z}_2),$$

where $D = P^{-1}C_2(P) \in \mathcal{M}_3(\mathbb{Z}_2)$ is invertible. By (17), we see that $\zeta(A)$ is singular whenever $A \in \mathcal{S}_3(\mathbb{Z}_2)$ is singular. For, suppose $\text{rank}\zeta(A_0) = 3$ for some singular matrix $A_0 \in \mathcal{S}_3(\mathbb{Z}_2)$. Then $\text{rank}C_2(A_0) \leq 1$, but $\text{rank}\zeta(C_2(A_0)) = \text{rank}(DC_2(\zeta(A_0))D^t) = 3$, which contradicts to the fact that ψ is rank-one nonincreasing. Further, the linearity of ζ and $\text{rank}\zeta(I_3) = 3$ thus ascertain that $\text{rank}\zeta(E_{ii}) = 1$ for $i = 1, 2, 3$, and $\text{rank}\zeta(E_{ii} + E_{jj}) = 2$ for all $1 \leq i \neq j \leq 3$. Since $\zeta(E_{11}) \in \mathcal{S}_3(\mathbb{Z}_2)$ is of rank one, there exists an invertible matrix $U_1 \in \mathcal{M}_3(\mathbb{Z}_2)$ such that $\zeta(E_{11}) = U_1 E_{11} U_1^t$. Let

$$\zeta(E_{22}) = U_1 \begin{pmatrix} a_1 & B_1 \\ B_1^t & D_1 \end{pmatrix} U_1^t$$

with $a_1 \in \mathbb{Z}_2$, $B_1 \in \mathcal{M}_{1,2}(\mathbb{Z}_2)$ and $D_1 \in \mathcal{S}_2(\mathbb{Z}_2)$. Clearly, $D_1 \neq 0$. This is because if $D_1 = 0$, then, since $\text{rank}\zeta(E_{22}) = 1$, we have $B_1 = 0$, and so, $\text{rank}\zeta(E_{11} + E_{22}) < 2$, a contradiction. Further, since $\text{rank}\zeta(E_{22}) = 1$, we conclude that D_1 is of rank one. Then there exists an invertible matrix $U_2 \in \mathcal{M}_2(\mathbb{Z}_2)$ such that $D_1 = U_2 \text{diag}(1, 0) U_2^t$, and so,

$$\begin{aligned} \zeta(E_{22}) &= U_1 \begin{pmatrix} a_1 & B_1 \\ B_1^t & U_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U_2^t \end{pmatrix} U_1^t \\ &= U_1 \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} a_1 & b_{11} & b_{12} \\ b_{11} & 1 & 0 \\ b_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U_2^t \end{pmatrix} U_1^t \end{aligned}$$

with $b_{11}, b_{12} \in \mathbb{Z}_2$. Since $\text{rank}\zeta(E_{22}) = 1$, it follows that $b_{12} = 0$ and $b_{11}^2 = a_1$. Let

$$U_3 = U_1 \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} 1 & b_{11} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{F}_2).$$

Clearly, U_3 is an invertible matrix such that $\zeta(E_{ii}) = U_3 E_{ii} U_3^t$ for $i = 1, 2$. Let

$$\zeta(E_{33}) = U_3 \begin{pmatrix} D_2 & B_2 \\ B_2^t & a_2 \end{pmatrix} U_3^t$$

with $a_2 \in \mathbb{Z}_2$, $B_2 \in \mathcal{M}_{2,1}(\mathbb{Z}_2)$ and $D_2 \in \mathcal{S}_2(\mathbb{Z}_2)$. By the facts that $\text{rank}\zeta(E_{33}) = 1$ and $\text{rank}\zeta(I_3) = 3$, we conclude that $a_2 = 1$, and so, $B_2 B_2^t = D_2$. Let

$$U = U_3 \begin{pmatrix} I_2 & B_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{F}_2).$$

It is clear that U is invertible and $\zeta(E_{ii}) = U E_{ii} U^t$ for $i = 1, 2, 3$.

Let i, j, k be three distinct integers satisfying $1 \leq i, j, k \leq 3$. We denote $S_{ij} = E_{ij} + E_{ji}$. Since $C_2(S_{ij}) = E_{4-k, 4-k}$, it follows from (17) that $C_2(\zeta(S_{ij})) = D\zeta(C_2(S_{ij}))D^t = D\zeta(E_{4-k, 4-k})D^t$ is of rank one, so $\text{rank}\zeta(S_{ij}) = 2$. Let $\zeta(S_{ij}) = U H_{ij} U^t$ with $H_{ij} \in \mathcal{S}_3(\mathbb{Z}_2)$ being of rank two. We want to claim that $H_{ij} = S_{ij}$. Since $S_{ij} + E_{ii} + E_{jj}$ is of rank one, we get $\text{rank}\zeta(S_{ij} + E_{ii} + E_{jj}) \leq 1$. Suppose that $\zeta(S_{ij} + E_{ii} + E_{jj}) = 0$. Then $\zeta(S_{ij}) = \zeta(E_{ii}) + \zeta(E_{jj}) = U(E_{ii} +$

$E_{jj})U^t$. Thus, $\zeta(S_{ij} + E_{ii} + E_{kk}) = U(E_{jj} + E_{kk})U^t$ is of rank two. Since $C_2(C_2(S_{ij} + E_{ii} + E_{kk})) = S_{ij} + E_{ii} + E_{kk}$, it follows from (17) that $\zeta(S_{ij} + E_{ii} + E_{kk}) = \zeta(C_2(C_2(S_{ij} + E_{ii} + E_{kk}))) = 0$. Hence, $\zeta(S_{ij}) = \zeta(E_{ii}) + \zeta(E_{kk})$, and so, $\zeta(E_{kk}) = \zeta(E_{jj})$, a contradiction. So, $\zeta(S_{ij} + E_{ii} + E_{jj}) = U(H_{ij} + E_{ii} + E_{jj})U^t$ is of rank one. Let $H_{ij} = (h_{pq}) \in \mathcal{S}_3(\mathbb{Z}_2)$. Suppose $h_{kk} \neq 0$ (i.e., $h_{kk} = 1$). Since $H_{ij} + E_{ii} + E_{jj}$ is of rank one, it follows that $h_{ii} + 1 = h_{ik}^2$, $h_{jj} + 1 = h_{jk}^2$ and $h_{ij} = h_{ik}h_{jk}$. Thus, we obtain $\text{rank}H_{ij} = 3$ for every $h_{ik}, h_{jk} \in \mathbb{Z}_2$, a contradiction. So, we conclude that $h_{kk} = 0$. Further, since $H_{ij} + E_{ii} + E_{jj}$ is of rank one, it follows that $h_{ik} = h_{jk} = 0$ and $(h_{ii} + 1)(h_{jj} + 1) = h_{ij}^2$. Suppose that $h_{ij} = 0$. Since $\text{rank}H_{ij} = 2$, we conclude that $h_{ii} = 1$ and $h_{jj} = 1$, and so, $H_{ij} = E_{ii} + E_{jj}$. This leads to a contradiction because $\zeta(S_{ij} + E_{ii} + E_{jj}) \neq 0$. So, we conclude that $h_{ij} = 1$, and so, $h_{ii} = h_{jj} = 0$. Hence, $H_{ij} = S_{ij}$, as claimed. Consequently, we have

$$\zeta(A) = UAU^t \quad \text{for all } A \in \mathcal{S}_3(\mathbb{Z}_2).$$

In view of (b), we obtain $\psi(A) = QAQ^t$ for all $A \in \mathcal{S}_3(\mathbb{Z}_2)$ where $Q = PU \in \mathcal{S}_3(\mathbb{Z}_2)$ is an invertible matrix. We are done. \square

By Lemma 2.5, and the structural result of rank-one nonincreasing additive maps between Hermitian matrix spaces in [9, Theorem 2.7], [10, Main Theorem, p. 603] and [11, Theorem 2.1 and Remark 2.4], we have the following result.

Theorem 2.14. *Let \mathbb{F} be a field which possesses a proper involution $-$. Let m and n be integers with $m, n \geq 3$. Then $\psi : \mathcal{H}_n(\mathbb{F}) \rightarrow \mathcal{H}_m(\mathbb{F})$ is a compound-commuting additive map if and only if $\psi = 0$, or $m = n$ and there exist a nonzero field homomorphism σ on \mathbb{F} and an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ with $C_{n-1}(P) = \mu P$ such that*

$$\psi(A) = \lambda P A^\sigma P^* \quad \text{for all } A \in \mathcal{H}_n(\mathbb{F}),$$

where $\lambda, \mu \in \mathbb{F}$ are nonzero scalars such that $\bar{\lambda} = \lambda$ and $\lambda^{n-2}\mu\bar{\mu} = 1$.

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