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$*$-NOETHERIAN DOMAINS AND THE RING $D[\mathrm{X}]_{N_*}$, II

Gyu Whan Chang

Abstract. Let D be an integral domain with quotient field K, $\mathbf{X}$ be a nonempty set of indeterminates over D, $*$ be a star operation on D, $N_* = \{f \in D[\mathbf{X}] | c(f)^* = D\}$, $*_w$ be the star operation on D defined by $I^{*_w} = ID[\mathbf{X}]_{N_*} \cap K$, and $[*]$ be the star operation on $D[\mathbf{X}]$ canonically associated to $*$ as in Theorem 2.1. Let A^g (resp., A^{*g}, $A^{[*]g}$) be the global (resp., $*$-global, $[*]$-global) transform of a ring A. We show that D is a $*_w$-Noetherian domain if and only if $D[\mathbf{X}]$ is a $[*]$-Noetherian domain. We prove that $D^{*g}[\mathbf{X}]_{N_*} = (D[\mathbf{X}]_{N_*})^g = (D[\mathbf{X}])^{[*]g}$; hence if D is a $*_w$-Noetherian domain, then each ring between $D[\mathbf{X}]_{N_*}$ and $D^{*g}[\mathbf{X}]_{N_*}$ is a Noetherian domain. Let $\widetilde{D} = \cap\{D_P | P \in *_w\text{-Max}(D) \text{ and } \mathrm{ht}P \geq 2\}$. We show that $D \subseteq \widetilde{D} \subseteq D^{*g}$ and study some properties of $\widetilde{D}$ and D^{*g}.

0. Introduction

Let D be an integral domain with quotient field K, $\mathbf{X}$ be a nonempty set of indeterminates over D, and $D[\mathbf{X}]$ be the polynomial ring over D. The *content* of a polynomial $f \in K[\mathbf{X}]$, denoted by $c(f)$, is the fractional ideal of D generated by the coefficients of f. An *overring* of D means a ring between D and K. Let $*$ be a star operation on D and D^g (resp., D^{*g}) be the global (resp., $*$-global) transform of D (Relevant definitions and notations are reviewed in Section 1).

Matijevic proved that if D is a Noetherian domain, then each overring R of D with $R \subseteq D^g$ is a Noetherian domain [14, Corollary]. If D is a Noetherian domain with $\dim(D) = 1$, then $D^g = K$, and hence Matijevic's result can be considered as a generalization of the Krull-Akizuki theorem that if D is a Noetherian domain with $\dim(D) = 1$, then each overring R of D is Noetherian and $\dim(R) \leq 1$ [13, Theorem 39]. Park generalized Matijevic's result as follows:

Theorem ([17, Theorem 1.5] or [3, Theorem 3.4(1)]). *If R is a t-linked overring of an SM domain D such that $R \subseteq D^{wg}$, then R is an SM domain.*

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Let R be a $*$-linked overring of D, and let $*_D$ be the star operation on R induced by $*$ as in Lemma 1.2. Chang extended Park's result to an arbitrary star operation $*$ on D as follows: *If D is a $*_w$-Noetherian domain and if R is a $*$-linked overring of D with $R \subseteq D^{*g}$, then R is a $*_D$-Noetherian domain* [4, Theorem 3.6(1)].

Let D be a Noetherian domain and $T = \cap\{D_M | M$ is a maximal ideal of D and ht$M \geq 2\}$. Wadsworth proved that each ring between D and T is Noetherian [18, Theorem 8]. However, in [1, Proposition 1], Anderson showed that $T \subseteq D^g$ and if $T = D^g$, then every maximal ideal of D of graded one has height one. Hence Wadsworth's result is a corollary of Matijevic's result and the ring T is a nontrivial example of overrings of D which are contained in D^g.

Let $N_* = \{f \in D[\mathbf{X}] | c(f)^* = D\}$, $\Lambda = \{P \in *_f\text{-Max}(D) |\ \text{ht}P \geq 2\}$, and $\widetilde{D} = \cap_{P\in\Lambda} D_P$. In this paper, we study a star operation $[*]$ on $D[\mathbf{X}]$ canonically associated to $*$, the $*$-global transforms and the ring $\widetilde{D}$. More precisely, in Section 1, we review basic facts and some recent results on star operations, Nagata rings, $*$-Noetherian domains, and $*$-global transforms. In Section 2, we introduce a star operation $[*]$ on $D[\mathbf{X}]$ such that $(ID[\mathbf{X}])^{[*]} = I^{*_w}[\mathbf{X}]$ for all nonzero fractional ideals I of D. Then we prove that D is a $*_w$-Noetherian domain if and only if $D[\mathbf{X}]$ is a $[*]$-Noetherian domain. We prove that $D^{*g}[\mathbf{X}]_{N_*} = (D[\mathbf{X}]_{N_*})^g = (D[\mathbf{X}])^{[*]g}$. As a corollary, we have that if D is a $*_w$-Noetherian domain, then each ring between $D[\mathbf{X}]_{N_*}$ and $D^{*g}[\mathbf{X}]_{N_*}$ is a Noetherian domain; in particular, each t-linked overring of $D[\mathbf{X}]$ that is contained in $D^{wg}[\mathbf{X}]_{N_v}$ is an SM-domain. Assume that D is a $*_w$-Noetherian domain. We show that $\widetilde{D}$ is $*$-linked over D and $\widetilde{D} \subseteq D^{*g}$. Also, we show that if $*_w = w$, then $\widetilde{D} = D^{wg}$ if and only if t-dim$(D) = 1$; t-Max$(\widetilde{D}) = \{PD_P \cap \widetilde{D} | P \in \Lambda\}$; if $\Lambda \neq \emptyset$, then t-dim$(D) = t$-dim$(\widetilde{D})$; and $\widetilde{D[\mathbf{X}]} = \widetilde{D}[\mathbf{X}]_{N_v} = \widetilde{D[\mathbf{X}]_{N_v}}$. Finally, we study an overring R of an SM-domain D such that each t-linked overring T of D with $T \subseteq R$ is an SM-domain.

1. Review of star operations, Nagata rings and related topics

Let D be an integral domain with quotient field K, $\mathbf{X}$ be a nonempty set of indeterminates over D, and $D[\mathbf{X}]$ be the polynomial ring over D. In this section, we review basic facts on star operations, $*$-Noetherian domains, Nagata rings and $*$-global transforms. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. A *star operation $*$ on D* is a mapping $I \mapsto I^*$ from $\mathbf{F}(D)$ into $\mathbf{F}(D)$ which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$:

(1) $(aD)^* = aD$ and $(aI)^* = aI^*$,
(2) $I \subseteq I^*$, and if $I \subseteq J$, then $I^* \subseteq J^*$, and
(3) $(I^*)^* = I^*$.

Given a star operation $*$ on D, we can use $*$ to construct two new star operations $*_f$ and $*_w$ on D. The $*_f$-operation is defined by $I^{*_f} = \cup\{(a_1, \ldots, a_n)^* | (0) \neq (a_1, \ldots, a_n) \subseteq I\}$ and the $*_w$-operation is defined by $I^{*_w} = \{x \in K | xJ \subseteq I$

for J a nonzero finitely generated ideal of D with $J^* = D\}$ for all $I \in \mathbf{F}(D)$. Clearly, $(*_f)_f = *_f$, $(*_w)_f = *_w = (*_f)_w$. An $I \in \mathbf{F}(D)$ is called a *$*$-ideal* if $I^* = I$. A $*$-ideal I is said to be *of finite type* if $I = (a_1, \ldots, a_n)^*$ for some $a_i \in I$. A $*$-ideal is called a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals of D. Let $*$-Max(D) denote the set of maximal $*$-ideals of D and Max(D) be the set of maximal ideals of D. It is known that $*_f$-Max$(D) \neq \emptyset$ if D is not a field, each maximal $*_f$-ideal is a prime ideal, a prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal, and each integral $*_f$-ideal is contained in a maximal $*_f$-ideal. An $I \in \mathbf{F}(D)$ is said to be *$*$-invertible* if $(II^{-1})^* = D$, where $I^{-1} = \{x \in K | xI \subseteq D\}$, while D is a *Prüfer $*$-multiplication domain* (P$*$MD) if each nonzero finitely generated ideal of D is $*_f$-invertible. It is well known that $I \in \mathbf{F}(D)$ is $*_f$-invertible if and only if I^{*_f} is of finite type and ID_P is principal for all $P \in *_f$-Max(D) [12, Proposition 2.6]. Also, we know that D is a P$*$MD if and only if D_P is a valuation domain for all $P \in *_f$-Max(D) [11, Theorem 1.1].

The simplest example of star operations is the d-operation. Other well-known examples of star operations are the v-, t-, and w-operations. The d-operation is just the identity function on $\mathbf{F}(D)$, i.e., $I^d = I$ for all $I \in \mathbf{F}(D)$; so $d = d_f = d_w$. The v-operation is defined by $I^v = (I^{-1})^{-1}$, whereas $t = v_f$ and $w = v_w$, i.e., $I^t = I^{v_f}$ and $I^w = I^{v_w}$ for all $I \in \mathbf{F}(D)$. If $*_1$ and $*_2$ are star operations on D, then we mean by $*_1 \leq *_2$ that $I^{*_1} \subseteq I^{*_2}$ for all $I \in \mathbf{F}(D)$. It is clear that $*_w \leq *_f \leq *$ and $d \leq * \leq v$ for any star operation $*$. Also, if $*_1 \leq *_2$, then $(*_1)_w \leq (*_2)_w$ and $(*_1)_f \leq (*_2)_f$; hence $d \leq *_w \leq w$ and $d \leq *_f \leq t$.

Let $*$ be a star operation on D. Put $N_* = \{f \in D[\mathbf{X}] | c(f)^* = D\}$; then $N_* = N_{*_f} = N_{*_w}$ and $N_* = D[\mathbf{X}] - \cup_{P \in *_f\text{-Max}(D)} P[\mathbf{X}]$. Hence $D[\mathbf{X}]_{N_*} = \{\frac{f}{g} | f \in D[\mathbf{X}]$ and $g \in N_*\}$, called the $(*-)$*Nagata ring* of D, is an overring of $D[\mathbf{X}]$. The ring $D[\mathbf{X}]_{N_*}$ has many interesting ring-theoretic properties. For example, each invertible ideal of $D[\mathbf{X}]_{N_*}$ is principal [12, Theorem 2.14]; D is a P$*$MD if and only if $D[\mathbf{X}]_{N_*}$ is a Prüfer domain, if and only if $D[\mathbf{X}]_{N_*}$ is a Bezout domain [5, Theorem 2.2]; and D is a Krull domain if and only if $D[\mathbf{X}]_{N_v}$ is a Dedekind domain, if and only if $D[\mathbf{X}]_{N_v}$ is a principal ideal domain [15].

Lemma 1.1. (1) $\text{Max}(D[\mathbf{X}]_{N_*}) = \{P[\mathbf{X}]_{N_*} | P \in *_f\text{-Max}(D)\}$.

(2) $*_w$-Max$(D) = *_f$-Max(D).

(3) $I^{*_w} = \cap_{P \in *_f\text{-Max}(D)} ID_P = ID[\mathbf{X}]_{N_*} \cap K$ *for all* $I \in \mathbf{F}(D)$.

Proof. (1) [12, Proposition 2.1]. (2) [2, Theorem 2.16]. (3) [4, Lemma 2.3]. □

As in [4], we say that an overring R of D is *$*$-linked* over D if $R[\mathbf{X}]_{N_*} \cap K = R$. It is known that R is $*$-linked over D if and only if $(Q \cap D)^{*_f} \subsetneq D$ for each prime t-ideal Q of R, if and only if $I^* = D$ implies $(IR)^v = R$ for each nonzero finitely generated ideal I of D [4, Proposition 3.2]. Next, we use the star operation $*$ on D to construct a new star operation $*_D$ on a $*$-linked overring R of D.

Lemma 1.2 ([4, Lemma 3.1]). *Let R be a $*$-linked overring of D, X be an indeterminate over D, and put $I^{*_D} = IR[X]_{N_*} \cap K$ for $I \in \mathbf{F}(R)$. Then the map $*_D : \mathbf{F}(R) \to \mathbf{F}(R)$, given by $I \mapsto I^{*_D}$, is a star operation on R and $(*_D)_w = *_D$.*

We say that D is a $*$-*Noetherian domain* if D satisfies the ascending chain condition on integral $*$-ideals of D; equivalently, if each $*$-ideal of D is of finite type. Hence Noetherian domains are just the d-Noetherian domains. A v-Noetherian domain is a Mori domain, while a w-Noetherian domain is a strong Mori domain (SM-domain). It is clear that if $*_1 \leq *_2$ are star operations, then $*_1$-Noetherian domains are $*_2$-Noetherian domains; hence Noetherian domains $\Rightarrow$ SM-domains $\Rightarrow$ Mori domains. Also, since $*_w \leq w$, a $*_w$-Noetherian domain is an SM-domain. Note that $I^{*_w}D_P = ID_P$ by Lemma 1.1(3); hence if D is a $*_w$-Noetherian domain, then D_P is Noetherian for all $P \in *_f\text{-Max}(D)$. The *global transform* of D is defined by $D^g = \{a \in K | aM_1 \cdots M_k \subseteq D$ where each M_i is a maximal ideal of $D\}$. As in [4], the $*$-*global transform* of D is the ring $D^{*g} = \{x \in K | xP_1 \cdots P_k \subseteq D$ for some $P_i \in *_f\text{-Max}(D)\}$. Clearly, $D^{*g} = D^{(*_f)g} = D^{(*_w)g}$ and the global transform D^g of D is just the d-global transform.

Lemma 1.3. *Let D be a $*$-Noetherian domain.*

1. *$(D[\mathbf{X}]_{N_*})^g \cap K = D^{*g}$.*
2. *D^{*g} is $*$-linked over D.*
3. *$D = D^{*g}$ if and only if each maximal $*_f$-ideal of D is not a t-ideal. In particular, $D \subsetneq D^{wg}$.*
4. *Let R be a $*$-linked overring of a $*_w$-Noetherian domain D, and let $*_D$ be the star operation on R as in Lemma 1.2. If $R \subseteq D^{*g}$, then R is a $*_D$-Noetherian domain, and hence R is an SM-domain.*
5. *If $*_1 \leq *_2$ are star operations on D, then $D^{(*_1)g} \subseteq D^{(*_2)g}$. In particular, $D^g \subseteq D^{*g} \subseteq D^{wg}$.*

Proof. (1) [4, Lemma 3.5]. (2) By (1), $D^{*g}[\mathbf{X}]_{N_*} \subseteq (D[\mathbf{X}]_{N_*})^g$. Hence $D^{*g} \subseteq D^{*g}[\mathbf{X}]_{N_*} \cap K \subseteq (D[\mathbf{X}]_{N_*})^g \cap K = D^{*g}$, and thus $D^{*g}[\mathbf{X}]_{N_*} \cap K = D^{*g}$. Thus D^{*g} is $*$-linked over D. (3) Assume to the contrary that there is a maximal $*_f$-ideal P of D with $P^t = P$; so $D \subsetneq P^{-1}$ because P is of finite type. But, since $P^{-1}P \subseteq D$, we have $P^{-1} \subseteq D^{*g}$. Thus $D \subsetneq D^{*g}$. Conversely, assume that each maximal $*_f$-ideal of D is not a t-ideal, and let $x \in D^{*g}$. Then there exist some maximal $*_f$-ideals $P_1, \ldots, P_n$ of D (not necessarily distinct) such that $xP_1 \cdots P_n \subseteq D$; so $x \in xD = x(P_1 \ldots P_n)^t = (xP_1 \cdots P_n)^t \subseteq D^t = D$. Hence $D^{*g} \subseteq D$, and thus $D = D^{*g}$. (4) [4, Theorem 3.6(1)]. (5) This follows because if $P \in (*_1)_f\text{-Max}(D)$, then either $P^{(*_2)_f} = D$ or $P \in (*_2)_f\text{-Max}(D)$. □

Let X be an indetermainate over D and $N_v = \{f \in D[X] | c(f)^v = D\}$. Let $D^{[w]} = \{x \in K | xI^w \subseteq I^w$ for some nonzero finitely generated ideal I of $D\}$. Then $D^{[w]}$, called the *w-integral closure* of D, is an integrally closed overring of D. It is known that $D^{[w]}$ is t-linked over D [8, Lemma 1.2]; if $\bar{D}$ is the integral closure of D, then $D^{[w]} = \bar{D}[X]_{N_v} \cap K = \cap_{P \in t\text{-Max(D)}} \bar{D}_{D \setminus P}$ [8, Theorem 1.3]; and $D^{[w]}$ is the smallest integrally closed t-linked overring of D [9, Proposition 2.13(b)].

A prime ideal P of D is said to be *strongly prime* if $xy \in P$ and $x, y \in K$ imply $x \in P$ or $y \in P$. The D is called a *pseudo valuation domain* (PVD) if each prime ideal of D is strongly prime; equivalently, if D is a quasi-local domain whose maximal ideal is strongly prime. Also, D is called a *locally PVD* (LPVD) if D_M is a PVD for each $M \in \text{Max}(D)$, while D is a *t-locally PVD* (t-LPVD) if D_P is a PVD for all $P \in t\text{-Max}(D)$. Clearly, the notion of PVDs is a generalization of valuation domains. Hence the notions of LPVDs and t-LPVDs can be considered as generalizations of Prüfer domains and PvMDs. Chang proved that $D[X]_{N_v}$ is an LPVD if and only if D is a t-LPVD and $D^{[w]}$ is a PvMD, if and only $D[X]$ is a t-LPVD [6, Theorem 3.8] and that $D[X]_{N_d}$ is an LPVD if and only if D is an LPVD and $\bar{D}$ is a Prüfer domain [6, Corollary 3.9].

2. $*$-Noetherian domains and $*$-global transforms

Throughout D denotes an integral domain with quotient field K, $*$ is a star operation on D, $\mathbf{X}$ is a nonempty set of indeterminates over D, and $N_* = \{f \in D[\mathbf{X}] | c(f)^* = D\}$.

Our first result gives a star operation $[*]$ on $D[\mathbf{X}]$, which is an extension of the $*_w$ to $D[\mathbf{X}]$ in the sense that $(I[\mathbf{X}])^{[*]} \cap K = I^{*_w}$ for each $I \in \mathbf{F}(D)$. This extension was first studied for $|\mathbf{X}| = 1$ by Chang and Fontana [7] in a more general setting of semistar operations. The proof of Theorem 2.1 is basically the same as that of [7, Theorem 2.3], and hence we omit the proof.

Theorem 2.1. *Let $\mathbf{X} \cup \{Y\}$ be a nonempty set of indeterminates over D, and let*

$$\mathbf{\Delta} = \{Q \in Spec(D[\mathbf{X}]) \ \ | \ \ Q \cap D = (0) \text{ with } htQ = 1 \text{ or } Q = (Q \cap D)[\mathbf{X}] \text{ and } (Q \cap D)^{*_f} \subsetneq D\}.$$

Set $\mathcal{S} = D[\mathbf{X}][Y] \setminus (\bigcup\{Q[Y] \mid Q \in \mathbf{\Delta}\})$ and define

$$A^{[*]} = A[Y]_{\mathcal{S}} \cap K(\mathbf{X}) \quad \text{for all } A \in F(D[\mathbf{X}]).$$

(1) *The mapping $[*] : \mathbf{F}(D[\mathbf{X}]) \to \mathbf{F}(D[\mathbf{X}])$, given by $A \mapsto A^{[*]}$, is a star operation on $D[\mathbf{X}]$ such that $[*] = [*]_f = [*]_w$.*
(2) $[*] = [*_f] = [*_w]$.
(3) $(ID[\mathbf{X}])^{[*]} \cap K = I^{*_w}$ *for all* $I \in \mathbf{F}(D)$.
(4) $(ID[\mathbf{X}])^{[*]} = I^{*_w} D[\mathbf{X}]$ *for all* $I \in \mathbf{F}(D)$.
(5) $[*]\text{-Max}(D[\mathbf{X}]) = \{Q \mid Q \in Spec(D[\mathbf{X}])$ *such that* $Q \cap D = (0)$, $htQ = 1$, *and* $(\sum_{g \in Q} c(g))^{*_f} = D\} \cup \{P[\mathbf{X}] \mid P \in *_f\text{-Max}(D)\}$.
(6) $[v]$ *is the w-operation on* $D[\mathbf{X}]$.

Corollary 2.2. *Let* $[*]$ *be the star operation on* $D[\mathbf{X}]$ *canonically associated to* $*$ *as in Theorem 2.1. If* Q *is a prime ideal of* $D[\mathbf{X}]$ *such that* $Q = fK[\mathbf{X}] \cap D[\mathbf{X}]$ *for some* $0 \neq f \in K[\mathbf{X}]$*, then* Q *is a maximal* $[*]$*-ideal if and only if there exists a polynomial* $f \in Q$ *such that* $c(f)^{*_f} = D$.

Proof. Suppose that Q is a maximal $[*]$-ideal of $D[\mathbf{X}]$. Then $(\sum_{g\in Q} c(g))^{*_f} = D$ by Theorem 2.1(5), and hence there are some $g_1, \ldots, g_n \in Q$ such that $(c(g_1) + \cdots + c(g_n))^{*_f} = D$. Let

$$f = g_1 + g_2 X^{\deg(g_1)+1} + \cdots + g_n X^{\deg(g_1)+\cdots+\deg(g_{n-1})+n-1},$$

where $X \in \mathbf{X}$. Then $f \in Q$ and $c(f)^{*_f} = D$. For the converse, note that $Q \cap D = (0)$, $\mathrm{ht}Q = 1$, and $D = c(f)^{*_f} \subseteq (\sum_{g\in Q} c(g))^{*_f} \subseteq D$ or $(\sum_{g\in Q} c(g))^{*_f} = D$. Thus Q is a maximal $[*]$-ideal by Theorem 2.1(5). □

Let Q be a prime ideal of $D[\mathbf{X}]$. It is clear that $(\sum_{g\in Q} c(g))^{*_f} = D$ if and only if $Q \nsubseteq P[\mathbf{X}]$ for all $P \in *_f\text{-Max}(D)$. Also, if $\mathbf{X}_1$ is a nonempty subset of $\mathbf{X}$, then $AD[\mathbf{X}] = A[\mathbf{X} - \mathbf{X}_1]$ for all ideals A of $D[\mathbf{X}_1]$. In the proof of Corollary 2.3, we use these facts without comments.

Corollary 2.3. *Let* $\mathbf{X}_1$ *be a nonempty subset of* $\mathbf{X}$*,* $\star_1$ *and* $\star$ *be the extensions of* $*$ *to* $D[\mathbf{X}_1]$ *and* $D[\mathbf{X}]$*, respectively, as in Theorem 2.1. If* $[\star_1]$ *is the star operation on* $D[\mathbf{X}]$ *canonically associated to* $\star_1$ *as in Theorem 2.1, then* $\star = [\star_1]$.

Proof. By Theorem 2.1(1) and Lemma 1.1(3), it suffices to show that

$$\star\text{-Max}(D[\mathbf{X}]) = [\star_1]\text{-Max}(D[\mathbf{X}]).$$

($\subseteq$) Let $Q \in \star\text{-Max}(D[\mathbf{X}])$; so either $Q \cap D = (0)$, $\mathrm{ht}Q = 1$ and $(\sum_{g\in Q} c(g))^{*_f} = D$ or $Q = P[\mathbf{X}]$ for some $P \in *_f\text{-Max}(D)$. If $Q = P[\mathbf{X}]$, then $P[\mathbf{X}_1] \in \star_1\text{-Max}(D[\mathbf{X}_1])$ and $P[\mathbf{X}] = P[\mathbf{X}_1][\mathbf{X} - \mathbf{X}_1]$. Thus $Q \in [\star_1]\text{-Max}(D[\mathbf{X}])$.

Next, assume that $Q \cap D = (0)$. If $Q \cap D[\mathbf{X}_1] \neq (0)$, then, since $\mathrm{ht}Q = 1$, we have $Q = (Q \cap D[\mathbf{X}_1])D[\mathbf{X}]$ and $\mathrm{ht}(Q \cap D[\mathbf{X}_1]) = 1$. Also, since $Q \nsubseteq P[\mathbf{X}]$ for all $P \in *_f\text{-Max}(D)$, $Q \cap D[\mathbf{X}_1] \nsubseteq P[\mathbf{X}_1]$. Hence $Q \cap D[\mathbf{X}_1] \in \star_1\text{-Max}(D[\mathbf{X}_1])$, and thus $Q \in [\star_1]\text{-Max}(D[\mathbf{X}])$. If $Q \cap D[\mathbf{X}_1] = (0)$, then, clearly, $Q \nsubseteq Q_0[\mathbf{X}]$ for all $Q_0 \in \star_1\text{-Max}(D[\mathbf{X}_1])$. Thus $Q \in [\star_1]\text{-Max}(D[\mathbf{X}])$.

($\supseteq$) Let $M \in [\star_1]\text{-Max}(D[\mathbf{X}])$. If $M \cap D[\mathbf{X}_1] \neq (0)$, then $M = (M \cap D[\mathbf{X}_1])D[\mathbf{X}]$ and $M \cap D[\mathbf{X}_1] \in [\star_1]\text{-Max}(D[\mathbf{X}])$; so either $(M \cap D[\mathbf{X}_1]) \cap D \in *_f\text{-Max}(D)$ or $(M \cap D[\mathbf{X}_1]) \cap D = (0)$, $\mathrm{ht}(M \cap D[\mathbf{X}_1]) = 1$ and $(\sum_{g\in M\cap D[\mathbf{X}_1]} c(g))^{*_f} = D$. Hence $M = (M \cap D)D[\mathbf{X}]$ and $M \cap D \in *_f\text{-Max}(D)$ or $M \cap D = (0)$, $\mathrm{ht}M = 1$ and $(\sum_{g\in M} c(g))^{*_f} = D$. Thus $M \in \star\text{-Max}(D[\mathbf{X}])$.

Next, assume that $M \cap D[\mathbf{X}_1] = (0)$. Then $M \cap D = (0)$, $\mathrm{ht}M = 1$, and $M \nsubseteq P[\mathbf{X}]$ for all $P \in *_f\text{-Max}(D)$; so $(\sum_{f\in M} c(f))^{*_f} = D$. Thus $M \in \star\text{-Max}(D[\mathbf{X}])$. □

We next prove that D is a $*_w$-Noetherian domain if and only if $D[\mathbf{X}]$ is a $[*]$-Noetherian domain, which was proved for $|\mathbf{X}| = 1$ by Chang and Fontana [7, Corollary 2.5] in a more general setting of semistar operations. This also recovers Park's result that D is an SM-domain (if and) only if $D[\mathbf{X}]$ is an SM-domain [16, Theorem 4.7] by Theorem 2.1(2) and (6).

Corollary 2.4. *Let $[*]$ be the star operation on $D[\mathbf{X}]$ canonically associated to $*$ as in Theorem 2.1. Then the following statements are equivalent.*

(1) *D is a $*_w$-Noetherian domain.*
(2) *Each prime $*_w$-ideal of D is of finite type.*
(3) *$D[\mathbf{X}]_{N_*}$ is a Noetherian domain.*
(4) *$D[\mathbf{X}]$ is a $[*]$-Noetherian domain.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) [4, Theorem 2.6].

(3) $\Rightarrow$ (4) Note that $[*]_w = [*]$ by Theorem 2.1(1); so by the equivalence of (1) and (2), it suffices to show that each prime $[*]$-ideal of $D[\mathbf{X}]$ is of finite type. Let Q be a prime $[*]$-ideal of $D[\mathbf{X}]$.

Case 1. $(\sum_{g\in Q} c(g))^{*_f} = D$. Then Q is a maximal $[*]$-ideal and $\mathrm{ht}Q = 1$ by Theorem 2.1(5) and there exists an $f \in Q$ such that $c(f)^{*_f} = D$ by Corollary 2.2. Also, note that $QK[\mathbf{X}] = hK[\mathbf{X}]$ for some $h \in Q$. Hence if we set $A = (f, h)D[\mathbf{X}]$, then $Q_M = A_M$ for all $M \in [*]\text{-Max}(D[\mathbf{X}])$. Thus $Q = A^{[*]}$ by Lemma 1.1(3).

Case 2. $(\sum_{g\in Q} c(g))^{*_f} \subsetneq D$. Then $QD[\mathbf{X}]_{N_*} \subsetneq D[\mathbf{X}]_{N_*}$, and hence by (3), there exists a finitely generated ideal B of $D[\mathbf{X}]$ such that $QD[\mathbf{X}]_{N_*} = BD[\mathbf{X}]_{N_*}$. Let $\Omega = \{M \in [*]\text{-Max}(D[\mathbf{X}]) | M \cap D = (0)\}$, and note that if $M \in \Omega$, then $\mathrm{ht}M = 1$; so the intersection $\cap_{M\in\Omega} D[\mathbf{X}]_M$ has finite character (note that $K[\mathbf{X}]$ is a UFD). Also, since $Q \nsubseteq M$ for all $M \in \Omega$, we can choose some $a, b \in Q$ such that $(a, b) \nsubseteq M$ for all $M \in \Omega$. Replacing B with (B, a, b), we may assume that $B \nsubseteq M$ or $QD[\mathbf{X}]_M = D[\mathbf{X}]_M = BD[\mathbf{X}]_M$ for all $M \in \Omega$. Thus $Q = \cap_{P\in *_f\text{-Max}(D)} QD[\mathbf{X}]_{P[\mathbf{X}]} \cap (\cap_{M\in\Omega} QD[\mathbf{X}]_M) = QD[\mathbf{X}]_{N_*} \cap (\cap_{M\in\Omega} QD[\mathbf{X}]_M) = BD[\mathbf{X}]_{N_*} \cap (\cap_{M\in\Omega} BD[\mathbf{X}]_M) = \cap_{P\in *_f\text{-Max}(D)} BD[\mathbf{X}]_{P[\mathbf{X}]} \cap (\cap_{M\in\Omega} BD[\mathbf{X}]_M) = B^{[*]}$ by Lemma 1.1(1) and (3).

(4) $\Rightarrow$ (1) Let I be a $*_w$-ideal of D; then $(I[\mathbf{X}])^{[*]} = I[\mathbf{X}]$ by Theorem 2.1(4). Hence there are some $f_1, \dots, f_n \in I[\mathbf{X}]$ such that $I[\mathbf{X}] = ((f_1, \dots, f_n)D[\mathbf{X}])^{[*]}$. Put $J = c(f_1) + \cdots c(f_n)$; then $J \subseteq I$ is finitely generated and $I[\mathbf{X}] = (J[\mathbf{X}])^{[*]} = J^{*_w}[\mathbf{X}]$. Thus $I = I[\mathbf{X}] \cap K = J^{*_w}[\mathbf{X}] \cap K = J^{*_w}$. □

Recall that if $f, g \in K[\mathbf{X}]$, then there exists a positive integer $m = m(f, g)$ such that $c(f)^{m+1}c(g) = c(f)^m c(fg)$ [10, Corollary 28.3]; so if $c(f)^* = D$, then $c(g)^* = c(fg)^*$. In particular, if $f_1, \dots, f_n \in D[\mathbf{X}]$, then $c(f_i)^* = D$ for $i = 1, \dots, n$ if and only if $c(f_1 \cdots f_n)^* = D$.

Theorem 2.5. *Let $[*]$ be the star operation on $D[\mathbf{X}]$ canonically associated to $*$ as in Theorem 2.1. Then $(D[\mathbf{X}]_{N_*})^g = D^{*g}[\mathbf{X}]_{N_*} = (D[\mathbf{X}])^{[*]g}$.*

Proof. Proof of $(D[\mathbf{X}]_{N_*})^g \subseteq D^{*g}[\mathbf{X}]_{N_*}$.

We first show that $(D[\mathbf{X}]_{N_*})^g \cap K[\mathbf{X}] \subseteq D^{*g}[\mathbf{X}]_{N_*}$. Let $g \in (D[\mathbf{X}]_{N_*})^g \cap K[\mathbf{X}]$. Then there are maximal $*_f$-ideals $P_1, \dots, P_n$ of D such that $gP_1 \cdots P_n \subseteq$

$gP_1[\mathbf{X}]_{N_*} \cdots P_n[\mathbf{X}]_{N_*} \subseteq D[\mathbf{X}]_{N_*}$ by Lemma 1.1(1). So if $0 \neq a \in P_1 \cdots P_n$, then $ag = \frac{h}{f}$ or $afg = h$ for some $h \in D[\mathbf{X}]$ and $f \in N_*$; hence $a(c(g)) \subseteq a(c(g)^*) = a(c(f)c(g))^* = a(c(fg)^*) = c(h)^* \subseteq D$. Since a is an arbitrary element of $P_1 \cdots P_n$, we have $c(g)P_1 \cdots P_n \subseteq D$, and hence $c(g) \subseteq D^{*g}$. Thus $g \in D^{*g}[\mathbf{X}] \subseteq D^{*g}[\mathbf{X}]_{N_*}$.

Next, let $u \in (D[\mathbf{X}]_{N_*})^g$. Then there are maximal $*_f$-ideals $P_1', \ldots, P_k'$ of D such that $uP_1'[\mathbf{X}] \cdots P_k'[\mathbf{X}] \subseteq uP_1'[\mathbf{X}]_{N_*} \cdots P_k'[\mathbf{X}]_{N_*} \subseteq D[\mathbf{X}]_{N_*}$ by Lemma 1.1(1). So for any $0 \neq b \in P_1' \cdots P_k'$, we have $ub = \frac{h_1}{f_1} \in D[\mathbf{X}]_{N_*}$, where $h_1 \in D[\mathbf{X}]$ and $f_1 \in N_*$. Hence $u = \frac{b^{-1}h_1}{f_1} \in K[\mathbf{X}]_{N_*}$ and $f_1 u = f_1 \frac{b^{-1}h_1}{f_1} = b^{-1}h_1 \in K[\mathbf{X}] \cap (D[\mathbf{X}]_{N_*})^g$. By the above paragraph, $f_1 u \in D^{*g}[\mathbf{X}]_{N_*}$, and since $f_1 \in N_*$, we have $u \in D^{*g}[\mathbf{X}]_{N_*}$. Thus $(D[\mathbf{X}]_{N_*})^g \subseteq D^{*g}[\mathbf{X}]_{N_*}$.

Proof of $D^{*g}[\mathbf{X}]_{N_*} \subseteq (D[\mathbf{X}])^{[*]g}$.

Note that if P is a maximal $*_f$-ideal of D, then $P[\mathbf{X}]$ is a maximal $[*]$-ideal of $D[\mathbf{X}]$ by Theorem 2.1(5). So $D^{*g} \subseteq (D[\mathbf{X}])^{[*]g}$, and hence $D^{*g}[\mathbf{X}] \subseteq (D[\mathbf{X}])^{[*]g}$. Hence it suffices to show that if $f \in N_*$, then $\frac{1}{f} \in (D[\mathbf{X}])^{[*]g}$.

Since $K[\mathbf{X}]$ is a UFD, we can write $f = f_1^{e_1} \cdots f_k^{e_k}$, where each $f_i \in K[\mathbf{X}]$, each e_i is a positive integer and $f_iK[\mathbf{X}]$ is a prime ideal of $K[\mathbf{X}]$ such that $f_iK[\mathbf{X}] \neq f_jK[\mathbf{X}]$ for $i \neq j$. If $g \in K[\mathbf{X}]$ such that $fg \in D[\mathbf{X}]$, then $c(g) \subseteq c(g)^* = (c(f)c(g))^* = c(fg)^* \subseteq D[\mathbf{X}]$; so $g \in D[\mathbf{X}]$. Hence $fK[\mathbf{X}] \cap D[\mathbf{X}] \subseteq fD[\mathbf{X}]$, and thus $fK[\mathbf{X}] \cap D[\mathbf{X}] = fD[\mathbf{X}]$. Also, $fD[\mathbf{X}] = fK[\mathbf{X}] \cap D[\mathbf{X}] = (f_1^{e_1}K[\mathbf{X}] \cap D[\mathbf{X}]) \cap \cdots \cap (f_k^{e_k}K[\mathbf{X}] \cap D[\mathbf{X}]) \subseteq f_iK[\mathbf{X}] \cap D[\mathbf{X}]$ for $i = 1, \ldots, k$. Since $c(f)^* = D$, each $f_iK[\mathbf{X}] \cap D[\mathbf{X}]$ is a maximal $[*]$-ideal of $D[\mathbf{X}]$ by Corollary 2.2. Also, since $(f_iK[\mathbf{X}] \cap D[\mathbf{X}])^{e_i} \subseteq (f_iK[\mathbf{X}] \cap D[\mathbf{X}])^{e_i}K[\mathbf{X}] \cap D[\mathbf{X}] = f_i^{e_i}K[\mathbf{X}] \cap D[\mathbf{X}]$, we have $(f_1K[\mathbf{X}] \cap D[\mathbf{X}])^{e_1} \cdots (f_kK[\mathbf{X}] \cap D[\mathbf{X}])^{e_k} \subseteq (f_1K[\mathbf{X}] \cap D[\mathbf{X}])^{e_1} \cap \cdots \cap (f_kK[\mathbf{X}] \cap D[\mathbf{X}])^{e_k} \subseteq fD[\mathbf{X}]$. Hence $\frac{1}{f}(f_1K[\mathbf{X}] \cap D[\mathbf{X}])^{e_1} \cdots (f_kK[\mathbf{X}] \cap D[\mathbf{X}])^{e_k} \subseteq \frac{1}{f}fD[\mathbf{X}] = D[\mathbf{X}]$, and thus $\frac{1}{f} \in (D[\mathbf{X}])^{[*]g}$.

Proof of $(D[\mathbf{X}])^{[*]g} \subseteq (D[\mathbf{X}]_{N_*})^g$.

Let $u \in (D[\mathbf{X}])^{[*]g}$. Recall that if Q is a maximal $[*]$-ideal of $D[\mathbf{X}]$ with $Q \cap D \neq (0)$, then $Q \cap D$ is a maximal $*_f$-ideal of D and $Q = (Q \cap D)[\mathbf{X}]$ by Theorem 2.1(5). Hence $uQ_1 \cdots Q_kP_1[\mathbf{X}] \cdots P_m[\mathbf{X}] \subseteq D[\mathbf{X}]$ for some maximal $[*]$-ideals $Q_1, \ldots, Q_k$ of $D[\mathbf{X}]$ with $Q_i \cap D = (0)$ and maximal $*_f$-ideals $P_1, \ldots, P_m$ of D. Also, by Corollary 2.2, there exists a polynomial $h_i \in Q_i$ such that $c(h_i)^* = D$. Let $h = h_1 \cdots h_k$; then $c(h)^* = D$, and hence $h \in N_*$. So $uhP_1[\mathbf{X}] \cdots P_m[\mathbf{X}] \subseteq D[\mathbf{X}]$ or $uP_1[\mathbf{X}] \cdots P_m[\mathbf{X}] \subseteq \frac{1}{h}D[\mathbf{X}] \subseteq D[\mathbf{X}]_{N_*}$. Hence $uP_1[\mathbf{X}]_{N_*} \cdots P_m[\mathbf{X}]_{N_*} \subseteq D[\mathbf{X}]_{N_*}$, and since each $P_i[\mathbf{X}]_{N_*}$ is a maximal ideal of $D[\mathbf{X}]_{N_*}$ by Lemma 1.1(1), we have $u \in (D[\mathbf{X}]_{N_*})^g$. □

Recall that each maximal ideal of $D[\mathbf{X}]_{N_v}$ is a t-ideal [12, Corollary 2.3]; hence $(D[\mathbf{X}]_{N_v})^{wg} = (D[\mathbf{X}]_{N_v})^g$. Also, $[w] = w$ on $D[\mathbf{X}]$ by Theorem 2.1(2) and (6). Thus the next result is the d- and w-operation versions of Theorem 2.5.

Corollary 2.6. (1) $D^g[\mathbf{X}]_{N_d} = (D[\mathbf{X}])^{[d]g} = (D[\mathbf{X}]_{N_d})^g$.
(2) $D^{wg}[\mathbf{X}]_{N_v} = (D[\mathbf{X}])^{wg} = (D[\mathbf{X}]_{N_v})^g = (D[\mathbf{X}]_{N_v})^{wg}$.

It is well known that if D is a Noetherian domain, then each ring between D and D^g is a Noetherian domain [14, Corollary]. Thus by Corollary 2.4 and Theorem 2.5, we have:

Corollary 2.7. *If D is a $*_w$-Noetherian domain, then each ring between $D[\mathbf{X}]_{N_*}$ and $D^{*g}[\mathbf{X}]_{N_*}$ is a Noetherian domain.*

The $*$-dimension of D, denoted by $*$-dim(D), is the number of prime $*$-ideals in a longest chain of prime $*$-ideals of D, or infinity if there is no such longest chain. If D is a rank one non-discrete valuation domain, then v-dim$(D) = 0$. However, if D is not a field, then $*_f$-dim$(D) \geq 1$; in particular, $*_f$-dim$(D) = 1$ if and only if each prime $*_f$-ideal of D is a maximal $*_f$-ideal.

Let $\Lambda = \{P \in *_f\text{-Max}(D) | \text{ ht}P \geq 2\}$, and let $\widetilde{D} = \cap_{P\in\Lambda} D_P$. Hence $\Lambda = \emptyset$ if and only if $*_f$-dim$(D) = 1$ (in this case, $\widetilde{D} = K$). It is known that if D is a Noetherian domain, then $R := \cap\{D_M | M \in \text{Max}(D) \text{ and ht}M \geq 2\}$ is a ring such that $D \subseteq R \subseteq D^g$ [1, Proposition 1]. We next study the relationship between D, $\widetilde{D}$, and D^{*g}.

Proposition 2.8. *Let D be a $*_w$-Noetherian domain, $\Lambda = \{P \in *_f\text{-Max}(D)|$ $\text{ht}P \geq 2\}$, and $\widetilde{D} = \cap_{P\in\Lambda} D_P$.*

(1) *$\widetilde{D}$ is $*$-linked over D.*
(2) *$\widetilde{D} \subseteq D^{*g}$. Hence if $*_D$ is the star operation on $\widetilde{D}$ as in Lemma 1.2, then $\widetilde{D}$ is a $*_D$-Noetherian domain.*
(3) *$\widetilde{D} \subsetneq D^{*g}$ if and only if there is $P \in \Lambda$ such that P is a t-ideal.*
(4) *$D \subsetneq \widetilde{D}$ if and only if there is a maximal $*_f$-ideal P of D with* ht$P = 1$.
(5) *$D \subsetneq \widetilde{D} = D^{*g}$ if and only if there is a maximal $*_f$-ideal P of D with* ht$P = 1$ *and each $Q \in \Lambda$ is not a t-ideal.*
(6) *If $*_f$-dim$(D) = 1$, then $\widetilde{D} = D^{*g} = K$.*
(7) *If $*_w = w$, then $\widetilde{D} = D^{wg}$ if and only if t-dim$(D) = 1$.*
(8) *If $*_w = w$ and $\Lambda \neq \emptyset$, then t-Max$(\widetilde{D}) = \{PD_P \cap \widetilde{D} | P \in \Lambda\}$ and t-dim$(D) = t$-dim$(\widetilde{D})$.*

Proof. (1) Clearly, if $P \in \Lambda$, then D_P is $*$-linked over D; so $D_P[X]_{N_*} \cap K = D_P$. Hence $\widetilde{D} \subseteq \widetilde{D}[X]_{N_*} \cap K \subseteq \cap_{P\in\Lambda}(D_P[X]_{N_*} \cap K) = \cap_{P\in\Lambda} D_P = \widetilde{D}$ or $\widetilde{D}[X]_{N_*} \cap K = \widetilde{D}$. Thus $\widetilde{D}$ is $*$-linked over D.

(2) Let $x \in \widetilde{D}$. Since $D \subseteq D^{*g}$, we may assume $x \notin D$, and so $(D : x) = \{a \in D | ax \in D\} \subsetneq D$. Note that $x(D : x)^v = (x(D : x))^v \subseteq D^v = D$; hence $(D : x)^v \subseteq (D : x)$, and thus $(D : x)^v = (D : x)$. Note also that, since D is a $*_w$-Noetherian domain, $(D : x)$ has a primary decomposition [4, Corollary 2.7].

Let $(D : x) = Q_1 \cap \cdots \cap Q_k$ be a primary decomposition of $(D : x)$ such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$. Since $(D : x)$ is a $*_w$-ideal, we may assume that each Q_i is a $*_w$-ideal. Then, since $x \in \widetilde{D} \subseteq D_P$ for all $P \in \Lambda$, we have $D_P = (D_P : xD_P) = (D : x)D_P = Q_1D_P \cap \cdots \cap Q_kD_P$. Hence each $\sqrt{Q_i}$ is a maximal $*_f$-ideal and $\mathrm{ht}(\sqrt{Q_i}) = 1$. Put $\sqrt{Q_i} = P_i$. Since D is a $*_w$-Noetherian domain, P_i is of finite type, i.e., $P_i = (c_1, \ldots, c_n)^{*_w}$ for some $c_i \in D$. Also, since $\sqrt{Q_i} = P_i$, there exists a positive integer e_i such that $P_i^{e_i} \subseteq (P_i^{e_i})^{*_w} \subseteq Q_i$. So $P_1^{e_1} \cdots P_k^{e_k} \subseteq Q_1 \cap \cdots \cap Q_k = (D : x)$, and hence $xP_1^{e_1} \cdots P_k^{e_k} \subseteq D$. Thus $x \in D^{*g}$.

Moreover, since $\widetilde{D}$ is $*$-linked over D by (1), $\widetilde{D}$ is a $*_D$-Noetherian domain by Lemma 1.3(4).

(3) Suppose that $\widetilde{D} \subsetneq D^{*g}$, and choose $x \in D^{*g} - \widetilde{D}$. Then there exist some maximal $*_f$-ideals $P_1, \ldots, P_n$ of D (not necessarily distinct) such that $xP_1 \cdots P_n \subseteq D$. Since $x \notin D$ and $(xP_1 \cdots P_n)^t \subseteq D$, we may assume that each P_i is a t-ideal. If $\mathrm{ht}P_i = 1$ for $i = 1, \ldots, n$, then $(P_1 \cdots P_n)D_P = D_P$, and hence $x \in xD_P = (xP_1 \cdots P_n)D_P \subseteq D_P$ for all $P \in \Lambda$; so $x \in \cap_{P \in \Lambda} D_P = \widetilde{D}$, a contradiction. Thus at least one of the P_i's is a t-ideal of height ≥ 2.

For the converse, let P be a maximal $*_f$-ideal of D such that $\mathrm{ht}P \geq 2$ and $P^t = P$. Then, since D is a $*_w$-Noetherian domain, we have $P^v = P$ or $D \subsetneq P^{-1}$. Choose an $x \in P^{-1} - D$, and note that $xP \subseteq D$ and P is a maximal t-ideal; so $x \in D^{*g}$ and $P = (D : x)$. Hence $(D_P : xD_P) = (D : x)D_P = PD_P \subsetneq D_P$, and thus $x \notin D_P$. Note that $\widetilde{D} \subseteq D_P$; so $x \notin \widetilde{D}$. Thus $\widetilde{D} \subsetneq D^{*g}$ by (2).

(4) Assume that $D \subsetneq \widetilde{D}$, and choose $x \in \widetilde{D} - D$. Then $x \in D_P$, and hence $(D : x) \nsubseteq P$ for all $P \in \Lambda$. But, since $x \notin D$ and $(D : x)^{*_f} = (D : x)^v = (D : x)$, there is a maximal $*_f$-ideal Q of D such that $(D : x) \subseteq Q$. Then, clearly, $\mathrm{ht}Q = 1$. Conversely, assume that there is a maximal $*_f$-ideal Q of D with $\mathrm{ht}Q = 1$. Then, since D is a $*_w$-Noetherian domain, $Q^v = Q$. So $D \subsetneq Q^{-1}$, and we can choose $x \in Q^{-1} - D$. Then $xQ \subseteq D$, and hence $x \in xD_P = xQD_P \subseteq D_P$ for all $P \in \Lambda$. Thus $x \in \cap_{P \in \Lambda} D_P = \widetilde{D}$.

(5) This is an immediate consequence of (3) and (4).

(6) This follows directly from (5) and the definition of $\widetilde{D}$.

(7) This is an immediate consequence of (5) and (6) and Lemma 1.3(3).

(8) For each $P \in \Lambda$, let $\widetilde{P} = PD_P \cap \widetilde{D}$. Then $\widetilde{D}_{\widetilde{P}} = D_P$, $\widetilde{P}\widetilde{D}_{\widetilde{P}} = PD_P$ and $\mathrm{ht}\widetilde{P} = \mathrm{ht}P \geq 2$. Next, note that $D \subsetneq P^{-1}$, and hence if $x \in P^{-1} - D$, then $P = x^{-1}D \cap D$; so $PD_P = x^{-1}D_P \cap D_P$, and thus PD_P is a t-ideal. Hence $\widetilde{P}\widetilde{D}_{\widetilde{P}} \cap \widetilde{D}$ is a t-ideal of $\widetilde{D}$ [12, Lemma 3.17]. Note also that, since D is an SM domain, the intersection $\widetilde{D} = \cap_{P \in \Lambda} D_P$ has finite character [3, Theorem 2.2(3)]. Let $\star$ be the star operation on $\widetilde{D}$ defined by $I^\star = \cap_{P \in \Lambda} ID_P$ [10, Theorem 32.5]. Let $Q \in t\text{-Max}(\widetilde{D})$. If $Q \nsubseteq PD_P \cap \widetilde{D}$ for all $P \in \Lambda$, then, since the intersection $\widetilde{D} = \cap_{P \in \Lambda} D_P$ has finite character, there are some $a, b \in Q$ such that $(a, b) \nsubseteq PD_P \cap \widetilde{D}$ for all $P \in \Lambda$. Hence $\widetilde{D} = (a, b)^\star \subseteq (a, b)^v \subseteq Q^t \subsetneq \widetilde{D}$, a contradiction. Hence $Q = PD_P \cap \widetilde{D}$ for some $P \in \Lambda$. Thus $t\text{-Max}(\widetilde{D}) = \{\widetilde{P} | P \in \Lambda\}$ (or see [20, Theorem 1]). Moreover, note that if D_1 is an SM-domain, then

t-dim(D_1) = sup$\{t$-dim$((D_1)_Q)|\ Q \in t$-Max$(D_1)\}$. Thus t-dim(D) = sup$\{t$-dim$(D_P)|P \in \Lambda\}$ = sup$\{t$-dim$(\widetilde{D}_{\widetilde{P}})|P \in \Lambda\} = t$-dim$(\widetilde{D})$. □

Let A be an integral domain, and let $\star$ be a star operation on A. In the next proposition, we denote by $\Gamma^{\star_f}(A)$ the ring $\cap\{A_P|P \in \star_f\text{-Max}(A)$ and ht$P \geq 2\}$. The proof of Proposition 2.8(1) shows that $\Gamma^{\star_f}(A)$ is $\star$-linked over A.

Proposition 2.9. *Let D be a $*_w$-Noetherian domain, $[*]$ be the star operation on $D[\mathbf{X}]$ as in Theorem 2.1, $R = \Gamma^{*_f}(D)$, $*_D$ be the star operation on R as in Lemma 1.2, and $N_{*_D}(R) = \{f \in R[\mathbf{X}]|(c(f)R)^{*_D} = R\}$.*

1. *$R[\mathbf{X}]_{N_*} = R[\mathbf{X}]_{N_{*_D}(R)} \subseteq \Gamma^d(D[\mathbf{X}]_{N_*}) = \Gamma^{[*]}(D[\mathbf{X}])$.*
2. *If $*_w = w$ and $N_v(R) = \{f \in R[\mathbf{X}]|(c(f)R)^v = R\}$, then $R[\mathbf{X}]_{N_v} = R[\mathbf{X}]_{N_v(R)} = \Gamma^d(D[\mathbf{X}]_{N_v}) = \Gamma^{[v]}(D[\mathbf{X}])$.*

Proof. Let $\Lambda = \{P \in *_f\text{-Max}(D)|\ \text{ht}P \geq 2\}$; so $R = \cap_{P\in\Lambda}D_P$, $D_P = R_{PD_P\cap R}$ and $D[\mathbf{X}]_{P[\mathbf{X}]} = R[\mathbf{X}]_{(PD_P\cap R)[\mathbf{X}]}$ for each $P \in \Lambda$.

(1) Let P be a maximal $*_f$-ideal of D. Then D_P is Noetherian, and hence ht$(P[\mathbf{X}])$ = ht$(PD_P[\mathbf{X}])$ = ht(PD_P) = ht$P < \infty$ [3, Lemma 1.2].

Claim 1. $R[\mathbf{X}]_{N_{*_D}(R)} \subseteq \Gamma^d(D[\mathbf{X}]_{N_*})$.

(Proof. Note that $\Gamma^d(D[\mathbf{X}]_{N_*}) = \cap_{P\in\Lambda}D[\mathbf{X}]_{P[\mathbf{X}]}$ by Lemma 1.1(1). So $R \subseteq \Gamma^d(D[\mathbf{X}]_{N_*}) \cap K$. Thus $R[\mathbf{X}] \subseteq \Gamma^d(D[\mathbf{X}]_{N_*})$. Next, let $f \in N_{*_D}(R)$. Then $(c(f)R)^{*_D} = R$, and hence $c(f)R[\mathbf{X}] \cap N_* \neq \emptyset$. But, since $PD_P[\mathbf{X}] \cap N_* = \emptyset$, we have $f \notin PD_P[\mathbf{X}]$ for all $P \in \Lambda$. Thus $\frac{1}{f} \in \cap_{P\in\Lambda}D[\mathbf{X}]_{P[\mathbf{X}]} = \Gamma^d(D[\mathbf{X}]_{N_*})$.)

Claim 2. $R[\mathbf{X}]_{N_*} = R[\mathbf{X}]_{N_{*_D}(R)}$.

(Proof. Clearly, $N_* \subseteq N_{*_D}(R)$, and thus $R[\mathbf{X}]_{N_*} \subseteq R[\mathbf{X}]_{N_{*_D}(R)}$. For the reverse containment, it suffices to show that if $f \in N_{*_D}(R)$, then $\frac{1}{f} \in R[\mathbf{X}]_{N_*}$. First, note that $R[\mathbf{X}]_{N_{*_D}(R)} \subseteq \Gamma^d(D[\mathbf{X}]_{N_*}) \subseteq (D[\mathbf{X}]_{N_*})^g = D^{*g}[\mathbf{X}]_{N_*}$ by Claim 1, Proposition 2.8(2) and Theorem 2.5. Hence $\frac{1}{f} \in D^{*g}[\mathbf{X}]_{N_*}$, and so $\frac{1}{f} = \frac{h}{g}$ for some $h \in D^{*g}[\mathbf{X}]$ and $g \in N_*$. So $g = fh$, and since R is $*$-linked over D by Proposition 2.8(1) and $(c(f)R)^{*_D} = R$, we have $c(h) \subseteq (c(h)R)^v = (c(fh)R)^v = (c(g)R)^v = R$. Hence $h \in R[\mathbf{X}]$, and thus $\frac{1}{f} \in R[\mathbf{X}]_{N_*}$.)

Claim 3. $\Gamma^d(D[\mathbf{X}]_{N_*}) = \Gamma^{[*]}(R[\mathbf{X}])$.

(Proof. Let Q be a maximal $[*]$-ideal of $D[\mathbf{X}]$. Then either $Q \cap D = (0)$ with ht$Q = 1$ or $Q \cap D$ is a maximal $*_f$-ideal of D and $Q = (Q \cap D)[\mathbf{X}]$ by Theorem 2.1(5). Thus the equality follows directly from Lemma 1.1(1).)

(2) By Proposition 2.8(8), t-Max$(R) = \{PD_P \cap R|P \in \Lambda\}$, and thus by Lemma 1.1(1), Max$(R[\mathbf{X}]_{N_v(R)}) = \{(PD_P \cap R)[\mathbf{X}]_{N_v(R)}|P \in \Lambda\}$. Thus

$$R[\mathbf{X}]_{N_v(R)} = \cap_{P\in\Lambda}D[\mathbf{X}]_{P[\mathbf{X}]} = \Gamma^d(D[\mathbf{X}]_{N_v}).$$

Also, the proof of Claim 3 of (1) above shows that $R[\mathbf{X}]_{N_v} = R[\mathbf{X}]_{N_v(R)}$. This completes the proof by (1). □

Let R be an overring of D. Following Wadsworth [18], we say that (D, R) is a Noetherian pair if every domain A with $D \subseteq A \subseteq R$ is Noetherian. So if D is a Noetherian domain, then (D, D^g) is a Noetherian pair. Also, if D is a $*_w$-Noetherian domain, then $(D[\mathbf{X}]_{N_*}, D^{*g}[\mathbf{X}]_{N_*})$ is a Noetherian pair by Corollary 2.7. As a generalization of the concept of an Noetherian pair, we will say that (D, R) is an *SM domain pair* if each t-linked overring T of D with $T \subseteq R$ is an SM domain. Clearly, a Noetherian pair is an SM domain pair and if D is an SM domain, then (D, D^{wg}) is an SM domain pair. Also, if each maximal ideal of D is a t-ideal, then each overring of D is t-linked over D. Hence, in this case, an SM domain pair is a Noetherian pair.

Proposition 2.10. *Let R be an overring of D and S a multiplicative subset of D.*

(1) *If (D, R) is an SM domain pair, then (D_S, R_S) is an SM domain pair.*
(2) *If D is an SM-domain, then $(D[\mathbf{X}], D^{wg}[\mathbf{X}]_{N_v})$ is an SM domain pair.*
(3) *If $(D[\mathbf{X}], R[\mathbf{X}])$ is an SM domain pair, then (D, R) is an SM domain pair.*
(4) *If $(D[\mathbf{X}]_{N_v}, R[\mathbf{X}]_{N_v})$ is an SM domain pair, then (D, R) is an SM domain pair.*

Proof. (1) Let A be a t-linked overring of D_S such that $A \subseteq R_S$. Then A is t-linked over D (for if I is a nonzero finitely generated ideal of D, then $I^{-1} = D$ $\Rightarrow (ID_S)^{-1} = D_S$ (cf. [9, Proposition 2.2(d)]) $\Rightarrow (IA)^{-1} = ((ID_S)A)^{-1} = A$), and hence $A \cap R$ is t-linked over D [9, Proposition 2.2(b)]. Since $D \subseteq A \cap R \subseteq R$, $A \cap R$ is an SM domain by assumption. Also, note that $(A \cap R)_S \subseteq A_S = A$ and $S \subseteq A \subseteq R_S$; so $(A \cap R)_S = A$. Thus A is an SM domain [19, Proposition 4.7].

(2) Note that $D[\mathbf{X}]$ is an SM domain by Theorem 2.1(2) and (6) and Corollary 2.4 and $(D[\mathbf{X}])^{wg} = D^{wg}[\mathbf{X}]_{N_v}$ by Corollary 2.6. Thus $(D[\mathbf{X}], D^{wg}[\mathbf{X}]_{N_v})$ is an SM domain pair.

(3) Let A be a t-linked overring of D such that $A \subseteq R$. Then $A[\mathbf{X}]$ is t-linked over $D[\mathbf{X}]$ [8, Lemma 1.6] and $A[\mathbf{X}] \subseteq R[\mathbf{X}]$. Hence $A[\mathbf{X}]$ is an SM domain by assumption, and thus A is an SM domain by Theorem 2.1(6) and Corollary 2.4. Thus (D, R) is an SM domain pair.

(4) Let T be a t-linked overring of D such that $T \subseteq R$; then $D[\mathbf{X}]_{N_v} \subseteq T[\mathbf{X}]_{N_v} \subseteq R[\mathbf{X}]_{N_v}$. Note that each maximal ideal of $D[\mathbf{X}]_{N_v}$ is a t-ideal; so $T[\mathbf{X}]_{N_v}$ is t-linked over $D[\mathbf{X}]_{N_v}$, and hence $T[\mathbf{X}]_{N_v}$ is an SM domain by assumption. Let $N_v(T) = \{g \in T[\mathbf{X}] | (c(g)T)^v = T\}$, then $N_v \subseteq N_v(T)$ since T is t-linked over D, and so $T[\mathbf{X}]_{N_v(T)} = (T[\mathbf{X}]_{N_v})_{N_v(T)}$. Hence $T[\mathbf{X}]_{N_v(T)}$ is an SM domain [19, Proposition 4.7], and thus T is an SM domain [3, Theorem 2.2]. □

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Department of Mathematics
University of Incheon
Incheon 406-772, Korea
E-mail address: whan@incheon.ac.kr