J. Korean Math. Soc. **48** (2011), No. 1, pp. 1–12
DOI 10.4134/JKMS.2011.48.1.001

ALGEBRAS WITH PSEUDO-RIEMANNIAN BILINEAR FORMS

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Abstract. The purpose of this paper is to study pseudo-Riemannian algebras, which are algebras with pseudo-Riemannian non-degenerate symmetric bilinear forms. We find that pseudo-Riemannian algebras whose left centers are isotropic play a curial role and show that the decomposition of pseudo-Riemannian algebras whose left centers are isotropic into indecomposable non-degenerate ideals is unique up to a special automorphism. Furthermore, if the left center equals the center, the orthogonal decomposition of any pseudo-Riemannian algebra into indecomposable non-degenerate ideals is unique up to an isometry.

1. Introduction

Let A be an algebra with a bilinear product $A \times A \to A$ denoted by $(a, b) \mapsto ab$. The purpose of this paper is to study the pairs (A, f) where f denotes a non-degenerate symmetric bilinear form on A satisfying

$$f(xy, z) + f(y, xz) = 0, \quad \forall x, y, z \in A. \tag{1}$$

In abuse of notation we will use the term pseudo-Riemannian algebra for denoting such a pair. There are some studies for A to be a Lie algebra [5], a fermionic Novikov algebra [4], another kind of Lie-admissible algebra [3] and so on.

The motivation to study pseudo-Riemannian algebras comes from the studies on Lie groups with left-invariant pseudo-metrics [1, 6]. In some senses, pseudo-Riemannian algebra is related to pseudo-Riemannian connection, which is a pseudo-metric connection such that the torsion is zero and parallel translation preserves the bilinear form on the tangent spaces [7].

The purpose of this paper is to study the decomposition about pseudo-Riemannian algebras. To begin with, we find that pseudo-Riemannian algebras

Received November 14, 2007; Revised March 3, 2010.
2010 *Mathematics Subject Classification.* Primary 17A30, 17D99.
Key words and phrases. pseudo-Riemannian algebra, indecomposable ideal, isometry, orthogonal decomposition.
This work was financially supported by National Natural Science Foundation of China (No. 10971103).

whose left centers are isotropic play a curial role (Theorem 3.2). And then we show that the decomposition of pseudo-Riemannian algebras whose left centers are isotropic into indecomposable non-degenerate ideals is unique up to a special automorphism (Theorem 4.4). It is interesting that there are decomposable pseudo-Riemannian algebras such that any decomposition into indecomposable non-degenerate ideals is not orthogonal (Remark 6.3). But there must be an orthogonal decomposition if the left center equals the center (Proposition 6.2). In this case, the orthogonal decomposition of a pseudo-Riemannian algebra into indecomposable non-degenerate ideals is unique up to an isometry (Theorem 6.6). As an application, we get that the orthogonal decomposition of a quadratic Lie algebra into irreducible non-degenerate ideals is unique up to an isometry (Corollary 6.10).

Throughout this paper, we assume that the algebras are of finite dimension over the complex number field.

2. Preliminaries

In this section, we list some definitions and propositions.

Definition. Let H be a subspace of A. If $AH \subseteq H$, then H is called a left ideal of A. If $HA \subseteq H$, then H is called a right ideal of A. If H is both a left ideal and a right ideal, then H is an ideal. The algebra A is called abelian if $A \neq 0$ and $xy = 0$ for any $x, y \in A$.

Definition. A bilinear form f on A is called pseudo-Riemannian if

$$f(xy, z) + f(y, xz) = 0, \quad \forall x, y, z \in A.$$

Definition. The pair (A, f) is called a pseudo-Riemannian algebra if f is an pseudo-Riemannian non-degenerate symmetric bilinear form on A.

Definition. Let (A, f) be a pseudo-Riemannian algebra and H a subspace of A. If $f(x, y) = 0$ for any $x, y \in H$, then H is called isotropic. If $f \mid_{H\times H}$ is non-degenerate, then H is called non-degenerate.

Definition. Let (A, f) be a pseudo-Riemannian algebra. If there exist non-trivial and non-degenerate ideals A_1 and A_2 such that $A = A_1 \oplus A_2$, then (A, f) is called decomposable, otherwise indecomposable. Furthermore, if $f(A_1, A_2) = 0$, then the decomposition $A = A_1 \oplus A_2$ is called an orthogonal decomposition.

Definition. The pair (A, f) is called irreducible if it has no nontrivial non-degenerate ideal.

Definition. Let (A, f) be a pseudo-Riemannian algebra. An automorphism π of A is called an isometry if π preserves the bilinear form, i.e.,

$$f(\pi(x), \pi(y)) = f(x, y), \quad \forall x, y \in A.$$

The following notation will be used in this paper. Let $H^\perp$ denote the subspace of A orthogonal to H with respect to f, i.e.,

$$H^\perp = \{x \in A \mid f(x,y) = 0, \quad \forall y \in H\}.$$

Let $LC(A)$ denote the left center of A, i.e.,

$$LC(A) = \{x \in A \mid yx = 0, \quad \forall y \in A\}.$$

Let $Z(A)$ denote the center of A, i.e.,

$$Z(A) = \{x \in A \mid xy = yx = 0, \quad \forall y \in A\}.$$

Proposition 2.1. *Let (A, f) be a pseudo-Riemannian algebra. Then $LC(A) = (AA)^\perp$. As a consequence, $\dim LC(A) + \dim AA = \dim A$.*

Proof. Assume that $x \in LC(A)$, i.e., $yx = 0$ for any $y \in A$. Then for any $y, z \in A$, $f(yx, z) = 0$. It follows that $f(x, yz) = 0$ for any $y, z \in A$. That is, $LC(A) \subseteq (AA)^\perp$. Similarly, $(AA)^\perp \subseteq LC(A)$. □

Proposition 2.2. *Let (A, f) be a pseudo-Riemannian algebra and H an ideal of A. Then $H^\perp$ is a left ideal and $HH^\perp = 0$.*

Proof. Since H is an ideal, we have

$$f(H, AH^\perp) = -f(AH, H^\perp) = 0.$$

It follows that $H^\perp$ is a left ideal. Since

$$f(A, HH^\perp) = -f(HA, H^\perp) = 0,$$

we have $HH^\perp = 0$ by the non-degeneracy of f. □

Proposition 2.3. *Let (A, f) be a pseudo-Riemannian algebra. Then there exists a decomposition $A = \bigoplus_{i=1}^{l} A_i$ of A into indecomposable non-degenerate ideals.*

Proof. It follows from a simple induction on $\dim A$. □

3. Pseudo-Riemannian algebras whose left centers are not isotropic

In this section, we focus on pseudo-Riemannian algebras whose left centers are not isotropic.

Proposition 3.1. *Let A be an abelian algebra. If f is a non-degenerate symmetric bilinear form on A, then (A, f) is a pseudo-Riemannian algebra. Furthermore, there exists an orthogonal decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ of A into indecomposable non-degenerate ideals such that $\dim A_i = 1, 1 \le i \le n$.*

Proof. Since A is abelian, we know that any subspace is an ideal. If f is a non-degenerate symmetric bilinear form on A, then there exists a sequence of non-degenerate ideals $A_i, 1 \le i \le n$ of dimension 1 such that the decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is orthogonal. Obviously, A_i is indecomposable and f satisfies the identity (1). □

Let (H, f_H) be an abelian pseudo-Riemannian algebra and (I, f_I) a pseudo-Riemannian algebra with the product $\circ$. Let

$$so(I) = \{A \in \mathrm{End}I \mid f_I(A(x), y) + f_I(x, A(y)) = 0\}.$$

Given a linear mapping $L : H \to so(I)$ denoted by $x \mapsto L_x$, define a product $*$ on vector space $A = H +_L I$ (direct sum as subspaces) by

$$\begin{aligned} &x * y = 0, \quad \forall x, y \in H,\\ &x * y = 0, \quad \forall x \in I, y \in H,\\ &x * y = x \circ y, \quad \forall x, y \in I,\\ &x * y = L_x(y), \quad \forall x \in H, y \in I, \end{aligned}$$

and define a symmetric bilinear form f on A by

$$\begin{aligned} &f(x, y) = f_H(x, y), \quad \forall x, y \in H,\\ &f(x, y) = f_I(x, y), \quad \forall x, y \in I,\\ &f(x, y) = 0, \quad \forall x \in H, y \in I. \end{aligned}$$

One can see that (A, f) is a pseudo-Riemannian algebra whose left center is not isotropic. On the other hand, we have:

Theorem 3.2. *Let (A, f) be a pseudo-Riemannian algebra whose left center is not isotropic. Then there exists a sequence of non-degenerate subalgebras of A such that*

$$A = A_0 \supset A_1 \supset \cdots \supset A_n,$$

where A_i is an ideal of A_{i-1}, the quotient algebra A_{i-1}/A_i is abelian for each $i \in \{1, 2, \dots, n\}$, and the left center of A_n is isotropic.

Proof. Since the left center $LC(A)$ of A is not isotropic, there exists a maximal subspace H_1 of $LC(A)$ such that $f\mid_{H_1 \times H_1}$ is non-degenerate. Let

$$A_1 = H_1^{\perp}.$$

Then for any $a \in A, h \in H_1, h' \in A_1^{\perp}$,

$$f(h, ah') = -f(ah, h') = 0.$$

It follows that A_1 is an ideal of A. The theorem follows by induction. □

4. Pseudo-Riemannian algebras whose left centers are isotropic

Theorem 3.2 shows that pseudo-Riemannian algebras whose left centers are isotropic play a crucial role.

Proposition 4.1. *Let (A, f) be a pseudo-Riemannian algebra whose left center is isotropic. Then (A, f) is decomposable if and only if there exist non-trivial ideals A_1 and A_2 of A such that $A = A_1 \oplus A_2$.*

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) Assume that there exist non-trivial ideals A_1 and A_2 of A such that $A = A_1 \oplus A_2$. It is enough to show that $f\mid_{A_1\times A_1}$ and $f\mid_{A_2\times A_2}$ are non-degenerate. Assume that $f\mid_{A_1\times A_1}$ is degenerate. Then there exists a non-zero element $x \in A_1$ such that $f(x, A_1) = 0$. If $x \in A_1A_1$, then

$$f(x, A) = 0$$

since $f(x, A_2) \subseteq f(A_1A_1, A_2) = f(A_1, A_1A_2) = 0$. Thus $x = 0$ since $f\mid_{A\times A}$ is non-degenerate. It is a contradiction, so $x \notin A_1A_1$. Since $LC(A)$ is isotropic, we have $LC(A) \subseteq LC(A)^{\perp} = AA$ by Proposition 2.1. Thus

$$x \notin LC(A).$$

Namely, there exists $y \in A_1$ such that $yx \neq 0$. Therefore there exists $z \in A$ such that $f(yx, z) \neq 0$ since $f\mid_{A\times A}$ is non-degenerate. Thus we have

$$f(x, yz) = -f(yx, z) \neq 0.$$

Since A_1 is an ideal of A and $y \in A_1$, we have $yz \in A_1$, which contradicts the choice of x. Namely, $f\mid_{A_1\times A_1}$ is non-degenerate. Similarly, $f\mid_{A_2\times A_2}$ is non-degenerate. □

The following is to show that the decomposition of any pseudo-Riemannian algebra whose left center is isotropic into non-degenerate indecomposable ideals is unique up to an automorphism.

Let (A, f) be a pseudo-Riemannian algebra whose left center is isotropic and let

$$A = A_1 \oplus \cdots \oplus A_n,$$
$$A = A'_1 \oplus \cdots \oplus A'_m$$

be decompositions of A. Here $A_i, A'_j, 1 \leq i \leq n, 1 \leq j \leq m$, are indecomposable non-degenerate ideals of A.

One can easily see that $A_1A_1 \neq 0$. In fact, assume that $A_1A_1 = 0$. Thus $A_1 \subseteq LC(A)$, which contradicts that $LC(A)$ is isotropic. Since $A_1A_1 = \bigoplus_{j=1}^{m} A_1A'_j$, we have $A_1A'_j \neq 0$ for some j. Without loss of generality, assume that $A_1A'_1 \neq 0$. Let $H_1 = \bigoplus_{j=2}^{n} A_j$ and $H'_1 = \bigoplus_{j=2}^{m} A'_j$, which are non-degenerate ideals of A by Proposition 4.1.

Lemma 4.2. *$A_1 \cap H'_1 = 0$ and $A'_1 \cap H_1 = 0$.*

Proof. Let $B_1 = A_1 \cap A'_1$ and $B_2 = A_1 \cap H'_1$. Clearly,

$$A_1A_1 = A_1A = A_1A'_1 \oplus A_1H'_1 \subseteq B_1 \oplus B_2.$$

(1) If $A_1 = B_1 \oplus B_2$, then both B_1 and B_2 are non-degenerate ideals of A_1, hence non-degenerate ideals of A. Since A_1 is indecomposable and $B_1 \neq 0$, we have $B_2 = 0$. That is, $A_1 \cap H'_1 = 0$.

(2) If $A_1 \neq B_1 \oplus B_2$, there exists $x \in A_1$ such that $x \notin B_1 \oplus B_2$. Then $x = x_1 + x_2$, where $x_1 \in A'_1, x_2 \in H'_1$. Using the other decomposition,

$$x_1 = x_1^1 + x_1^2, \quad x_2 = x_2^1 + x_2^2,$$

where $x_1^1, x_2^1 \in A_1, x_1^2, x_2^2 \in H_1$. So

$$x = x_1^1 + x_1^2 + x_2^1 + x_2^2.$$

Then $x = x_1^1 + x_2^1$ and $x_1^2 + x_2^2 = 0$. One can easily check that

$$A_1 x_1^1 \subseteq A_1 A_1', \quad x_1^1 A_1 \subseteq A_1' A_1;$$

$$A_1 x_2^1 \subseteq A_1 H_1', \quad x_2^1 A_1 \subseteq H_1' A_1.$$

If $x_1^1 \notin B_1 \oplus B_2$, let

$$B_1^{(1)} = B_1 + \mathbb{C} x_1^1, \quad B_2^{(1)} = B_2.$$

If $x_1^1 \in B_1 \oplus B_2$, then $x_2^1 \notin B_1 \oplus B_2$. Let

$$B_1^{(1)} = B_1, \quad B_2^{(1)} = B_2 + \mathbb{C} x_2^1.$$

It is clear that both $B_1^{(1)}$ and $B_2^{(1)}$ are ideals of A_1 and $B_1^{(1)} \cap B_2^{(1)} = 0$. If

$$A_1 = B_1^{(1)} \oplus B_2^{(1)},$$

using similar argument as in (1), $B_2^{(1)} = 0$. In particular, $A_1 \cap H_1' = 0$.

If $A_1 \neq B_1^{(1)} \oplus B_2^{(1)}$, since $\dim A_1 < \infty$, repeating the discussion in (2), we may choose $B_1^{(k)}$ and $B_2^{(k)}$ such that

$$A_1 = B_1^{(k)} \oplus B_2^{(k)},$$

where both $B_1^{(k)}$ and $B_2^{(k)}$ are ideals of A_1. Using similar argument as in (1), $B_2^{(k)} = 0$. In particular, $A_1 \cap H_1' = 0$. Similarly, $A_1' \cap H_1 = 0$. □

Lemma 4.3. *The projection $\pi_1 : A_1 \to A_1'$ is an isomorphism and preserves the bilinear form.*

Proof. Since $\ker \pi_1 \subseteq A_1 \cap H_1' = 0$, we have that π_1 is injective. Thus $\dim A_1 \leq \dim A_1'$. Similarly, $\dim A_1' \leq \dim A_1$. Therefore $\dim A_1' = \dim A_1$. For any $x, y \in A_1$, it is clear that $\pi_1(xy) = \pi_1(x)\pi_1(y)$, i.e., π_1 is an isomorphism from A_1 to A_1'. For any $x \in A_1$, $x = x_1 + x_2$, where $x_1 \in A_1', x_2 \in H_1'$. It is clear that $A_1' x_2 = 0$ and

$$H_1' x_2 = H_1' x \subseteq H_1' \cap A_1 = 0.$$

Thus $x_2 \in LC(A)$. Therefore $f(x,x) = f(x_1,x_1) + 2f(x_1,x_2)$. Let $x_1 = h_1 + h_2$, where $h_1 \in H_1', h_2 \in (H_1')^\perp$. Furthermore $h_1 \in LC(H_1') \subseteq LC(A)$ by

$$H_1' h_1 = H_1'(x_1 - h_2) = 0.$$

It follows that

$$f(x,x) = f(x_1,x_1) = f(\pi_1(x), \pi_1(x)).$$

Namely, π_1 keeps the bilinear from. □

Furthermore, we have

$$A_1A_1 = A_1A_1' = A_1'A_1 = A_1'A_1',$$
$$A_1H_1' = H_1'A_1 = A_1'H_1 = H_1A_1' = 0.$$

Repeating the above discussion for $j = 2, 3, \dots, n$, we have:

Theorem 4.4. *Let (A, f) be a pseudo-Riemannian algebra whose left center is isotropic and let*

$$A = A_1 \oplus \cdots \oplus A_n,$$
$$A = A_1' \oplus \cdots \oplus A_m'$$

be decompositions of A. Here $A_i, A_j', 1 \le i \le n, 1 \le j \le m$, are indecomposable non-degenerate ideals of A. Then we have

(1) $n = m$.
(2) *Changing the subscripts if necessary, we can get*

$$\dim A_j = \dim A_j',$$
$$A_jA_j = A_jA_j' = A_j'A_j = A_j'A_j',$$
$$A_jA_k' = A_j'A_k = 0, \quad j \ne k.$$

(3) *The projections $\pi_i : A_i \to A_i', 1 \le i \le n$ are isomorphisms and preserve the bilinear form, so $\pi = (\pi_1, \dots, \pi_n)$ is an automorphism of A.*

5. Direct sum of two pseudo-Riemannian algebras whose left centers are isotropic

The decomposition of pseudo-Riemannian algebras whose left centers are isotropic into indecomposable non-degenerate ideals is unique up to an automorphism. But the decomposition is not necessarily orthogonal. A natural question is: How to construct a new one by two pseudo-Riemannian algebras whose left centers are isotropic?

Theorem 5.1. *Let (A_1, f_1) and (A_2, f_2) be pseudo-Riemannian algebras whose left centers are isotropic, $A = A_1 \oplus A_2$ and f a symmetric bilinear form on A such that $f|_{A_1 \times A_1} = f_1$ and $f|_{A_2 \times A_2} = f_2$. If*

$$f(A_1A_1, A_2) = f(A_2A_2, A_1) = 0,$$

then (A, f) is a pseudo-Riemannian algebra whose left center is isotropic.

Proof. Since the left center of A_1 is isotropic, by Proposition 2.1, we have

$$LC(A_1) \subseteq LC(A_1)^{\perp} = A_1A_1.$$

Thus there exists a basis $\{e_1, \dots, e_{i_1}, e_{i_1+1}, \dots, e_m, e_{m+1}, \dots, e_{m+i_1}\}$ of A_1 such that

$$f_1(e_i, e_j) = \delta_{ij}, \quad i_1 + 1 \le i, j \le m,$$
$$f_1(e_i, e_{m+j}) = \delta_{ij}, \quad 1 \le i, j \le i_1,$$
$$f_1(e_i, e_j) = 0, \quad 1 \le i, j \le i_1,$$

$$f_1(e_i, e_j) = 0, \quad m+1 \le i, j \le m+i_1,$$

where $LC(A_1) = L(e_1, \dots, e_{i_1})$ and $A_1A_1 = L(e_1, \dots, e_m)$. Here $L(e_1, \dots, e_i)$ means the subspace spanned by $e_1, \dots, e_i$.

Similarly, there exists a basis $\{h_1, \dots, h_{i_2}, h_{i_2+1}, \dots, h_n, h_{n+1}, \dots, h_{n+i_2}\}$ of A_2 such that

$$\begin{aligned} &f_2(h_i, h_j) = \delta_{ij}, \quad i_2+1 \le i, j \le n, \\ &f_2(h_i, h_{n+j}) = \delta_{ij}, \quad 1 \le i, j \le i_2, \\ &f_2(h_i, h_j) = 0, \quad 1 \le i, j \le i_2, \\ &f_2(h_i, h_j) = 0, \quad n+1 \le i, j \le n+i_2, \end{aligned}$$

where $LC(A_2) = L(h_{n+1}, \dots, h_{n+i_2})$ and $A_2A_2 = L(h_{i_2+1}, \dots, h_{n+i_2})$. Since $f(A_1A_1, A_2) = f(A_2A_2, A_1) = 0$, we have that the matrix of f with respect to the basis $\{e_1, \dots, e_{m+i_1}, h_1, \dots, h_{n+i_2}\}$ is

$$G = \begin{pmatrix} 0 & 0 & B & & & & \\ 0 & C & 0 & & & & \\ B & 0 & 0 & F' & & & \\ & & F & 0 & 0 & D \\ & & & 0 & E & 0 \\ & & & D & 0 & 0 \end{pmatrix},$$

where $C = I_{m-i_1}$, $E = I_{n-i_2}$, $B = I_{i_1}$ and $D = I_{i_2}$. For any matrix F, $\det G \neq 0$. It follows that f is a non-degenerate symmetric bilinear form satisfying the identity (1). Thus (A, f) is a pseudo-Riemannian algebra whose left center is isotropic. □

Remark 5.2. Let (A, f) be a pseudo-Riemannian algebra. If $A = A_1 \oplus A_2$, then it is easy to see that $f(A_1A_1, A_2) = f(A_2A_2, A_1) = 0$.

Remark 5.3. Assume that $Z(A) \neq 0$. Therefore $Z(A_1) \neq 0$ or $Z(A_2) \neq 0$. Without loss of generality, assume that $Z(A_1) \neq 0$. Let $a_{ij} = ent_{ij}(F)$. Assume that $e_k \in Z(A_1)$ for some $k \in \{1, 2, \dots, i_1\}$. Then for any $0 < i \le i_2$, let

$$h_i' = h_i - a_{ik}e_k.$$

Let $A_2' = L(h_1', \dots, h_{i_2}', h_{i_2+1}, \dots, h_{n+i_2})$. It is easy to check that A_2' is an ideal of A and $A = A_1 \oplus A_2'$. But

$$f(h_i', e_{k+m}) = 0.$$

Proposition 5.4. *Let notations be as above. If $LC(A_1) = 0$ or $LC(A_2) = 0$, then the decomposition $A = A_1 \oplus A_2$ is orthogonal.*

Proof. If $LC(A_1) = 0$, then $A_1 = A_1A_1$. It follows that

$$f(A_1, A_2) = f(A_1A_1, A_2) = 0.$$

Similarly, $f(A_1, A_2) = 0$ if $LC(A_2) = 0$. □

6. Pseudo-Riemannian algebras whose left centers equal the centers

In this section, we focus on pseudo-Riemannian algebras whose left centers equal the centers. Similar to Theorem 3.2, we have:

Theorem 6.1. *Let (A, f) be a pseudo-Riemannian algebra whose left center equals the center. If the left center is not isotropic, then there exist non-degenerate ideals A_1 and A_2 such that $A = A_1 \oplus A_2$, where $f(A_1, A_2) = 0$, $A_1A_1 = 0$ and the left center of A_2 is isotropic.*

Proposition 6.2. *Let (A, f) be a decomposable pseudo-Riemannian algebra whose left center equals the center. If the left center is isotropic, then there exist non-degenerate ideals A_1 and A_2 such that the decomposition $A = A_1 \oplus A_2$ is orthogonal.*

Proof. Since A is decomposable, we have $A = A_1 \oplus A_2$, where $f|_{A_i\times A_i}, i = 1, 2$ are non-degenerate. Therefore $A = A_1 + A_1^{\perp}$ and $A_1A_1^{\perp} = 0$. Let

$$x = x_1 + x_2,$$

where $x \in A_1^{\perp}, x_1 \in A_1, x_2 \in A_2$. Since both A_1 and A_2 are ideals, we have

$$f(yx_1, z) = -f(x_1, yz) = f(x_2, yz) = -f(yx_2, z) = 0$$

for any $y, z \in A_1$. Thus $A_1x_1 = 0$ since $f|_{A_1\times A_1}$ is non-degenerate. Namely $x_1 \in LC(A) = Z(A)$. Then $xy = (x_1 + x_2)y = 0$ for any $y \in A_1$, i.e.,

$$A_1^{\perp}A_1 = 0.$$

It follows that $A_1^{\perp}$ is an ideal. Similarly, $A_2^{\perp}$ is an ideal. □

Remark 6.3. Let notations be as in Remark 5.3. Let A_1 and A_2 be indecomposable pseudo-Riemannian algebras such that

$$LC(A_1) \neq Z(A_1) \text{ and } LC(A_2) \neq Z(A_2).$$

Suppose that $e_i \in LC(A_1), e_i \notin Z(A_1)$ and $h_{n+j} \in LC(A_2), h_{n+j} \notin Z(A_2)$. Here $i \in \{1, 2, \dots, i_1\}$ and $j \in \{1, 2, \dots, i_2\}$. Let F be a matrix such that $ent_{ij}(F) \neq 0$. Then A is a decomposable pseudo-Riemannian algebra without orthogonal decomposition.

Similar to the proof of Theorem 4.4, in terms of Proposition 6.2, we have:

Theorem 6.4. *Let (A, f) be a pseudo-Riemannian algebra whose left center equals the center and whose left center is isotropic, and let*

$$A = A_1 \oplus \cdots \oplus A_n,$$

$$A = A_1' \oplus \cdots \oplus A_m'$$

be orthogonal decompositions of A. Here $A_i, A_j', 1 \le i \le n, 1 \le j \le m$, are indecomposable non-degenerate ideals of A. Then we have

(1) $n = m$.

(2) *Changing the subscripts if necessary, we can get*

$$\dim A_j = \dim A'_j,$$
$$A_j A_j = A_j A'_j = A'_j A_j = A'_j A'_j,$$
$$A_j A'_k = A'_j A_k = 0, \quad j \neq k.$$

(3) *The projections* $\pi_i : A_i \to A'_i, 1 \leq i \leq n$ *are isomorphisms and preserve the bilinear form, so* $\pi = (\pi_1, \ldots, \pi_n)$ *is an isometry of* A*, that is, the decomposition is unique up to an isometry.*

Theorem 6.5. *Let* (A, f) *be a pseudo-Riemannian algebra whose left center equals the center and whose left center is not isotropic. If the decomposition* $A = A_1 \oplus A_2$ *is orthogonal such that* A_1 *and* A_2 *are non-degenerate,* $LC(A_1)$ *is isotropic and* $A_2 \subseteq LC(A)$*, then the decomposition is unique up to an isometry.*

Proof. Let $A = A'_1 \oplus A'_2$ be another such decomposition. Then we have

$$AA = A_1 A_1 = A'_1 A'_1 = A_1 A'_1.$$

Since the left center of A_1 is isotropic, by Proposition 2.1, we have

$$LC(A_1) \subseteq LC(A_1)^{\perp} = A_1 A_1 = A'_1 A'_1.$$

Since $LC(A) = Z(A)$, we have $LC(A_1) \subseteq LC(A) \cap A'_1 A'_1 = LC(A'_1)$. Similarly $LC(A'_1) \subseteq LC(A'_1)$. Namely

$$LC(A_1) = LC(A'_1).$$

By Proposition 2.1, we have $\dim A_1 = \dim A'_1$, and then $\dim A_2 = \dim A'_2$.

Let $\{e_1, \ldots, e_k, \ldots, e_n, \ldots, e_{n+k}\}$ be a basis of A_1 such that $LC(A_1) = L(e_1, \ldots, e_k)$, $A_1 A_1 = L(e_1, \ldots, e_n)$, and

$$f(e_i, e_j) = \delta_{ij}, \quad k+1 \leq i, j \leq n,$$
$$f(e_i, e_{n+j}) = \delta_{ij}, \quad 1 \leq i, j \leq k,$$
$$f(e_i, e_j) = 0, \quad 1 \leq i, j \leq k,$$
$$f(e_i, e_j) = 0, \quad n+1 \leq i, j \leq n+k.$$

Now consider the projections

$$\pi_1 : A_1 \to A'_1,$$
$$\pi_2 : A_2 \to A'_2,$$

which are isomorphisms. We have $\pi_1 \mid_{A_1 A_1} = id$ and $f(\pi_1(e_i), \pi_1(e_j)) = f(e_i, e_j)$ for $1 \leq i \leq n+k$ and $1 \leq j \leq n$.

Assume that $e_p = e_{p_3} + e_{p_4}$ for $n+1 \leq p \leq n+k$, where $e_{p_3} \in A'_1$ and $e_{p_4} \in A'_2$. For $n+1 \leq q \leq n+k$, we have

$$0 = f(e_p, e_q) = f(e_{p_3}, e_{q_3}) + f(e_{p_4}, e_{q_4}).$$

Let $b_{pq} = f(e_{p_4}, e_{q_4})$ for $p \neq q$, $2b_{pp} = f(e_{p_4}, e_{p_4})$ and $e'_{p_3} = e_{p_3} + \sum_{l=p}^{n+k} b_{pl} e_{l-n}$, it is easy to see that

$$f(e'_{p_3}, e'_{p_3}) = f(e_{p_3}, e_{p_3}) + 2b_{pp} = 0, \quad n+1 \leq p \leq n+k;$$

$$f(e'_{p_3}, e'_{q_3}) = f(e_{p_3}, e_{q_3}) + b_{pq} = 0, \quad n+1 \leq p \leq q \leq n+k.$$

Define $\pi'_1 : A_1 \to A'_1$ by

$$\pi'_1(e_j) = e_j, \quad 1 \leq j \leq n;$$
$$\pi'_1(e_j) = e'_{j_3}, \quad n+1 \leq j \leq n+k.$$

It is easy to check that π'_1 is also an isomorphism from A_1 onto A'_1 and preserves the bilinear form. Then $\pi = (\pi'_1, \pi_2)$ is an isometry of A. □

Thanks to Theorems 6.4 and 6.5, we have:

Theorem 6.6. *Let (A, f) be a pseudo-Riemannian algebra whose left center equals the center. Then the orthogonal decomposition of A into indecomposable non-degenerate ideals is unique up to an isometry.*

If the algebra is anti-commutative, i.e.,

$$ab = -ba, \quad \forall a, b \in A,$$

then $LC(A) = Z(A)$ and

$$f(ab, c) = -f(b, ac) = f(b, ca) = f(a, bc), \quad \forall a, b, c \in A. \tag{2}$$

Lemma 6.7 ([2])**.** *Let (A, f) be an anti-commutative pseudo-Riemannian algebra. If H is an ideal of A, then $H^\perp$ is an ideal of A. Furthermore, assume that H is non-degenerate, then $H^\perp$ is also non-degenerate and $A = H \oplus H^\perp$.*

It follows that:

Proposition 6.8. *Let (A, f) be an anti-commutative pseudo-Riemannian algebra. Then A is indecomposable if and only if A is irreducible.*

Thus, we have:

Theorem 6.9. *Let (A, f) be an anti-commutative pseudo-Riemannian algebra. Then the orthogonal decomposition of A into irreducible non-degenerate ideals is unique up to an isometry.*

By Theorem 6.9 and the identity (2), we have the following result on the uniqueness of the decomposition of quadratic Lie algebras.

Corollary 6.10 ([8])**.** *Let $\mathfrak{g}$ be a quadratic Lie algebra. Then the orthogonal decomposition of $\mathfrak{g}$ into irreducible non-degenerate ideals is unique up to an isometry.*

Acknowledgments. The authors would like to express their thanks to the referee for the helpful corrections and suggestions.

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