MEROMORPHIC SOLUTIONS OF SOME
q-DIFFERENCE EQUATIONS

BaoQin Chen and ZongXuan Chen

Abstract. We consider meromorphic solutions of q-difference equations of the form

$$\sum_{j=0}^{n} a_j(z)f(q^j z) = a_{n+1}(z),$$

where $a_0(z), \ldots, a_{n+1}(z)$ are meromorphic functions, $a_0(z)a_n(z) \not\equiv 0$ and $q \in \mathbb{C}$ such that $0 < |q| \leq 1$. We give a new estimate on the upper bound for the length of the gap in the power series of entire solutions for the case $0 < |q| < 1$ and $n = 2$. Some growth estimates for meromorphic solutions are also given in the cases $0 < |q| < 1$ and $|q| = 1$. Moreover, we investigate zeros and poles of meromorphic solutions for the case $|q| = 1$.

1. Introduction and main results

Recently many papers (see [1-4, 6-10, 13]) focused on complex difference equations and q-difference equations. Many meaningful results have been obtained. In this paper, we are concerned about meromorphic solutions of q-difference equations of the form

$$(1.1) \qquad \sum_{j=0}^{n} a_j(z)f(q^j z) = a_{n+1}(z),$$

where $a_0(z), \ldots, a_{n+1}(z)$ are meromorphic functions, $a_0(z)a_n(z) \not\equiv 0$ and $q \in \mathbb{C}$ such that $0 < |q| \leq 1$. Throughout this paper, we assume that the reader is familiar with the standard notations and the fundamental results of Nevanlinna theory (see [11, 14, 18]). In addition, we use $\rho(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

We firstly recall a result proved by Bergweiler, Ishizaki and Yanagihara in [3].

Theorem A ([3]). *Let $a_0(z), \ldots, a_{n+1}(z)$ be polynomials without common zeros and $0 < |q| < 1$. Suppose (1.1) possesses a transcendental entire solution $f(z)$. Then there is some j, $1 \le j \le n$, such that $\deg a_0(z) < \deg a_j(z)$.*

In a recent paper [5], we gave estimates on the upper bound for the length of the gap in the power series of entire solutions of (1.1). In fact, we proved the following results.

Theorem B ([5]). *Let $0 < |q| < 1$ and $a_j(z) = b_j z^{d_j}$, $j = 0, \ldots, n$ where b_j are constants, d_j are nonnegative integer numbers such that $b_0 b_n \ne 0$, and let $a_{n+1}(z)$ be a polynomial. Suppose that (1.1) has a transcendental entire solution*

$$(1.2) \qquad f(z) = \sum_{k=0}^{\infty} \alpha_{v_k} z^{v_k} \quad (\alpha_{v_k} \ne 0 \text{ are constants}).$$

Then there is some $k_0 > 0$ such that for any $k > k_0$,

$$v_{k+1} - v_k \le \max_{0 \le i < j \le n} |d_j - d_i|.$$

Theorem C ([5]). *Let $0 < |q| < 1$ and $a_0(z), \ldots, a_{n+1}(z)$ be polynomials. Suppose that (1.1) has a transcendental entire solution denoted by (1.2). Then there is some $k_0 > 0$ such that for any $k > k_0$,*

$$v_{k+1} - v_k \le \max\{\deg a_j : j = 0, \ldots, n\}.$$

In what follows, we say $f(z) = \sum_{n=0}^{\infty} \alpha_{\lambda_n} z^{\lambda_n}$ ($\alpha_{\lambda_n} \ne 0$ are constants) has a Fabry gap (see [12]), if

$$\frac{\lambda_n}{n} \to \infty \qquad (n \to \infty).$$

Corollary D ([5]). *Let $0 < |q| < 1$ and $a_0(z), \ldots, a_{n+1}(z)$ be polynomials. Suppose that (1.1) possesses a transcendental entire solution $f(z)$. Then $f(z)$ must not have a Fabry gap.*

We improve Theorem B for the case $n = 2$ here by proving the following result.

Theorem 1.1. *Let $0 < |q| < 1$ and $a_j(z) = b_j z^{d_j}$, $j = 0, 1, 2$, where b_j are constants, d_j are nonnegative integer numbers such that $b_0 b_2 \ne 0$, and let $a_3(z)$ be a polynomial such that $a_3(0) \ne 0$. Suppose that*

$$(1.3) \qquad a_2(z) f(q^2 z) + a_1(z) f(qz) + a_0(z) f(z) = a_3(z)$$

has a transcendental entire solution denoted by (1.2). Then there is some $k_0 > 0$ such that for any $k > k_0$,

$$v_{k+1} - v_k \le \max_{j=1,2} |d_j - d_0|.$$

Next, we recall some results in the case that $0 < |q| < 1$ and the coefficients $a_0(z), \ldots, a_{n+1}(z)$ are not all rational functions. Theorem E below follows from Theorem 3.5 and Corollary 3.6 in [13].

Theorem E ([13]). *Suppose that the coefficients $a_0(z), \ldots, a_{n+1}(z)$ in (1.1) are meromorphic and of finite order $\leq \rho$ and $0 < |q| < 1$. Then all meromorphic solutions of (1.1) are of finite order $\leq \rho$. In addition, if $\rho(a_{n+1}) > \rho(a_j)$ for all $j = 0, \ldots, n$, then $\rho(f) = \rho(a_{n+1})$.*

The following two special cases of the coefficients $a_0(z), \ldots, a_{n+1}(z)$ were investigated in [5].

Case 1: $a_j(z) = P_j(z)e^{p_j z} + Q_j(z)$, $j = 0, \ldots, n+1$, where $p_j = d_j e^{i\theta}$, $\theta \in [0, 2\pi)$, $P_j(z)$, $Q_j(z)$ are meromorphic functions with $\max\{\rho(P_j), \rho(Q_j)\} < 1$ and there is some $s \in \{0, \ldots, n+1\}$ such that $d_s > \max\{d_j : j \neq s, 0 \leq j \leq n+1\}$;

Case 2: $a_j(z) = P_j(z)e^{p_j z} + Q_j(z)e^{-q_j z} + C_j(z)$, $j = 0, \ldots, n+1$, where p_j, $q_j \geq 0$, $P_j(z)$, $Q_j(z)$, $C_j(z)$ are meromorphic functions with $\max\{\rho(P_j), \rho(Q_j), \rho(C_j)\} < 1$ and there is some $s \in \{0, \ldots, n+1\}$ such that $P_s(z)Q_s(z) \not\equiv 0$, $p_s q_s \neq 0$ while $P_0(z) = \cdots = P_{s-1}(z) = P_{s+1}(z) = \cdots = P_{n+1}(z) \equiv 0$, or $Q_0(z) = \cdots = Q_{s-1}(z) = Q_{s+1}(z) = \cdots = Q_{n+1}(z) \equiv 0$.

Theorem F ([5]). *If $0 < |q| < 1$ and the coefficients of (1.1) satisfy any one of Cases 1 and 2 listed above, then each non-zero meromorphic solution of (1.1) is of order $\rho(f) = 1$.*

Some cases similar to Case 1 had been investigated in [13] and some meaningful results were proved. We find that there always exists some dominating coefficient in all these considerations. As a continuation, we give the following result which is a generalization of Theorem F.

Theorem 1.2. *Suppose that the coefficients $a_0(z), \ldots, a_{n+1}(z)$ in (1.1) are meromorphic and of finite order $\leq \rho$ and $0 < |q| < 1$. If for any given $\varepsilon > 0$, there exist some l, $0 \leq l \leq n+1$, and an unbounded domain $D \subset \mathbb{C}$, such that for all $z \in D$,*

$$|a_l(z)| \geq \exp\{\alpha r^{\rho - \varepsilon}\},$$

$$|a_j(z)| \leq \exp\{\beta r^{\rho - \varepsilon}\}, \quad j = 0, 1, \ldots, l-1, l+1, \ldots, n+1,$$

where $\alpha > \beta > 0$ are positive real numbers, then each non-zero meromorphic solution of (1.1) is of order $\rho(f) = \rho$.

To describe the growth of entire solutions more precisely, we consider its type and prove the following results.

Theorem 1.3. *Let $0 < |q| < 1$ and the coefficients $a_0(z), \ldots, a_{n+1}(z)$ in (1.1) be entire and of finite order $\leq \rho$ such that among those coefficients having the maximal order $\rho := \max_{0 \leq j \leq n+1} \rho(a_j)$, exactly one denoted by $a_l(z)$ has its type strictly greater than the others, then each non-zero entire solution of (1.1) is of order $\rho(f) = \rho$. Moreover, if $l = n+1$, then we have $\tau(f) \geq \tau(a_{n+1}) - \max\{\tau(a_j) : \rho(a_j) = \rho, j \neq n+1\}$.*

Corollary 1.4. *Under the assumptions of Theorem 1.3, if $\tau(a_{n+1}) = \infty$, then each non-zero entire solution of (1.1) is of order $\rho(f) = \rho$ and of type $\tau(f) = \infty$.*

Example. $f(z) = e^z$ satisfies

$$e^{\frac{1}{4}z} f(\frac{1}{4}z) - f(\frac{1}{2}z) + e^{\frac{1}{2}z} f(z) = e^{\frac{3}{2}z}.$$

This is an example of Theorem 1.3, where $q = \frac{1}{2}$ and $n = 2$. It shows that $\tau(f) = \tau(a_{n+1}) - \max\{\tau(a_j) : \rho(a_j) = \rho, j \neq n+1\}$ may hold in Theorem 1.3.

We prove Theorem 1.1 and Theorem 1.3 in Section 2 and Section 3 respectively. We omit the proof of Theorem 1.2 as it can be proved with the same idea as in the proof of Theorem 1.3. And we give some results and their proofs for the case $|q| = 1$ in Sections 4–6.

2. Proof of Theorem 1.1

Firstly, by (1.2), for each $j \in \{0, 1, 2\}$, we have

$$a_j(z) f(q^j z) = a_j(z) \sum_{k=0}^{\infty} \alpha_{v_k} (q^j z)^{v_k} = b_j z^{d_j} \sum_{k=0}^{\infty} \alpha_{v_k} (q^j z)^{v_k}$$

$$= b_j \alpha_{v_0} q^{j v_0} z^{v_0 + d_j} + b_j \alpha_{v_1} q^{j v_1} z^{v_1 + d_j} + \cdots$$

$$+ \mathbf{b_j} \alpha_{\mathbf{v_{(q_k-1)}}} \mathbf{q^{j v_{(q_k-1)}}} \mathbf{z^{v_{(q_k-1)}+d_j}}$$

$$(2.1) \qquad + \mathbf{b_j} \alpha_{\mathbf{v_{q_k}}} \mathbf{q^{j v_{q_k}}} \mathbf{z^{v_{q_k}+d_j}} + \mathbf{b_j} \alpha_{\mathbf{v_{(q_k+1)}}} \mathbf{q^{j v_{(q_k+1)}}} \mathbf{z^{v_{(q_k+1)}+d_j}} + \cdots .$$

Since $a_3(0) \neq 0$, we can see that $a_j(z), j = 0, 1, 2, 3$, do not have common zeros. By Theorem A, we have $\max\{d_1, d_2\} > d_0$. What's more, from Theorem B, we see that our conclusion holds for the cases: **Case 1.** $d_2 = d_1 > d_0$, **Case 2.** $d_2 > d_1 \geq d_0$, and **Case 3.** $d_1 > d_2 \geq d_0$. Thus we only need to discuss two cases: **Case 4.** $d_2 > d_0 > d_1$ and **Case 5.** $d_1 > d_0 > d_2$.

Case 4. $d_2 > d_0 > d_1$. Suppose that the assertion does not hold, by Theorem B, then for any given $k > 0$, there is some $q_k > k$ such that

$$\max\{d_2 - d_0, d_0 - d_1\} < v_{(q_k+1)} - v_{q_k} \leq d_2 - d_1.$$

If $d_2 - d_0 = \max\{d_2 - d_0, d_0 - d_1\}$, then we can choose an infinitely sequence $\{v_{q_k}\} \subset \{v_k\}$, such that for any q_k,

$$(2.2) \qquad v_{(q_k+1)} + d_1 \leq v_{q_k} + d_2 < v_{(q_k+1)} + d_0.$$

If $v_{(q_k+1)} + d_1 = v_{q_k} + d_2$, then we have

$$v_{q_k} + d_1 < v_{q_k} + d_0 < v_{q_k} + d_2$$

$$(2.3) \qquad = v_{(q_k+1)} + d_1 < v_{(q_k+1)} + d_0 < v_{(q_k+1)} + d_2.$$

Substituting (2.1) into (1.3), then arranging the power series of $a_2(z) f(q^2 z) + a_1(z) f(qz) + a_0(z) f(z)$ anew in accordance with the monotone nondecreasing degree of z, we get that

$$a_3(z) = a_2(z) f(q^2 z) + a_1(z) f(qz) + a_0(z) f(z)$$

$$(2.4) \qquad = \cdots + \mathbf{b_0} \alpha_{\mathbf{v_{q_k}}} \mathbf{z^{v_{q_k}+d_0}} + \cdots .$$

We notice the boldface terms in (2.1) for $j = 0, 1, 2$. By (2.2) and (2.3), we see that in (2.4), there is exactly one term $b_0 \alpha_{v_{q_k}} z^{v_{q_k}+d_0}$ has degree $v_{q_k} + d_0$.

For sufficiently large q_k, $v_{q_k} + d_0 > \deg a_3$, by (2.4), we see that the coefficient of $z^{v_{q_k}+d_0}$ equals 0. That is $b_0 \alpha_{v_{q_k}} = 0$, a contradiction.

Thus we have $v_{(q_k+1)} + d_1 < v_{q_k} + d_2$. Since $d_2 - d_0 = \max\{d_2 - d_0, d_0 - d_1\} \geq d_0 - d_1$, and by (2.2), we obtain

$$v_{(q_k+1)} + d_1 = v_{(q_k+1)} - v_{q_k} + v_{q_k} + d_1 > v_{q_k} + d_2 - d_0 + d_1$$
$$= v_{q_k} + d_0 + (d_1 + d_2 - 2d_0) \geq v_{q_k} + d_0.$$

Then we have

$$v_{q_k} + d_1 < v_{q_k} + d_0 < v_{(q_k+1)} + d_1 < v_{q_k} + d_2 < v_{(q_k+1)} + d_0.$$

Then we can deduce the same contradiction that $b_0 \alpha_{v_{q_k}} = 0$ by an argument similar to the above.

If $d_0 - d_1 = \max\{d_2 - d_0, d_0 - d_1\}$, we can deduce the same contradiction without difficulty. This completes the proof of Case 4.

Case 5. $d_1 > d_0 > d_2$. By reasoning similar to that in the proof of Case 4, we can prove Case 5 and finish our proof of Theorem 1.1.

3. Proof of Theorem 1.3

Lemma 3.1 ([15]). *Let $f(z)$ be a meromorphic function with $\rho(f) = \beta < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ with finite measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, and r sufficiently large,*

$$\exp\{-r^{\beta+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\beta+\varepsilon}\}.$$

Lemma 3.2 ([16]). *Let $f(z)$ be a nonconstant entire function with $\rho(f) = \rho < \infty$, $0 < \tau(f) = \tau < \infty$. Then for any given $\beta < \tau$, there exists a set $E \subset (1, \infty)$ with infinite logarithmic measure, such that for all z satisfying $|z| = r \in E$, we have,*

$$\log M(r, f) > \beta r^\rho.$$

Proof of Theorem 1.3. Let $f(z)$ be a non-zero entire solution of (1.1). By Theorem E, we have $\rho(f) \leq \rho$. We next proceed to show that $\rho(f) = \rho$. Suppose that $\rho(f) = \sigma < \rho$, then $\rho(\frac{1}{f(q^j z)}) = \rho(f(q^j z)) = \sigma$ for $j = 0, \ldots, n$.

Denote $I := \{j \in \{0, \ldots, n+1\} | \rho(a_j) = \rho\}$, $\tau := \max\{\tau(a_j) : j \in I \setminus \{l\}\}$, and $\tilde{\rho} := \max\{\rho(a_j) : j \in \{0, \ldots n+1\} \setminus I\} < \rho$. From Lemma 3.1, for any given ε_1 ($0 < 2\varepsilon_1 < \min\{\rho - \sigma, \rho - \tilde{\rho}\}$), there exists a set $E_1 \subset (1, +\infty)$ with finite measure, such that for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_1$, and r sufficiently large, we have

$$(3.1) \qquad \left| \frac{1}{f(q^l z)} \right| \leq \exp\{(|q|^l r)^{\sigma+\varepsilon_1}\} \leq \exp\{r^{\sigma+\varepsilon_1}\},$$

$$(3.2) \qquad |f(q^j z)| \leq \exp\{(|q|^j r)^{\sigma+\varepsilon_1}\} \leq \exp\{r^{\sigma+\varepsilon_1}\}$$

for each $j \in \{0, \ldots, n+1\} \setminus \{l\}$, and

$$(3.3) \qquad |a_j(z)| \leq \exp\{r^{\tilde{\rho}+\varepsilon_1}\} \leq \exp\{r^{\rho-\varepsilon_1}\}$$

for each $j \in \{0, \ldots, n+1\} \setminus I$.

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be two constants such that $\tau < \alpha_1 < \alpha_2 < \tau(a_l)$. By Lemma 3.2 and the definition of the type of an entire function, we see that there exists a set $E_2 \subset (1, +\infty)$ with infinite logarithmic measure, such that for all z satisfying $|z| = r \in E_2$, and r sufficiently large, we have

$$(3.4) \qquad M(r, a_l(z)) > \exp\{\alpha_2 r^\rho\}, \text{ and}$$

$$(3.5) \qquad M(r, a_j(z)) < \exp\{\alpha_1 r^\rho\}, \qquad j \in I \setminus \{l\}.$$

If $l \neq n+1$, combining these inequalities (3.1)-(3.5) with (1.1), we get

$$
\begin{aligned}
&|a_l(z)| \\
&= \left|\frac{1}{f(q^l z)}\right| \left|-\sum_{j \in I \setminus \{l, n+1\}} a_j(z) f(q^j z) - \sum_{j \in \{0,\ldots,n\} \setminus I} a_j(z) f(q^j z) + a_{n+1}(z)\right| \\
&\leq \left|\frac{1}{f(q^l z)}\right| \left(\sum_{j \in I \setminus \{l, n+1\}} M(r, a_j(z)) |f(q^j z)| + \sum_{j \in \{0,\ldots,n\} \setminus I} |a_j(z)| |f(q^j z)|\right) \\
&\quad + \left|\frac{1}{f(q^l z)}\right| |a_{n+1}(z)| \\
&\leq (n+1) \exp\{\alpha_1 r^\rho + 2r^{\sigma+\varepsilon_1}\}
\end{aligned}
$$

for all $z = re^{i\theta}$ satisfying $|z| = r \in E_2 \setminus E_1$, and $M(r, a_l(z)) = |a_l(z)|$. This then implies that

$$\exp\{(\alpha_2 - \alpha_1) r^\rho - 2r^{\sigma+\varepsilon_1}\} \leq n+1,$$

holds for sufficiently large $r \in E_2 \setminus E_1$, which is a contradiction.

If $l = n+1$, we can use the same method to deduce a similar contradiction. The first assertion now follows immediately.

We next show that $\tau(f) \geq \tau(a_{n+1}) - \tau$ when $l = n+1$. Assume contrary to the assertion that $\tau(f) < \tau(a_{n+1}) - \tau$.

From Lemma 3.1, for any given ε_2 $(0 < 4\varepsilon_2 < \min\{\tau(a_{n+1}) - \tau - \tau(f), \rho - \tilde{\rho}\})$, there exists a set $E_3 \subset (1, +\infty)$ with finite measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, and r sufficiently large, we have

$$(3.6) \qquad |a_j(z)| \leq \exp\{r^{\tilde{\rho}+\varepsilon_2}\} \leq \exp\{r^{\rho-\varepsilon_2}\} \qquad j \in \{0, \ldots, n\} \setminus I.$$

By Lemma 3.2 and again the definition of the type of an entire function, we see that there exists a set $E_4 \subset (1, +\infty)$ with infinite logarithmic measure, such that for all $z = re^{i\theta}$ satisfying $|z| = r \in E_4$, $|a_{n+1}(re^{i\theta})| = M(r, a_{n+1})$ and r sufficiently large, we have

$$(3.7) \qquad M(r, a_{n+1}(z)) > \exp\{(\tau(a_{n+1}) - \varepsilon_2) r^\rho\},$$

(3.8) $\qquad M(r, f(q^j z)) < \exp\{(\tau(f) + \varepsilon_2)(|q|^j r)^\rho\} \leq \exp\{(\tau(f) + \varepsilon_2)r^\rho\}$

for $j = 0, \ldots, n$, and

(3.9) $\qquad M(r, a_j(z)) < \exp\{(\tau + \varepsilon_2)r^\rho\}, \qquad j \in I \setminus \{n+1\}.$

Combining inequalities (3.6)-(3.9) with (1.1), we get

$$
\begin{aligned}
|a_{n+1}(z)| &= \left| \sum_{j \in I \setminus \{n+1\}} a_j(z)f(q^j z) + \sum_{j \in \{0,\ldots,n\} \setminus I} a_j(z)f(q^j z) \right| \\
&\leq \sum_{j \in I \setminus \{n+1\}} M(r, a_j(z))M(r, f(q^j z)) + \sum_{j \in \{0,\ldots,n\} \setminus I} |a_j(z)|M(r, f(q^j z)) \\
&\leq (n+1) \exp\{(\tau + \tau(f) + 2\varepsilon_2)r^\rho\}
\end{aligned}
$$

for all $z = re^{i\theta}$ satisfying $|z| = r \in E_4 \setminus E_3$, and $M(r, a_{n+1}(z)) = |a_{n+1}(z)|$. This then implies that

$$
\exp\{[\tau(a_{n+1}) - \tau - \tau(f) - 3\varepsilon_2]r^\rho\} \leq n+1,
$$

holds for sufficiently large $r \in E_4 \setminus E_3$, which is a contradiction. Hence we have proven that $\tau(f) \geq \tau(a_{n+1}) - \tau$. $\qquad\square$

4. The case of $|q| = 1$

In the previous sections, we assumed that q was a non-zero complex constant with $0 < |q| < 1$. The following mainly deals with the corresponding results if $|q| = 1$. Moreover, a further result about the solutions of (1.1) is also given.

Theorem G ([5])**.** *Suppose $a_0(z), \ldots, a_{n+1}(z)$ are polynomials and q is a constant with $|q| = 1$ and $q \neq 1$. If (1.1) possesses a nonconstant meromorphic solution $f(z)$, then*
(i) *if $f(z)$ is a rational function of the form*

$$
f(z) = \frac{p_0 + p_1 z + \cdots + p_m z^m}{q_0 + q_1 z + \cdots + q_t z^t},
$$

then $t - m \leq \max\{\deg a_j : j = 0, \ldots, n\} - \deg a_{n+1}$.
(ii) *if there is some $l \in \{0, \ldots, n\}$, such that*

$$
\deg a_l > \max_{j \in \{0,\ldots,n\} \setminus \{l\}} \{\deg a_j\},
$$

and $f(z)$ is a transcendental function, then $f(z)$ has infinitely many poles.

Now we get the Theorem 4.1 as a continuation of Theorem 5.1 in [13] below.

Theorem 4.1. *Suppose the coefficients $a_0(z), \ldots, a_{n+1}(z)$ of (1.1) are meromorphic, $a_0(z)a_n(z) \not\equiv 0$, and $q = e^{i\theta}, \theta = \lambda\pi, \lambda \in (0, 2) \setminus \mathbb{Q}$. If (1.1) possesses a nonconstant meromorphic solution $f(z)$, then the non-zero poles of $f(z)$ are of the form $\{\frac{z}{q^k} | a_0(z) = 0 \text{ or } a_j(z) = \infty, j = 1, \ldots, n+1, k \in \mathbb{N} \cup \{0\}\}$.*

From Theorem G and Theorem 4.1, the following consequence obviously holds.

Corollary 4.2. *Suppose the coefficients $a_0(z), \ldots, a_{n+1}(z)$ of (1.1) are polynomials, $a_0(z)a_n(z) \not\equiv 0$, and $q = e^{i\theta}, \theta = \lambda\pi, \ \lambda \in (0, 2) \setminus \mathbb{Q}$. If (1.1) possesses a nonconstant meromorphic solution $f(z)$, then $f(z)$ has at most finitely many non-zero poles, which are of the form $\{\frac{z}{q^k} | \ a_0(z) = 0, \ k \in \mathbb{N} \cup \{0\}\}$. Moreover, if there is some $l, \ 0 \le l \le n$, such that*

$$\deg a_l > \max_{j \in \{0, \ldots, n\} \setminus \{l\}} \{\deg a_j\},$$

then $f(z)$ must be a rational function.

We give some examples to show that there exist some equations of the form (1.1), which satisfy Theorem 4.1 or Corollary 4.2.

Examples. (1) $f(z) = \frac{e^z}{(z-1)(qz-1)}$ is a meromorphic function and satisfies the equation

$$a_1(z)f(qz) + a_0(z)f(z) = a_2(z),$$

where

$$q = e^{i\sqrt{2}\pi}, \quad a_0(z) = -(z-1)e^{qz}, \quad a_1(z) = (q^2z-1)(e^z+1), \quad a_2(z) = \frac{e^{qz}}{qz-1};$$

(2) $f(z) = \frac{1}{z+1}$ is a rational function and satisfies the equation

$$a_1(z)f(qz) + a_0(z)f(z) = a_2(z),$$

where

$$q = e^{i\sqrt{2}\pi}, \quad a_0(z) = z^2(z+1), \quad a_1(z) = qz+1, \quad a_2(z) = z^2+1;$$

(3) $f(z) = z+1$ is a polynomial and satisfies the equation

$$a_2(z)f(q^2z) + a_1(z)f(qz) + a_0(z)f(z) = a_3(z),$$

where

$$q = e^{i\sqrt{2}\pi}, \quad a_0(z) = 1-z, \quad a_1(z) = qz-1, \quad a_2(z) = z^2, \quad a_3(z) = q^2z^2(z+1).$$

In the theorems above, we investigated the poles of meromorphic solutions of (1.1). Now we give a result on the zeros of meromorphic solutions of (1.1).

Theorem 4.3. *Suppose the coefficients $a_0(z), \ldots, a_{n+1}(z)$ of (1.1) are polynomials, $a_0(z)a_n(z) \not\equiv 0$, and $q = e^{i\theta}, \theta = \lambda\pi, \ \lambda \in (0, 2) \setminus \mathbb{Q}$. If (1.1) possesses a nonconstant finite order meromorphic solution $f(z)$, then $\lambda(f) = \rho(f)$.*

As counterparts of Theorem 1.2, Theorem 1.3 and Corollary 1.4, we give the following results. Proofs of Theorem 4.4 and Theorem 4.5 are similar to the proof of Theorem 1.3 and thus omitted.

Theorem 4.4. *Suppose that the coefficients $a_0(z), \ldots, a_{n+1}(z)$ in (1.1) are meromorphic and of finite order $\le \rho$ and $|q| = 1$. If for any given $\varepsilon > 0$, there exist some $l, \ 0 \le l \le n+1$, and an unbounded domain $D \subset \mathbb{C}$, such that for all $z \in D$,*

$$|a_l(z)| \ge \exp\{\alpha r^{\rho-\varepsilon}\},$$

and

$$|a_j(z)| \leq \exp\{\beta r^{\rho-\varepsilon}\}, \quad j = 0, 1, \ldots, l-1, l+1, \ldots, n+1,$$

where $\alpha > \beta > 0$ are positive real numbers, then each non-zero meromorphic solution of (1.1) is of order $\rho(f) \geq \rho$.

Theorem 4.5. *Let $|q| = 1$ and the coefficients $a_0(z), \ldots, a_{n+1}(z)$ in (1.1) be entire and of finite order $\leq \rho$ such that among those coefficients having the maximal order $\rho := \max_{0 \leq j \leq n+1} \rho(a_j)$, exactly one denoted by $a_l(z)$ has its type strictly greater than the others. Then each non-zero entire solution of (1.1) is of order $\rho(f) \geq \rho$. Moreover, if $l = n+1$ and $\rho(f) = \rho$, then we have $\tau(f) \geq \tau(a_{n+1}) - \max\{\tau(a_j) : \rho(a_j) = \rho, j \neq n+1\}$.*

Corollary 4.6. *Under the assumptions of Theorem 4.5, if $\tau(a_{n+1}) = \infty$, then each non-zero entire solution of (1.1) satisfying $\rho(f) = \rho$ is of type $\tau(f) = \infty$.*

We recall the following examples given in [5] to show that the growth estimate of meromorphic solutions of (1.1) in Theorems 4.5 is sharp, where examples (1) and (3) show that $\rho(f) > \rho$ may hold in Theorems 4.5. What's more, Example (2) shows that the type estimate is also sharp.

Examples. (1) $e^z f(qz) - e^{3z} f(z) = ze^{4z} + z, \qquad q = -1, \ f(z) = e^{z^2 - z} - ze^z$;
(2) $e^z f(qz) + (e^z - e^{-z})f(z) = e^{2z}, \qquad q = -1, \ f(z) = e^z$;
(3) $e^z f(q^2 z) + e^z f(qz) - (e^z + e^{-z})f(z) = 0, \qquad q = -1, \ f(z) = e^{z^2 + z}$.

5. Proof of Theorem 4.1

Set $B = \{z| \ a_j(z) = \infty, \ j = 1, \ldots, n+1\}$. Suppose that there exists a pole z_0 of $f(z)$ such that for any $b_s \in B$, z_0 is not of the form $\frac{b_s}{q^k}$ (for any $k \in \mathbb{N} \cup \{0\}$). That is

$$(5.1) \qquad a_j(q^k z_0) \neq \infty, \ j = 1, \ldots, n+1, \ \forall \ k \in \mathbb{N} \cup \{0\}.$$

As $q = e^{i\theta}, \theta = \lambda\pi, \ \lambda \in (0,2) \setminus \mathbb{Q}$, we see that for any $k, \ l \in \mathbb{N} \cup \{0\}, \ k \neq l$,

$$(k-l)\theta \not\equiv 0 \pmod{2\pi},$$

which yields that

$$q^k \neq q^l.$$

Since $f(z)$ is a meromorphic function, we see that $f(z)$ has at most finitely many poles on the circle $|z| = |z_0|$. Then $f(z)$ has at most finitely many poles which are of the form $q^{k_m} z_0, \ m = 0, \ldots, t$. such that $0 \leq k_0 \leq k_1 \leq \cdots \leq k_t$. That is

$$(5.2) \qquad f(q^{k_t} z_0) = \infty, \ f(q^{k_t + j} z_0) \neq \infty, \ j = 1, \ldots, n+1.$$

From (1.1), we have

$$(5.3) \qquad a_0(q^{k_t} z_0)f(q^{k_t} z_0) = -\sum_{j=1}^{n} a_j(q^{k_t} z_0)f(q^{k_t + j} z_0) + a_{n+1}(q^{k_t} z_0).$$

Then by (5.1)-(5.3), we conclude that $a_0(q^{k_t}z_0) = 0$, which means that, there exists a point z^* such that

$$a_0(z^*) = 0, \ z_0 = \frac{z^*}{q^{k_t}}.$$

Thus Theorem 4.1 is proved.

6. Proof of Theorem 4.3

Lemma 6.1 ([17]). *Let $f_j(z)$ $(j = 1, \ldots, n, \ n \geq 2)$ be meromorphic functions, $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions, and satisfy*
  (i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
  (ii) *when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not a constant; and*
  (iii) *when $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g^h - g^k})\}, \ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Proof of Theorem 4.3. Assume contrary to the assertion that $\lambda(f) < \rho(f)$. By Corollary 4.2, we see that $f(z)$ has at most finitely many non-zero poles, then $\lambda(\frac{1}{f}) = 0 < \rho(f)$. Thus, since $f(z)$ is a meromorphic function of finite order, then we can set

(6.1) $f(z) = P(z)e^{p(z)},$

where $P(z)$ is a meromorphic function such that $\rho(P) < \rho(f)$, and $p(z)$ is a polynomial such that $d = \deg(p) = \rho(f)$. Now set

$$p(z) = \alpha_d z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_0,$$

where $\alpha_d \neq 0, \alpha_{d-1}, \ldots, \alpha_0$ are constants.
  Substituting (6.1) into (1.1), we get

(6.2) $\sum_{j=0}^{n} a_j(z)P(q^j z)e^{p(q^j z)} - a_{n+1}(z) = 0.$

As $q = e^{i\theta}, \theta = \lambda\pi, \ \lambda \in (0, 2)\backslash\mathbb{Q}$, we see that for any $0 \leq k < l \leq n$, $q^k \neq q^l$. Clearly,

$$p(q^k z) - p(q^l z) = \alpha_d(q^{kd} - q^{ld})z^d + \cdots + \alpha_1(q^k - q^l)z,$$

and hence $\deg(p(q^k z) - p(q^l z)) = d$.
  Since $a_0(z), \ldots, a_{n+1}(z)$ of (1.1) are polynomials and by $\rho(P) < d$, we have, for $0 \leq j \leq n, \ 0 \leq k < l \leq n, \ 0 \leq h \leq n$,

$$T(r, a_j(z)P(q^j z)) = o\{T(r, e^{p(q^k z) - p(q^l z)})\} = o\{T(r, e^{p(q^h z)})\}, \text{ and}$$

$$T(r, a_{n+1}(z)) = o\{T(r, e^{p(q^k z) - p(q^l z)})\} = o\{T(r, e^{p(q^h z)})\}.$$

Applying Lemma 6.1 to (6.2) yields

$$a_j(z) \equiv 0, \ j = 0, \dots, n+1.$$

This is a contradiction. Thus Theorem 4.3 is proved.   □

References

[1] M. Ablowitz, R. G. Halburd, and B. Herbst, *On the extension of the Painlevé property to difference equations*, Nonlinearity **13** (2000), no. 3, 889–905.

[2] W. Bergweiler, K. Ishizaki, and N. Yanagihara, *Meromorphic solutions of some functional equations*, Methods Appl. Anal. **5** (1998), no. 3, 248–258.

[3] _____, *Growth of meromorphic solutions of some functional equations I*, Aequationes Math. **63** (2002), no. 1-2, 140–151.

[4] W. Bergweiler and J. K. Langley, *Zeros of differences of meromorphic functions*, Math. Proc. Cambridge Philos. Soc. **142** (2007), no. 1, 133–147.

[5] B. Q. Chen, Z. X. Chen, and S. Li, *Properties on solutions of some q-difference equations*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 10, 1877–1886.

[6] Z. X. Chen and K. H. Shon, *On zeros and fixed points of differences of meromorphic functions*, J. Math. Anal. Appl. **344** (2008), no. 1, 373–383.

[7] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in thecomplex plane*, Ramanujan J. **16** (2008), no. 1, 105–129.

[8] G. Gundersen, J. Heittokangas, I. Laine, J. Rieppo, and D. G. Yang, *Meromorphic solutions of generalized Schröder equations*, Aequationes Math. **63** (2002), no. 1-2, 110–135.

[9] R. G. Halburd and R. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), no. 2, 477–487.

[10] _____, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 2, 463–478.

[11] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[12] _____, *Angular value distribution of power series with gaps*, Proc. Lond. Math. Soc. (3) **24** (1972), no. 4, 590–624.

[13] J. Heittokangas, I. Laine, J. Rieppo, and D. G. Yang, *Meromorphic solutions of some linear functional equations*, Aequationes Math. **60** (2000), no. 1-2, 148–166.

[14] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[15] S. Saks and A. Zygmund, *Analytic Functions*, Monografie Mat. (Engl. Transl.) Tom 28, Warsaw, 1952.

[16] J. Tu and C. F. Yi, *On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order*, J. Math. Anal. Appl. **340** (2008), no.1, 487–497.

[17] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Math. Appl., vol. 557, Kluwer Academic Publishers Group. Dordrecht, 2003.

[18] L. Yang, *Value Distribution Theory and New Research*, Science Press, Beijing, 1982.

BaoQin Chen
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631, P. R. China
E-mail address: chenbaoqin_chbq@126.com

ZongXuan Chen
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631, P. R. China
E-mail address: chzx@vip.sina.com