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LOCAL WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL IN H^s

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Abstract. We establish the local well-posedness for the Cauchy problem of the nonlinear Schrödinger equation with harmonic potential in $H^s(\mathbb{R}^n)$, where $s \in \mathbb{R},\ s > 0$.

1. Introduction and preliminaries

In this paper we study the Cauchy problem for the nonlinear Schrödinger equation with harmonic potential

$$i\partial_t u + \frac{1}{2}\triangle u = \frac{1}{2}\omega^2|x|^2u + \lambda|u|^\alpha u,\ u(0,x) = \phi(x). \tag{1.1}$$

Here $u = u(t,x)$ is a complex valued function defined on $[0,T) \times \mathbb{R}^n$ for some $T > 0, \lambda$ is a real number, $\omega > 0, \alpha > 0$. The initial condition ϕ is a complex valued function defined on $\mathbb{R}^n$. This model has applications in many problems, especially in Bose-Einstein condensates (BECs). By Duhamel's formula [5], (1.1) is equivalent to the integral equation below

$$u(t) = S(t)\phi - i\lambda\int_0^t S(t-\tau)[|u(\tau)|^\alpha u(\tau)]d\tau, \tag{1.2}$$

where $S(t)$ is the unitary group $e^{\frac{it}{2}(\triangle+\omega^2|x|^2)}$ determined by the linear Schrödinger equation, i.e., when $\lambda = 0$.

For the Cauchy problem (1.1) or the integral equation (1.2), Oh [7] show the local well-posedness in $H = \{u \in H^1(\mathbb{R}^n), xu \in L^2(\mathbb{R}^n)\}$: Let $\phi \in H$, then there exists a solution u of the Cauchy problem (1.1) in $C([0,T);H)$ for some $T \in [0,\infty)$, and $T = \infty$ or $T < \infty$ and $\lim_{t\to T}\|\nabla u(t)\|_{L^2} = \infty$. Furthermore

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$u(t,x)$ satisfies the following two conservations

$$\|u(t)\|_{L^2} = \|\phi\|_{L^2} \tag{1.3}$$

and

$$E(u(t)) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\omega^2\|xu\|_{L^2}^2 + \frac{2\lambda}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2} = E(\phi). \tag{1.4}$$

By Carles [3], when $\alpha < 4/n$, the solution of Cauchy problem (1.1) exists globally; when $4/n \leqslant \alpha \leqslant 4/(n-2)$, $(4/n < \alpha < \infty (n = 1, 2))$ there is blow-up solution to exist for the problem (1.1); and when $\alpha > 4/(n-2)$, the integral term in (1.2) seems to be too singular, H^1 theory has been limited.

In [6], authors study the Cauchy problem (1.1) without harmonic potential

$$i\partial_t u + \frac{1}{2}\triangle u = \lambda |u|^\alpha u,\ u(0,x) = \phi(x) \tag{1.5}$$

in $H^s(\mathbb{R}^n)$. They show the local well-posedness for (1.5) as $0 \leqslant s < n/2,\ 0 < \alpha < 4/(n-2s)$.

In this paper, we try to extend the existing theory of (1.1) to include all $\alpha > 0$. We will consider local well-posedness for the Cauchy problem (1.1) in $H^s(\mathbb{R}^n) \cap \{xu \in L^2(\mathbb{R}^n)\}$. As we will see, the existence result follows from a Fixed point argument of Kato and relies heavily on Strichartz estimates corresponded to linear Schrödinger equations with harmonic potential given by Carles in [3, 2] (see Lemma 2.1 below). By following the H^1 argument, we also show the following blow-up condition and blow-up rate of the local solution in $H^s(\mathbb{R}^n) \cap \{xu \in L^2(\mathbb{R}^n)\}$ (see Theorem 3.4 below).

We define a space Σ by

$$\Sigma := \{u \in H^s(\mathbb{R}^n),\ xu \in L^2(\mathbb{R}^n)\}$$

with the inner product

$$(u,v) = (u,v)_{H^s(\mathbb{R}^n)} + |x|^2(u,v)_{L^2(\mathbb{R}^n)}$$

for all $u, v \in \Sigma$. The norm of Σ is denoted by $\|\cdot\|_\Sigma$, thus Σ becomes a Hilbert space, continuously embedded in $H^s(\mathbb{R}^n)$.

Definition 1.1 ([4]). The pair (q,r) is admissible if $2 \leqslant r < 2n/(n-2s)$ and $2/q = n(1/2 - 1/r)$ (If $n = 1$ or 2, $2 \leqslant r < \infty$ is allowed).

Note that if (q,r) is a admissible pair, then $2 < q \leqslant \infty$. The following two conditions describe the relationship between α and s needed for our arguments:

$$0 \leqslant s < n/2, \tag{1.6}$$

$$0 < \alpha \leqslant 4/(n-2s). \tag{1.7}$$

Moreover, since we are working in space of order s differentiability, we need the nonlinear map $f(u) = \lambda|u|^\alpha u$ to have a certain amount of regularity. This will sometimes be expressed by the condition:

$$[s] < \alpha. \tag{1.8}$$

Finally, for α and s just specified, there is a particular admissible pair (γ, ρ), defined by

$$\gamma = \frac{4(\alpha+2)}{\alpha(n-2s)},\ \ \rho = \frac{\alpha+2}{1+\alpha s/n}. \tag{1.9}$$

The rest of this paper is organized as follows. In Section 2, we show $L^q(0,T;\dot{B}^s_{r,b})$ estimates for inhomogenous linear Schrödinger equation and the estimates for the nonlinear map $f(u) = \lambda |u|^\alpha u$ between Besov spaces. In Section 3, we give the main result of this paper. In Section 4, we prove the result given in Section 3.

2. Estimates

In this section, we show the estimates for the linear Schrödinger equation and estimates for the nonlinear term. We consider the homogeneous linear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\triangle u = \frac{1}{2}\omega^2|x|^2 u,\ \ u(0,x) = \phi(x) \tag{2.1}$$

and the inhomogeneous linear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\triangle u = \frac{1}{2}\omega^2|x|^2 u + g,\ \ u(0,x) = \phi(x). \tag{2.2}$$

Where $g = g(t)$. The solution of (2.1) is $u(t) = S(t)\phi = e^{\frac{it}{2}(\triangle + \omega^2|x|^2)}\phi$, the solution of (2.2) is $u(t) = -i\mathcal{G}g(t)$, where

$$\mathcal{G}g(t) = \int_0^t S(t-\tau)g(t)d\tau. \tag{2.3}$$

Lemma 2.1 ([3, 2])**.** *Let (q,r) be any admissible pair, and I be any interval contained in $[0,\pi/2\omega]$. Then it holds that*:

(1) *$S(t)$ is unitary on L^2, i.e., $\|S(t)\|_{L^2\to L^2} = 1$, and we have $S(t)^\star = S(-t)$, where $S(t)^\star$ is the dual operator of $S(t)$.*

(2) *If $0 < t \leqslant \pi/\omega$, then $S(t) : L^1 \to L^\infty$ satisfies $\|S(t)\|_{L^1\to L^\infty} \leqslant 1/|\sin \omega t|^{n/2}$.*

(3) *If $\phi \in L^2$, then $S(\cdot)\phi \in L^q(I;L^r(\mathbb{R}^n))$; there exists a constant C such that*

$$\|S(\cdot)\phi\|_{L^q(I;L^r)} \leqslant C\|\phi\|_{L^2}. \tag{2.4}$$

(4) *If $g \in L^{\gamma'}(I;L^{\rho'}(\mathbb{R}^n))$ for some admissible pair (γ,ρ), then $\mathcal{G}g \in L^q(0,T;$ $L^r(\mathbb{R}^n)) \cap C([0,T];L^2(\mathbb{R}^n))$ for some $T \in (0, \frac{\pi}{2\omega}]$; there exists a constant C such that*

$$\|\int_{I\cap\{\tau\leqslant t\}} S(t-\tau)d\tau\|_{L^q(I;L^r)} \leqslant C\|g\|_{L^{\gamma'}(I;L^{\rho'})}. \tag{2.5}$$

In order to get the estimates with respect to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ and the homogeneous Besove space $\dot{B}^s_{r,2}(\mathbb{R}^n)$, we need following lemma.

Lemma 2.2 ([8], Rizse potential estimates)**.** *Let* $0 < \alpha < n,\ 1 < p < q < \infty,\ 1/q = 1/p - \alpha/n$. *Define Riesz potential by*

$$(-\triangle)^{-\alpha/2}(f) = \frac{1}{\gamma(\alpha)} \int_{R^n} |x-y|^{-n+\alpha} f(y)dy,$$

where $\gamma(\alpha) = \pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)/\Gamma(n/2 - \alpha/2)$. *If* $f \in L^p$, *then*

$$\|(-\triangle)^{-\alpha/2}(f)\|_{L^q} \leqslant C\|f\|_{L^p}.$$

By Lemma 2.2, we can get:

Proposition 2.3. *If* $t \neq 0,\ |t| \leqslant \pi/2\omega$, *then* $S(t) : \dot{B}^s_{r',2} \to \dot{B}^s_{r,2}$ *is a bounded map. Furthermore, there exists a constant* C *such that*

$$\|S(t)\phi\|_{\dot{B}^s_{r,2}} \leqslant C|t|^{-2/q}\|\phi\|_{\dot{B}^s_{r',2}}$$

for every $\phi \in \dot{B}^s_{r',2}(\mathbb{R}^n)$.

Proof. By Lemma 2.1(2), we know that $S(t) :\ L^{r'} \to L^r$ is a bounded map. When $|t| \leqslant \pi/2\omega$, we get $|\sin \omega t| \geqslant \frac{2\omega}{\pi}|t|$, it yields that $\|S(t)\phi\|_{L^r} \leqslant C|t|^{-2/q}\|\phi\|_{L^{r'}}$, where $-\frac{2}{q} = -n(\frac{1}{2} - \frac{1}{r})$. Since $\dot{H}^{s,r}(R^n) = (-\triangle)^{-s/2}L^r(\mathbb{R}^n)$, from Lemma 2.2, we have

$$\begin{aligned}\|S(t)\phi\|_{\dot{H}^{s,r}} = \|(-\triangle)^{-s/2}S(t)\phi\|_{L^r} &\leqslant C\|S(t)\phi\|_{L^{nr/(n+rs)}}\\ &\leqslant C|t|^{-2/q}\|\phi\|_{L^{nr/(rn-n-rs)}}.\end{aligned}$$

By Sobolev embedding theorem:

$$\dot{H}^{s,r'}(\mathbb{R}^n) \hookrightarrow L^{nr'/(n-r's)}(\mathbb{R}^n) = L^{nr/(rn-n-rs)}(\mathbb{R}^n),$$

we get

$$\|S(t)\phi\|_{\dot{H}^{s,r}} \leqslant C|t|^{-2/q}\|\phi\|_{\dot{H}^{s,r'}}.$$

Then by interpolation, the result follows. Indeed, by Theorem 6.3.1 in [1], $(\dot{H}^{s_0,r}, \dot{H}^{s_1,r})_{\theta,2} = \dot{B}^s_{r,2}$, (and likewise with r replaced by r') where $s_0 \neq s_1,\ 0 < \theta < 1,\ s = (1-\theta)s_0 + \theta s_1$, we can get $\|S(t)\phi\|_{\dot{B}^s_{r,2}} \leqslant C|t|^{-2/q}\|\phi\|_{\dot{B}^s_{r',2}}$. □

Using the same method as in [6], we can prove the following proposition.

Proposition 2.4. *Let* $s \in \mathbb{R}$, *and let* (q, r) *be any admissible pair,* $t,\ T \in (0, \pi/2\omega]$.

(1) *If* $\phi \in \dot{H}^s$, *then* $S(\cdot)\phi \in L^q(0, T; \dot{B}^s_{r,2})$; *and there exists a constant* C *such that*

$$\|S(\cdot)\phi\|_{L^q(0,T;\dot{B}^s_{r,2})} \leqslant C\|\phi\|_{\dot{H}^s}. \tag{2.6}$$

(2) *If* $g \in L^{\gamma'}(0, T; \dot{B}^s_{\rho',2})$ *for some admissible pair* (γ, ρ) *and some* $T > 0$ $(T \in (0, \frac{\pi}{2\omega}])$, *then* $\mathcal{G}g \in L^q(0, T; \dot{B}^s_{r,2}) \cap C([0, T]; \dot{H}^s)$; *and there exists a constant* C *such that*

$$\|\mathcal{G}g\|_{L^q(0,T;\dot{B}^s_{r,2})} \leqslant C\|g\|_{L^{\gamma'}(0,T;\dot{B}^s_{\rho',2})}. \tag{2.7}$$

Now we recall the estimates for the nonlinear term $f(u)=\lambda|u|^\alpha u$ by Cazenave and Weissler in [6].

Lemma 2.5 ([6])**.** *Suppose that α and $s>0$ satisfy* (1.6), (1.7); *and also if α is not an even integer,* (1.8) *is satisfied. Let (γ,ρ) be the admissible pair given by* (1.9)*, and define ρ^* by*

$$\rho^* = \frac{1}{\rho} - \frac{s}{n}. \tag{2.8}$$

Let $m=[s]+1$, so that $s<m$, and (*in the case where* (1.8) *is satisfied*) *$m<\alpha+1$. Suppose $f:\mathbb{C}\to\mathbb{C}$ is m times real continuously differentiable, and satisfies $|f^{(k)}(u)|\leqslant C|u|^{\alpha+1-k}$, $0\leqslant k\leqslant m$, where $|f^{(k)}(u)|$ denotes the norm of the real k-linear mapping $f^{(k)}(u)$. If α is an even integer, we suppose also that $f^{(k)}(u)\equiv 0$ in case $\alpha+1<k\leqslant m$. Then f maps the homogeneous Besov space $\dot{B}^s_{\rho,2}(\mathbb{R}^n)$ into the homogeneous Besov space $\dot{B}^s_{\rho',2}(\mathbb{R}^n)$ and satisfies the inequality*

$$\|f(u)\|_{\dot{B}^s_{\rho',2}} \leqslant C\{\|u\|_{\dot{B}^s_{\rho,2}}\}^{\alpha+1}.$$

Proposition 2.6. *Let $u,\ v\in L^\gamma(0,T;B^s_{\rho,2}(\mathbb{R}^n))$. Set $\delta=1-(\alpha+2)/\gamma$. Let (q,r) be any admissible pair. Then*

$$\begin{aligned} &\|\mathcal{G}f(u)-\mathcal{G}f(v)\|_{L^q(0,T;L^r)} \\ &\leqslant CT^\delta\{\|u\|^\alpha_{L^\gamma(0,T;\dot{B}^s_{\rho,2})}+\|v\|^\alpha_{L^\gamma(0,T;\dot{B}^s_{\rho,2})}\}\|u-v\|_{L^\gamma(0,T;L^\rho)} \end{aligned} \tag{2.9}$$

and

$$\|\mathcal{G}f(u)\|_{L^q(0,T;\dot{B}^s_{r,2})} \leqslant CT^\delta\|u\|^{\alpha+1}_{L^\gamma(0,T;\dot{B}^s_{\rho,2})}. \tag{2.10}$$

Remark 2.7. It follows $\gamma=4(\alpha+2)/\alpha(n-2s)$, $\delta=1-(\alpha+2)/\gamma=1-\alpha(n-2s)/4$ that $\delta\geqslant 0$, with $\delta=0$ if and only if $\alpha=4/(n-2s)$.

Proof of Proposition 2.6. We first prove that

$$|f(u)-f(v)|\in L^{\gamma'}(0,T;L^{\rho'}(\mathbb{R}^n)).$$

Since $f(u)=\lambda|u|^\alpha u$, by a direct computation, we have $|f(u)-f(v)|\leqslant C(|u|^\alpha+|v|^\alpha)|u-v|$, so

$$\begin{aligned} \|f(u)-f(v)\|_{L^{\rho'}} &\leqslant \|C(|u|^\alpha+|v|^\alpha)|u-v|\|_{L^{\rho'}} \\ &\leqslant C(\|u\|^\alpha_{L^{\rho^*}}+\|v\|^\alpha_{L^{\rho^*}})\|u-v\|_{L^\rho}. \end{aligned}$$

By Sobolev embedding theorem $\dot{B}^s_{\rho,2}(\mathbb{R}^n)\hookrightarrow L^{\rho^*}(\mathbb{R}^n)$, we can get that

$$\|f(u)-f(v)\|_{L^{\rho'}}\leqslant \|C(|u|^\alpha_{\dot{B}^s_{\rho,2}}+|v|^\alpha_{\dot{B}^s_{\rho,2}})\|u-v\|_{L^\rho}.$$

It yields that

$$\begin{aligned} &\|f(u)-f(v)\|_{L^{\gamma'}(0,T;L^{\rho'})} \\ &\leqslant C(|u|^\alpha_{L^\gamma(0,T;B^s_{\rho,2})}+|v|^\alpha_{L^\gamma(0,T;B^s_{\rho,2})})\|u-v\|_{L^p(0,T;L^\rho)}, \end{aligned} \tag{2.11}$$

where $1/p = [4-\alpha(n-2s)]/4 + 1/\gamma,\ p<\gamma$. Applying Hölder's inequality on $\|u-v\|_{L^p(0,T;L^\rho)}$, we have

$$\|u-v\|_{L^p(0,T;L^\rho)} = \Big(\int_0^T \|u-v\|_{L^\rho} dt\Big)^{1/p} \leqslant T^\delta \|u-v\|_{L^\gamma(0,T;L^\rho)}. \tag{2.12}$$

From Lemma 2.1, (2.11), (2.12) we can get (2.8). (2.9) follows from Lemma 2.5 and Proposition 2.4(2). Indeed,

$$\begin{aligned}\|\mathcal{G}f(u)\|_{L^q(0,T;\dot B^s_{r,2})} &\leqslant C\|f(u)\|_{L^{\gamma'}(0,T;\dot B^s_{\rho',2})} = C(\textstyle\int_0^T \|f(u)\|^{\gamma'}_{\dot B^s_{\rho',2}} dt)^{1/\gamma'}\\ &\leqslant C(\textstyle\int_0^T \|u\|^{(\alpha+1)\gamma'}_{\dot B^s_{\rho,2}} dt)^{1/\gamma'}\\ &\leqslant CT^\delta \|u\|^{\alpha+1}_{L^\gamma(0,T;\dot B^s_{\rho,2})}.\end{aligned}$$

□

3. Main result

Theorem 3.1. *Suppose that α and $s>0$ satisfy* (1.6), (1.7) *and also, if α is not an even integer,* (1.8) *is satisfied. Let (γ,ρ) be the admissible pair given by* (1.9). *Then, for every $\phi\in\Sigma$, there exists a solution $u\in C\big([0,T^*];\Sigma\big)\cap L^\gamma_{loc}\big(0,T^*;B^s_{\rho,2}(\mathbb{R}^n)\big)$ of the integral equation* (1.2) *for some $T^*=T^*(\phi)\in(0,\pi/2\omega]$* (*maximal existence time*). *Moreover T^* satisfies: $T^*=\pi/2\omega$ or else $T^*<\pi/2\omega$ and $\lim_{t\to T^*}\|u(t)\|_{\dot H^s}=\infty$. Furthermore, this solution has the following additional properties:*

(1) *u is unique in $L^\gamma(0,T;B^s_{\rho,2})$ for every $T<T^*$.*

(2) *u satisfies* (1.3) *for every $t<T^*$.*

(3) *If $s\geqslant 1$, then u satisfies* (1.4) *for every $t<T^*$.*

(4) *u depends continuously on ϕ in the following sense. There exists $0<T<T^*(\phi)$ such that if $\{\phi_k\}$ is a sequence in Σ with $\phi_k\to\phi$ in Σ, then, for sufficiently large k, $T<T^*(\phi_k)$ and the solutions u_k* (*of* (1.2) *with ϕ replaced by ϕ_k*) *form a bounded sequence in $L^q\big(0,T;B^s_{r,2}(\mathbb{R}^n)\cap\{xu\in L^2(\mathbb{R}^n)\}\big)$. Moreover, $u_k\to u$ in $L^q\big(0,T;L^r(\mathbb{R}^n)\cap\{xu\in L^2(\mathbb{R}^n)\}\big)$ for every admissible pair (q,r). In particular, $u_k\to u$ in $C\big([0,T];H^{s-\varepsilon}(\mathbb{R}^n)\cap\{xu\in L^2(\mathbb{R}^n)\}\big)$ for every $\varepsilon>0$.*

Remark 3.2. When $s\to n/2$, the upper bound of α tends to $+\infty$, thus the existing theory can be extended to include all $\alpha>0$.

Remark 3.3. Here T^* gets its maximal value $\frac{\pi}{2\omega}$. The standard argument to extend T^* to $+\infty$ needs $\|u(T^*)\|_{\dot H^s}<\infty$, but the conversation laws we got can't afford this.

By following the H^1 argument, we show the following blow-up condition and blow-up rate of the local solution in Σ.

Theorem 3.4. *Suppose $\alpha,\ s$ satisfy the hypotheses of Theorem 3.1, $\phi\in\Sigma$ is a nonzero initial date.*

(1) *Suppose $s\geqslant 1,\ n\geqslant 3,\ \lambda<0,\ 4/n\leqslant\alpha\leqslant 4/(n-2s)$. Let ϕ satisfies*

$$\frac{1}{2}\|\nabla\phi\|^2_{L^2} + \frac{2\lambda}{\alpha+2}\|\phi\|^{\alpha+2}_{L^{\alpha+2}} \leqslant 0,$$

then the solution of integral equation (1.2) *blows-up at* $t^* \leqslant \pi/2\omega$, *i.e.*,

$$\lim_{t\to t^*} \|u(t)\|_{H^s} = \infty.$$

(2) *If* $\alpha < 4/(n-2s)$, $T^* < \pi/2\omega$, *then* $\lim_{t\to t^*} \|u(t)\|_{\dot{H}^s} = \infty$, *and there exists a constant* C *such that*

$$\|u\|_{\dot{H}^s} \geqslant \frac{C}{(T^*-t)^{1/\alpha-(n-2s)/4}}.$$

4. Proof of main results

Throughout this section α and s satisfy the hypotheses of Theorem 3.1, i.e., α and $s>0$ satisfy (1.6), (1.7) and also, if α is not an even integer, (1.8) is satisfied. Moreover (γ,ρ) is the admissible pair given by (1.9) and ρ^* is defined by (2.8). The integral equation (1.2), with the pure power term replaced by a nonlinear term $f(u) = \lambda|u|^\alpha u$, can be written

$$u(t) = S(t)\phi - i\mathcal{G}f(u(t)). \tag{4.1}$$

We mainly prove the existence and uniqueness of the local solution of integral equation (4.1) in Σ with a fixed point argument. Since initial date $\phi \in \Sigma$, $\Sigma \hookrightarrow H^s(\mathbb{R}^n)$ with the injection being continuous, we first prove the existence and uniqueness of the local solution in $H^s(\mathbb{R}^n)$ with initial condition $\phi \in \Sigma \subset H^s(\mathbb{R}^n)$, then we prove this solution belongs to Σ by researching properties of the solution.

Proposition 4.1. *For any* $\phi \in \Sigma$, *there exists* $T>0$ ($T \in (0, \frac{\pi}{2\omega}]$) *and a solution* $u \in L^\gamma(0,T; B^s_{\rho,2}(\mathbb{R}^n))$ *of integral equation* (4.1). *Furthermore* u *belongs to* $C([0,T]; H^s(\mathbb{R}^n)) \cap L^q_{loc}(0,T; B^s_{r,2}(\mathbb{R}^n))$, *and* u *is unique in* $L^\gamma(0,T; B^s_{\rho,2}(\mathbb{R}^n))$.

Proof. Let $M>0$ (finite). We set

$$\mathcal{D} = \mathcal{D}(T,M) = \{u \in L^\gamma(0,T; B^s_{\rho,2}(\mathbb{R}^n)) : \|u\|_{L^\gamma(0,T;\dot{B}^s_{\rho,2})} \leqslant M\}$$

equipped with the distance

$$d(u,v) = \|u-v\|_{L^\gamma(0,T;L^\rho)}.$$

Note that by Lemma 2.1 and Proposition 2.4, $\mathcal{D}$ is never empty. Indeed, $u(t) = S(t)\eta$ is in $\mathcal{D}(T,M)$ if $\eta \in H^s(\mathbb{R}^n))$ and $\|\eta\|_{\dot{H}^s}$ is sufficiently small. We can claim that $\mathcal{D}$ is a complete metric space. In what follows, we wish to find conditions on T and M which imply that $\mathcal{F}$, given by

$$\mathcal{F}(u) = S(\cdot)\phi - i\mathcal{G}f(u)$$

is a strict contraction on $\mathcal{D}$.

From Lemma 2.1 and Proposition 2.4 and Proposition 2.6 we see that if $u \in \mathcal{D}$, then $\mathcal{F}(u) \in L^\gamma(0,T; B^s_{\rho,2})$. Moreover it follows from formula (2.10) that if

$$\|S(\cdot)\phi\|_{L^\gamma(0,T;\dot{B}^s_{\rho,2})} + CT^\delta M^{\alpha+1} \leqslant M, \tag{4.2}$$

then also $\mathcal{F}(u) \in \mathcal{D}$. It follows from formula (2.9) that if

$$CT^{\delta} M^{\alpha} \leqslant 1, \tag{4.3}$$

then $\mathcal{F}$ is a strict contraction on $\mathcal{D}$. By making one of the constant larger if necessary, we may assume that the constants in (4.2) and (4.3) are the same. Thus, if $\phi \neq 0$, (4.3) is a consequence of (4.2). By part (1) of Proposition 2.4, the following inequality implies, but is not equivalent to (4.2)

$$C\|\phi\|_{\dot{H}^s} + CT^{\delta} M^{\alpha+1} \leqslant M, \tag{4.4}$$

where we may take both constant to be the same.

If $\alpha < 4/(n-2s)$, then $\delta > 0$. Given any $\phi \in H^s$ and any $M > C\|\phi\|_{\dot{H}^s}$, there exists $T > 0$ depending only on the H^s norm of ϕ and on M, such that (4.4) is verified. If $\alpha = 4/(n-2s)$, we have $\delta = 0$. It follows from part (1) of Proposition 2.4 that given any $\phi \in H^s$ and $M > 0$ satisfying $CM^{\alpha} < 1$, there exists $T > 0$ such that (4.2) is verified. In both case, for such a T there exists a unique fixed point in $\mathcal{D}$ of the mapping $\mathcal{F}$, i.e., a solution of (4.1) in $L^{\gamma}(0,T; B^s_{\rho,2}(\mathbb{R}^n))$.

Next we will prove that if $u \in L^{\gamma}(0,T; \dot{B}^s_{\rho,2}(\mathbb{R}^n))$ satisfies (4.1), then $u \in L^q(0,T; B^s_{r,2}) \cap C([0,T]; H^s(\mathbb{R}^n))$, and u is unique in $L^{\gamma}(0,T; B^s_{\rho,2}(\mathbb{R}^n))$. Indeed, $u = \mathcal{F}(u) = S(\cdot)\phi - i\mathcal{G}f(u)$,

$$\begin{aligned} \|\mathcal{F}(u)\|_{L^q(0,T;B^s_{r,2})} &\leqslant \|S(t)\|_{L^q(0,T;B^s_{r,2})} + \|\mathcal{G}u_k\|_{L^q(0,T;B^s_{r,2})} \\ &\leqslant C\|\phi\|_{H^s} + CT^{\delta}\|u_k\|^{\alpha+1}_{L^{\gamma}(0,T;B^s_{\rho,2})}. \end{aligned}$$

From part (2) of Proposition 2.4 and proof of Proposition 2.6,

$$\begin{aligned} u \in L^{\gamma}(0,T; B^s_{\rho,2}(\mathbb{R}^n)) &\Longrightarrow f(u) \in L^{\gamma'}(0,T; B^s_{\rho',2}(\mathbb{R}^n)) \\ &\Longrightarrow \mathcal{G}f(u) \in C([0,T]; H^s(\mathbb{R}^n)). \end{aligned}$$

Now we prove uniqueness by contradiction.

Assume that $u, v \in L^{\gamma}(0,T; B^s_{\rho,2}(\mathbb{R}^n))$ are solutions of (4.1) that satisfying $u(t) \neq v(t)$ for some $t \in [0,T]$. Let $t_0 = \inf_{t\in[0,T]}\{u(t) \neq v(t)\}$. Since both u and v are continuous into $H^s(R^n)$, let $t_k = t_0 - 1/k$, then $t_k \to t_0$ when $k \to \infty$. It follows from the definition of t_0 that $u(t_k) = v(t_k)$, since both u and v are continuous, we have $u(t_0) = v(t_0)$ as $k \to \infty$. Denote $u(t_0) = v(t_0) = \psi$, then $U(t) = u(t+t_0)$ and $V(t) = v(t+t_0)$ both satisfy the equation

$$w = S(\cdot)\psi - i\mathcal{G}f(w)$$

on $[0, T-t_0]$. Choosing $(q,r) = (\gamma,\rho)$ in formula (2.9), we see that for all $t \in [t_0, T]$,

$$\begin{aligned} &\|u-v\|_{L^{\gamma}(t_0,t;L^{\rho})} \\ &= \|\mathcal{G}f(u) - \mathcal{G}f(v)\|_{L^{\gamma}(t_0,t;L^{\rho})} \\ &\leqslant C(t-t_0)^{\delta}\{\|u\|^{\alpha}_{L^{\gamma}(t_0,t;\dot{B}^s_{\rho,2})} + \|u\|^{\alpha}_{L^{\gamma}(t_0,t;\dot{B}^s_{\rho,2})}\}\|u-v\|_{L^{\gamma}(t_0,t;L^{\rho})}. \end{aligned}$$

For $t > t_0$, but sufficiently close to t_0, it follows that

$$C(t-t_0)^{\delta}\{\|u\|^{\alpha}_{L^{\gamma}(t_0,t;\dot{B}^s_{\rho,2})} + \|u\|^{\alpha}_{L^{\gamma}(t_0,t;\dot{B}^s_{\rho,2})}\} < 1,$$

and so that $\|u-v\|_{L^{\gamma}(t_0,t;L^{\rho})} = 0$. This contradicts the choice of t_0, and thus uniqueness is proved. □

Proposition 4.2. *There exists $0 < T < T^*(\phi)$ such that if $\{\phi_k\}$ is a sequence in $H^s(\mathbb{R}^n)$ with $\phi_k \to \phi$ in $H^s(\mathbb{R}^n)$, then, for sufficiently large k, $T < T^*(\phi_k)$ and the solutions u_k (of (4.1) with ϕ replaced by ϕ_k) form a bounded sequence in $L^q(0,T;B^s_{r,2}(\mathbb{R}^n))$. Moreover, $u_k \to u$ in $L^q(0,T;L^r(\mathbb{R}^n))$ for every admissible pair (q,r). In particular, $u_k \to u$ in $C([0,T];H^{s-\varepsilon}(R^n))$ for every $\varepsilon > 0$.*

Proof. We prove proposition by three steps.
Step 1. Continuous dependence. We may first assume $\alpha < 4/(n-2s)$, let $\phi_k \to \phi$ in $H^s(\mathbb{R}^n)$, since $\|\phi_k\|_{\dot{H}^s} \leqslant 2\|\phi\|_{\dot{H}^s}$ for k sufficiently large, we see that

$$C\|\phi_k\|_{\dot{H}^s} + CT^{\delta}M^{\alpha+1} \leqslant 2C\|\phi\|_{\dot{H}^s} + CT^{\delta}M^{\alpha+1} \leqslant M.$$

This implies that the solutions u_k (corresponding solutions with ϕ replaced by ϕ_k) belong to $\mathcal{D}(T,M)$, where $T = T(\|\phi\|_{\dot{H}^s})$. If $\alpha = 4/(n-2s)$, then $\delta = 0$, we see that (4.4) is verified for all $T \in (0, \pi/2\omega]$. We also have $u_k \in \mathcal{D}(T,M)$ where $T = T(\|\phi\|_{\dot{H}^s})$. We denote $T_{\max}(\phi) = T^*(\phi)$, then by fixed point argument, we have

$$\|u\|_{L^{\gamma}(0,T;\dot{B}^s_{\rho,2})} \leqslant M \ (T < T^*(\phi)),$$

and

$$\|u_k\|_{L^{\gamma}(0,T;\dot{B}^s_{\rho,2})} \leqslant M \ (T < T^*(\phi))$$

for k sufficiently large. Thus we have

$$\begin{aligned} d(u_k,u) &= \|u_k - u\|_{L^{\gamma}(0,T;L^{\rho})} \\ &= \|\mathcal{F}(u_k) - \mathcal{F}(u)\|_{L^{\gamma}(0,T;L^{\rho})} \\ &\leqslant \|S(t)\phi_k - S(t)\phi\|_{L^{\gamma}(0,T;L^{\rho})} + \|\mathcal{G}f(u_k) - \mathcal{G}f(u)\|_{L^{\gamma}(0,T;L^{\rho})} \\ &\leqslant C\|\phi_k - \phi\|_{L^2} + CT^{\delta}M^{\alpha}\|u_k - u\|_{L^{\gamma}(0,T;L^{\rho})}, \end{aligned}$$

where the last inequality follows from Lemma 2.1 and formula (2.9).
By (4.4), we can get $CT^{\delta}M^{\alpha} \leqslant 1$, so

$$d(u_k,u) \leqslant C\|\phi_k - \phi\|_{L^2} \leqslant C\|\phi_k - \phi\|_{\dot{H}^s},$$

which implies that $u_k \to u$ in $L^{\gamma}(0,T;L^{\rho}(\mathbb{R}^n))$. It follows from Lemma 2.1 that

$$\begin{aligned} \|u_k - u\|_{L^q(0,T;L^r)} &\leqslant C\|\phi_k - \phi\|_{L^2} + C\|f(u_k) - f(u)\|_{L^{\gamma'}(0,T;L^{\rho'})} \\ &\leqslant C\|\phi_k - \phi\|_{L^2} + CT^{\delta}M^{\alpha}\|u_k - u\|_{L^{\gamma}(0,T;L^{\rho})}, \end{aligned}$$

which implies that $u_k \to u$ in $L^q(0,T;L^r(\mathbb{R}^n))$.
Step 2. Boundedness.

$$\|u_k\|_{L^q(0,T;\dot{B}^s_{r,2})} = \|\mathcal{F}u_k\|_{L^q(0,T;\dot{B}^s_{r,2})}$$

$$\leqslant \|S(t)\|_{L^q(0,T;\dot{B}^s_{r,2})} + \|\mathcal{G}u_k\|_{L^q(0,T;\dot{B}^s_{r,2})}.$$

By Proposition 2.4 and formula (2.10), for sufficiently large k we have

$$\begin{aligned}\|u_k\|_{L^q(0,T;\dot{B}^s_{r,2})} &\leqslant C\|\phi_k\|_{\dot{H}^s} + CT^\delta\|u_k\|^{\alpha+1}_{L^\gamma(0,T;\dot{B}^s_{\rho,2})}\\ &\leqslant C\|\phi\|_{\dot{H}^s} + CT^\delta\|u_k\|^{\alpha+1}_{L^\gamma(0,T;\dot{B}^s_{\rho,2})}.\end{aligned}$$

This implies that u_k form a bounded sequence in $L^q(0,T;\dot{B}^s_{r,2}(\mathbb{R}^n))$.

Step 3. Let $(q,r)=(\infty,2)$, the convergence in $C\big([0,T];H^{s-\varepsilon}(\mathbb{R}^n)\big)$ follows from the convergence in $L^\infty(0,T;L^2(\mathbb{R}^n))$, the boundedness in $L^\infty(0,T;H^s(R^n))$, and the elementary interpolation estimate $\|u\|_{H^{s-\varepsilon}} \leqslant \|u\|_{H^s}^{\frac{s-\varepsilon}{s}}\|u\|_{L^2}^{\frac{\varepsilon}{s}}$. □

Definition 4.3 ([1]). Let $\rho\in\mathcal{S}'$, ρ is called a *Fourier* multiplier on L^p if the convolution $(F^{-1}\rho)*f\in L^p$ for all $f\in\mathcal{S}$, and if $\sup_{\|f\|_{L^p}=1}\|(F^{-1}\rho)*f\|_{L^p}$ is finite. The linear spaces of all such ρ is denoted by M_p; the norm on M_p is the above supremum, written $\|\cdot\|_{M_p}$

Lemma 4.4 ([1]). *Let M_p be defined by Definition 4.3. Then*

$$M_2 = L^\infty \ (\textit{equal norm}).$$

Proposition 4.5. (1) *If $u\in H^s(\mathbb{R}^n)$, and $1\leqslant s<n/2$, then $xu\in L^2(\mathbb{R}^n)$.*

(2) *If $0<s<1$ and $u\in C\big([0,T];H^s(\mathbb{R}^n)\big)$ satisfies* (4.1)*, then $xu(t)\in L^2(\mathbb{R}^n)$.*

Proof. It is well known that $(1+|\xi|^2)^{s/2}\in\mathcal{S}'$. Since $\mathcal{S}\subset H^s$, for any $u\in\mathcal{S}\subset H^s$, it follows from the definition of $H^s(R^n)$ that $F^{-1}(1+|\xi|^2)^{s/2}*u\in L^2$. By Definition 4.3 and Lemma 4.4

$$(1+|\xi|^2)^{s/2}\in M_2 = L^\infty.$$

It yields that

$$\begin{aligned}\|F^{-1}(1+|\xi|^2)^{s/2}*\widehat{u}\|_{L^2} &= \|(1+|\xi|^2)^{s/2}\widehat{\widehat{u}}\|_{L^2}\\ &\leqslant \|(1+|\xi|^2)^{s/2}\|^{1/2}_{L^\infty}\|\widehat{\widehat{u}}\|_{L^2}\\ &= \|(1+|\xi|^2)^{s/2}\|^{1/2}_{L^\infty}\|u\|_{L^2}\\ &\leqslant C\|(1+|\xi|^2)^{s/2}\|^{1/2}_{L^\infty}\|u\|_{H^s},\end{aligned}$$

which implies that $\widehat{u}\in H^s$. If $1\leqslant s<n/2$, then $\nabla\widehat{u}\in H^{s-1}\hookrightarrow L^2$, we have

$$\|xu\|_{L^2} = \|\widehat{xu}\|_{L^2} = C\|\nabla\widehat{u}\|_{L^2} \leqslant C\|\nabla\widehat{u}\|_{H^{s-1}}.$$

This implies that $xu\in L^2(\mathbb{R}^n)$.

If $0<s<1$, then $\alpha<4/(n-2s)<4/(n-2)$. By following H^1 argument, for any $\phi\in H$, there exists $T>0$, such that $xu\in C([0,T);L^2(\mathbb{R}^n))$. Approaching $\phi\in\Sigma$ by a sequence $\{\phi_k\}\in H$ and take $(q,r)=(\infty,2)$ in Proposition 4.2, we have $u_k\to u$ in $L^2(\mathbb{R}^n)$, it yields that $\widehat{u_k}\to\widehat{u}$ in $L^2(\mathbb{R}^n)$. Since $\phi_k\in H$, we have $u_k(t)\in H^1(\mathbb{R}^n)$ and $xu_k(t)\in L^2(\mathbb{R}^n)$. It yields that $\|\nabla\widehat{u_k(t)}\|_{L^2} =$

$C\|\widehat{xu_k(t)}\|_{L^2} < \infty$ uniformly with respect to k. By the weak compactness in $L^2(\mathbb{R}^n)$, there exists $u_\alpha \in L^2(\mathbb{R}^n)$ such that $\nabla\widehat{u_k(t)} \to u_\alpha$ in $L^2(\mathbb{R}^n)$. We can prove that $\nabla\widehat{u(t)} = u_\alpha \in L^2(\mathbb{R}^n)$, i.e., $xu(t) \in L^2(\mathbb{R}^n)$. □

Proof of Theorem 3.1. It follows from Proposition 4.1 and Proposition 4.5 that for every $\phi \in \Sigma$, there exists a solution $u \in C([0,T];\Sigma) \cap L^\gamma_{loc}\big(0,T^*;B^s_{\rho,2}(\mathbb{R}^n)\big)$ of the integral equation (1.2) for some $T^* = T^*(\phi) \in (0,\pi/2\omega]$. Here T^* is the maximal existence time satisfies $T^* = \pi/2\omega$ or $T^* < \pi/2\omega$ and $\|u(t)\|_{\dot{H}^s} = \infty$, as $t \to T^*$. Indeed, if $T^* < \pi/2\omega$ and $\lim_{t\to T^*}\|u(t)\|_{\dot{H}^s} < \infty$, then there exist $M < \infty$ and a sequence $t_j \uparrow T^*$ such that $\|u(t_j)\|_{\dot{H}^s} < M$. Let k satisfies $t_k + T(M) > T^*$, we take $u(0) = u(t_k)$, by fixed point argument, we can extend T^* to $t_k + T(M)$, this contradicts the choice of T^*.

So far we have proved the first statement of Theorem 3.1, combine Proposition 4.2 with Proposition 4.5, part (4) has been proved. At last, the conversation laws (1.3) and (1.4) can be proved by using the same arguments as in the proof of conversation laws in Theorem 1.1 by Cazenave and Weissler in [4].

This completes the proof of Theorem 3.1. □

Lemma 4.6 ([3]). *Let $\phi \in H = \{u \in H^1(\mathbb{R}^n) : xu \in L^2(\mathbb{R}^n)\}$ be nonzero, and if $n \geqslant 3$, assume $\lambda < 0$, $4/n \leqslant \alpha \leqslant 4/(n-2)$, then under the condition*

$$\frac{1}{2}\|\nabla\phi\|^2_{L^2} + \frac{2\lambda}{\alpha+2}\|\phi\|^{\alpha+2}_{L^{\alpha+2}} \leqslant 0,$$

u blows up at time $t^* \leqslant \pi/2\omega$, and

$$\lim_{t\to t^*}\|\nabla u(t)\|_{L^2} = \infty,\ \lim_{t\to t^*}\|u(t)\|_{L^{\alpha+2}} = \infty.$$

Proof of Theorem 3.4. Notice that conservation laws are verified in Σ as $s \geqslant 1$, by the same method as in the proof of Lemma 4.6, we get $\lim_{t\to t^*}\|u(t)\|_{L^{\alpha+2}} = \infty$, then follows from Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{\alpha+2}(\mathbb{R}^n)$ in the case $s \geqslant 1$ that $\lim_{t\to t^*}\|u(t)\|_{H^s} = \infty$. This proves the first statement of Theorem 3.4.

When $T^* < \pi/2\omega$, in view of Theorem 3.1 that $\lim_{t\to t^*}\|u(t)\|_{\dot{H}^s} = \infty$. If we consider $u(t),\ t < T^*$ as the initial value, it follows from inequality (4.4) and fixed point argument that if for some $M > 0$

$$C\|u(t)\|_{\dot{H}^s} + C(T-t)^\delta M^{\alpha+1} \leqslant M,$$

then $T < T^*$. Thus, for any $M > 0$,

$$C\|u(t)\|_{\dot{H}^s} + C(T^*-t)^\delta M^{\alpha+1} > M.$$

Choosing for example $M = 2C\|u(t)\|_{\dot{H}^s}$, we have

$$(T^*-t)^\delta\|u(t)\|^\alpha_{\dot{H}^s} > C,$$

which easily implies that

$$\|u\|_{\dot{H}^s} \geqslant \frac{C}{(T^*-t)^{1/\alpha-(n-2s)/4}}.$$

This completes the proof of Theorem 3.4. □

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