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A DETERMINANT FORMULA FOR CONGRUENT ZETA FUNCTIONS OF REAL ABELIAN FUNCTION FIELDS

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Abstract. In this paper we give a determinant formula for congruent zeta functions of real Abelian function fields. We also give some examples of congruent zeta functions when the conductor of real Abelian function field is monic irreducible.

1. Introduction

Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$. Let ∞ be the place of k associated to $1/T$, which is called the infinite one of k. Write $\mathbb{A}^+ = \{1 \neq M \in \mathbb{A} : M \text{ is monic}\}$ and $\mathbb{A}^+_{\mathrm{irr}} = \{P \in \mathbb{A}^+ : P \text{ is irreducible}\}$. For any $M \in \mathbb{A}^+$, write K_M for the Mth cyclotomic function field and K_M^+ for the maximal real subfield of K_M. In this paper, by an *Abelian function field*, we always mean a finite Abelian extension F of k which is contained in a cyclotyomic function field K_M, and F is said to be *real* if ∞ splits completely in F. Let $N = N(F) \in \mathbb{A}^+$ be the conductor of F, that is, K_N is the smallest cyclotomic function field containing F. For such a field F, there exists a polynomial $P_F(X) \in \mathbb{Z}[X]$ such that

$$\zeta(s, F) = \frac{P_F(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $\zeta(s, F)$ is the congruence zeta function of F, and $P_F(1)$ is equal to the divisor class number h_F of F. In recent paper [5], Shiomi has expressed the polynomial $P_{K_M^+}(X)$ as determinant of matrix $D_{K_M^+}(X)$ with entries in $\mathbb{Z}[X]$ up to some polynomial $J_{K_M^+}(X)$. Since $h_{K_M^+} = P_{K_M^+}(1)$, this determinant formula for $P_{K_M^+}(X)$ can be regarded as generalization of that for class number $h^+_{K_M}$ ([1], [3]).

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The aim of this paper is to give a determinant formula for the polynomial $P_F(X)$ for arbitrary real Abelian function field F. We also give some examples of the polynomials $P_F(X)$ when F is a real Abelian function field of monic irreducible conductor and $q = 3$.

2. Zeta and *L*-functions

Let $\zeta(s, F)$ be the congruence zeta function of a real Abelian function field F given by

$$\zeta(s, F) = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{N\mathfrak{p}^s}\Big)^{-1},$$

where $\mathfrak{p}$ runs over all primes of F. It is well known that there exists a polynomial $P_F(X) \in \mathbb{Z}[X]$ of degree $2g$, where g is the genus of F, such that

$$\zeta(s, F) = \frac{P_F(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}. \tag{2.1}$$

Moreover, the polynomial $P_F(X)$ satisfies $P_F(0) = 1$ and $P_F(1) = h_F$, where h_F is the divisor class number of F. Let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F and $\zeta(s, \mathcal{O}_F)$ be the zeta function of $\mathcal{O}_F$ given by

$$\zeta(s, \mathcal{O}_F) = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{N\mathfrak{p}^s}\Big)^{-1},$$

where $\mathfrak{p}$ runs over all prime ideals of $\mathcal{O}_F$. Since ∞ splits completely in F, the functions $\zeta(s, F)$ and $\zeta(s, \mathcal{O}_F)$ satisfy the following equality

$$\zeta(s, F) = \zeta(s, \mathcal{O}_F)(1 - q^{-s})^{-[F:\mathrm{k}]}. \tag{2.2}$$

Let X_F be the group of primitive Dirichlet characters of $\mathbb{A}$ associated to F. For $\chi \in X_F$, let $L(s, \chi)$ be the L-function associated to χ given by

$$L(s, \chi) = \prod_{P \in \mathbb{A}^+_{\mathrm{irr}}} \big(1 - \chi(P) q^{-s \deg P}\big)^{-1}.$$

Then we have

$$\zeta(s, \mathcal{O}_F) = \prod_{\chi \in X_F} L(s, \chi). \tag{2.3}$$

Let $\chi_0 \in X_F$ denote the trivial character. Since $L(s, \chi_0) = (1 - q^{1-s})^{-1}$, from (2.1), (2.2) and (2.3), we get

$$\prod_{\chi_0 \neq \chi \in X_F} L(s, \chi) = (1 - q^{-s})^{[F:\mathrm{k}]-1} P_F(q^{-s}). \tag{2.4}$$

For any $\chi \in X_F$, let $F_\chi \in \mathbb{A}^+$ be the conductor of χ and $\tilde{\chi} = \chi \circ \pi_\chi$, where $\pi_\chi : (\mathbb{A}/N\mathbb{A})^* \to (\mathbb{A}/F_\chi\mathbb{A})^*$ is the canonical homomorphism. Then we have

$$L(s, \tilde{\chi}) = L(s, \chi) \prod_{Q \in \mathbb{A}^+_{\mathrm{irr}},\, Q | N} \big(1 - \chi(Q) q^{-s \deg Q}\big). \tag{2.5}$$

Thus, by (2.4) and (2.5), we have

$$\prod_{\chi_0 \neq \chi \in X_F} L(s, \tilde{\chi}) = (1 - q^{-s})^{[F:\mathrm{k}]-1} P_F(q^{-s}) J_F(q^{-s}), \tag{2.6}$$

where $J_F(X)$ is the polynomial given by

$$J_F(X) = \prod_{\chi_0 \neq \chi \in X_F} \prod_{Q \in \mathbb{A}^+_{\mathrm{irr}},\, Q|N} \big(1 - \chi(Q) X^{\deg Q}\big).$$

Finally we give some remarks on the polynomial $J_F(X)$. They satisfy the following equality (cf. [5, Proposition 3.1])

$$J_F(X) = \prod_{Q \in \mathbb{A}^+_{\mathrm{irr}},\, Q|N} \frac{(1 - X^{f_Q \deg Q})^{g_Q}}{1 - X^{\deg Q}},$$

where f_Q is the residue class degree of Q in F/k and g_Q is the number of primes over Q in F. Hence we see that $J_F(X) \in \mathbb{Z}[X]$ and in particular, $J_F(X) = 1$ if N is a power of some $Q \in \mathbb{A}^+_{\mathrm{irr}}$.

3. A determinant formula for $P_F(X)$

Let F be a real Abelian function field with conductor N. Let

$$\mathcal{R}_N = (\mathbb{A}/N\mathbb{A})^*/\mathbb{F}_q^*.$$

For $\alpha \in (\mathbb{A}/N\mathbb{A})^*$, there exists a unique polynomial $A_\alpha \in \mathbb{A}$ such that $\deg A_\alpha < \deg N$ and $A_\alpha + N\mathbb{A} = \alpha$. Write $\mathrm{sgn}_N(\alpha) \in \mathbb{F}_q^*$ for the leading coefficient of A_α, $\deg_N(\alpha) = \deg A_\alpha$ and $c^\lambda(\alpha) = \lambda^{-1}(\mathrm{sgn}_N(\alpha))$ for any character λ of $\mathbb{F}_q^*$. We can easily see that $\deg_N$ is a function over $\mathcal{R}_N$. Let $\mathcal{H}$ be the subgroup of $\mathcal{R}_N$ which is isomorphic to $\mathrm{Gal}(K_N^+/F)$ under the canonical isomorphism $\mathcal{R}_N \cong \mathrm{Gal}(K_N^+/\mathrm{k})$. For each $\sigma \in \mathrm{Gal}(F/\mathrm{k})$, choose $\beta_\sigma \in (\mathbb{A}/N\mathbb{A})^*$ which corresponds to σ under $(\mathbb{A}/N\mathbb{A})^* \to \mathcal{R}_N \to \mathcal{R}_N/\mathcal{H} \cong \mathrm{Gal}(F/\mathrm{k})$. Choose a subset $\Omega_\mathcal{H} \subseteq (\mathbb{A}/N\mathbb{A})^*$ which are mapped bijectively onto $\mathcal{H}$ under the homomorphism $(\mathbb{A}/N\mathbb{A})^* \to \mathcal{R}_N$. For each $\sigma \in \mathrm{Gal}(F/\mathrm{k})$, define a polynomial $f_\sigma(X) \in \mathbb{Z}[X]$ by

$$f_\sigma(X) := \sum_{\alpha \in \Omega_\mathcal{H}} X^{\deg_N(\beta_\sigma \alpha)}.$$

Since $\{\beta_\sigma \alpha : \sigma \in \mathrm{Gal}(F/\mathrm{k}), \alpha \in \Omega_\mathcal{H}\}$ forms a complete system of representatives of $\mathcal{R}_N$, we can see easily that $f_\sigma(X)$ is independent of the choices of $\Omega_\mathcal{H}$ and β_σ. Define a matrix $D_F(X)$ by

$$D_F(X) := \left(\frac{f_{\sigma\tau^{-1}}(X) - f_\sigma(X)}{1 - X} \right)_{1 \neq \sigma, \tau \in \mathrm{Gal}(F^+/\mathrm{k})}.$$

Theorem 3.1. *With notations as above, we have*

$$\det D_F(X) = P_F(X) J_F(X). \tag{3.1}$$

Proof. For $\chi_0 \neq \chi \in X_F$, as in the proof of [5, Theorem 3.1], we have

$$\begin{aligned} L(s,\tilde{\chi}) &= \sum_{\sigma\in\mathrm{Gal}(F/\mathrm{k})}\sum_{\alpha\in\Omega_{\mathcal{H}}} \tilde{\chi}(\beta_\sigma\alpha)q^{-s\deg_N(\beta_\sigma\alpha)} \\ &= \sum_{\sigma\in\mathrm{Gal}(F/\mathrm{k})} \tilde{\chi}(\beta_\sigma)\Big(\sum_{\alpha\in\Omega_{\mathcal{H}}} q^{-s\deg_N(\beta_\sigma\alpha)}\Big) = \sum_{\sigma\in\mathrm{Gal}(F/\mathrm{k})} \tilde{\chi}(\beta_\sigma)f_\sigma(q^{-s}). \end{aligned}$$

Thus, by the Frobenius determinant formula,

$$\begin{aligned} \prod_{\chi_0\neq\chi\in X_F} L(s,\tilde{\chi}) &= \prod_{\chi_0\neq\chi\in X_F}\sum_{\sigma\in\mathrm{Gal}(F/\mathrm{k})} \tilde{\chi}(\beta_\sigma)f_\sigma(q^{-s}) \\ &= \det\left(f_{\sigma\tau^{-1}}(q^{-s}) - f_\sigma(q^{-s})\right)_{1\neq\sigma,\tau\in\mathrm{Gal}(F/\mathrm{k})}. \end{aligned} \tag{3.2}$$

So, by (2.6), we get

$$\det D_F(q^{-s}) = P_F(q^{-s})\cdot J_F(q^{-s}).$$

Putting $X = q^{-s}$, we get the desired result. □

Now we assume that F has a monic irreducible conductor $P \in \mathbb{A}^+_{\mathrm{irr}}$ of degree d. Fix a primitive root Q of P with $\deg Q < d$. For each integer $i \geq 0$, let Q_i be the unique polynomial such that $Q^i \equiv Q_i \bmod P$ and $\deg Q_i < d$. Let $r = \frac{q^d-1}{q-1}$ and $n = [F:\mathrm{k}]$. Under the isomorphism $(\mathbb{A}/P\mathbb{A})^* \to \mathrm{Gal}(K_P/\mathrm{k}), A + P\mathbb{A} \mapsto \sigma_A$, we have $\mathrm{Gal}(K_P/K_P^+) = \{\sigma_{Q_{ri}} : 0 \leq i < q-1\}$ and $\mathrm{Gal}(K_P/F) = \{\sigma_{Q_{ni}} : 0 \leq i < \frac{q^d-1}{n}\}$. Hence we may take $\Omega_{\mathcal{H}} = \{Q_{ni} : 0 \leq i < \frac{r}{n}\}$. For each integer $i \geq 0$, let σ_i be the restriction of σ_{Q_i} to F. Then $\mathrm{Gal}(F/\mathrm{k}) = \{\sigma_i : 0 \leq i < n\}$. For each $0 \leq i < n$, we may take $\beta_{\sigma_i} = Q_i$, so

$$f_{\sigma_i}(X) = \sum_{h=0}^{\frac{r}{n}-1} X^{\deg_P(Q_iQ_{nh})} = \sum_{h=0}^{\frac{r}{n}-1} X^{\deg Q_{i+nh}}.$$

Note that $\sigma_j^{-1} = \sigma_{n-j}$ for $0 \leq j < n$. Then, for each $1 \leq i,j < n$, we have

$$f_{ij}(X) := f_{\sigma_i\sigma_j^{-1}}(X) - f_{\sigma_i}(X) = \sum_{h=0}^{\frac{r}{n}-1} X^{\deg Q_{i+n-j+nh}} - X^{\deg Q_{i+nh}}.$$

Since F has a monic irreducible conductor P, $J_F(X) = 1$, so we get the following.

Proposition 3.2. *Let F be a real Abelian function field with a monic irreducible conductor P. For each $1 \leq i,j < n = [F:\mathrm{k}]$, let $f_{ij}(X)$ be as above. Then we have*

$$D_F(X) = \left(\frac{f_{ij}(X)}{1-X}\right)_{1\leq i,j<n}$$

and $P_F(X) = \det D_F(X)$.

TABLE 1. $P_F(X)$ for the subfield F of K_P with $[F : \mathrm{k}] = 13$

P	$P_F(X)$	$h(F)$
T^3+2T+1	$1+9X+42X^2+144X^3+399X^4+900X^5+1691X^6$ $+2700X^7+3591X^8+3888X^9+3402X^{10}+2187X^{11}+729X^{12}$	3^9
T^3+2T+2	$1+9X+42X^2+144X^3+399X^4+900X^5+1691X^6$ $+2700X^7+3591X^8+3888X^9+3402X^{10}+2187X^{11}+729X^{12}$	3^9
T^3+T^2+2	$1+9X+42X^2+131X^3+308X^4+601X^5+1067X^6$ $+1803X^7+2772X^8+3537X^9+3402X^{10}+2187X^{11}+729X^{12}$	$53\cdot 313$
T^3+2T^2+1	$1+9X+42X^2+131X^3+308X^4+601X^5+1067X^6$ $+1803X^7+2772X^8+3537X^9+3402X^{10}+2187X^{11}+729X^{12}$	$53\cdot 313$

4. Examples

Let F be a real Abelian function field with a monic irreducible conductor P. In this section, we give some examples of the congruence zeta function $P_F(X)$ of F using Proposition 3.2. As the k-isomorphism $T \mapsto T+\alpha$ with $\alpha \in \mathbb{F}_q^*$ sends a monic irreducible polynomial to another one, it suffices to consider only the monic irreducible polynomials up to these k-isomorphisms.

Example 4.1. Assume $q = 3$. There are four monic irreducible polynomials T^3+2T+1, T^3+2T+2, T^3+T^2+2 and T^3+2T^2+1 of degree 3 up to the above k-isomorphisms. Since $[K_P : \mathrm{k}] = 26$, there is only one non-trivial real subfield F of K_P, which is of degree 13 over k. The table of $P_F(X)$ and $h(F) = P_F(1)$ for these polynomials are given in Table 1.

Example 4.2. Assume $q = 3$. There are six monic irreducible polynomials of degree 4 up to the above k-isomorphisms. We only give the table for T^4+T+2 in Table 2. Note that for each positive divisor of 40 there is only one real subfield F of K_P for the same degree. In the case $[F : \mathrm{k}] = 8$, we have the matrix $D_F(X)$ as follows;

$$\begin{pmatrix} X^2+1 & -X^2+X & 2X^2 & -X & -X^2 & X^2+X & X^2 \\ -2X^2-X & X+1 & X^2+X & 2X^2-X & -X^2-X & X & 2X^2+X \\ -X^2-X & -3X^2 & 2X^2+X+1 & X^2 & X^2-X & 0 & X^2+X \\ 0 & -2X^2 & -X^2 & 2X^2+1 & 0 & 2X^2 & X^2 \\ -X^2 & -X^2+X & 0 & -X^2-X & X^2+1 & X^2+X & 3X^2 \\ -X^2-X & -2X^2+X & X^2+X & -X & -2X^2-X & 2X^2+X+1 & 2X^2+X \\ X^2-X & -2X^2 & X & X^2 & -X^2-X & -X^2 & 3X^2+X+1 \end{pmatrix}.$$

In Table 2, $f_1(X)$ is given by

$$\begin{aligned} &1+16X+133X^2+760X^3+3326X^4+11764X^5+\cdots \\ &+17175641679X^{36}+6198727824X^{37}+1162261467X^{38} \end{aligned}$$

and $f_2(X)$ is given by

$$\begin{aligned} &1+36X+663X^2+8320X^3+79951X^4+626884X^5+\cdots \\ &+298538229605731251669X^{76}+486306618362277715204X^{77} \\ &+405255515301897626 7X^{78}. \end{aligned}$$

One can see that our data coincide with one in [2].

TABLE 2. $P_F(X)$ for the real subfield F of K_P with $P = T^4 + T + 2$

$[F:k]$	$P_F(X)$	$h(F)$
2	$1+3X^2$	2^2
4	$1+4X+11X^2+24X^3+33X^4+36X^5+27X^6$	$2^3 \cdot 17$
5	$1+X-2X^2-5X^3+X^4-15X^5-18X^6+27X^7+81X^8$	71
8	$1+4X+19X^2+56X^3+153X^4+356X^5+715X^6+1344X^7+2145X^8$ $+3204X^9+4131X^{10}+4536X^{11}+4617X^{12}+2916X^{13}+2187X^{14}$	$2^4 \cdot 17 \cdot 97$
10	$1+6X+18X^2+40X^3+71X^4+70X^5-43X^6-358X^7-1064X^8-2220X^9-3192X^{10}$ $-3222X^{11}-1161X^{12}+5670X^{13}+17253X^{14}+29160X^{15}+39366X^{16}+39366X^{17}+19683X^{18}$	$2^2 \cdot 71 \cdot 491$
20	$f_1(X)$	$2^3 \cdot 11^2 \cdot 17$ $\cdot 71 \cdot 491 \cdot 541$
40	$f_2(X)$	$2^4 \cdot 11^2 \cdot 17 \cdot 41 \cdot 71 \cdot 97$ $\cdot 491 \cdot 541 \cdot 881 \cdot 1564361$

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