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DECOMPOSITION THEOREMS OF LIE OPERATOR ALGEBRAS

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Abstract. In this paper, we introduce a notion of Lie operator algebras which as a generalization of ordinary Lie algebras is an analogy of operator groups. We discuss some elementary properties of Lie operator algebras. Moreover, we also prove a decomposition theorem for Lie operator algebras.

1. Introduction

A (right) operator group is a triple (G, Ω, α) consisting of a group G, a set Ω called the operator domain and a function $\alpha : G\times\Omega \to G$ such that the mapping $x \mapsto \alpha(x, \omega)$ is an endomorphism of G for each $\omega \in \Omega$. Usually, we write $\omega(x)$ for $\alpha(x, \omega)$ and speak of the Ω-group G if the function α is understood. An operator group with empty operator domain is just an ordinary group. As a generalization of groups, operator groups have been studied intensively and effected many research papers (see [1, 2, 3, 4, 5, 9, 11, 12]).

The classical Krull-Remark-Schmidt theorem states that an Ω-group G satisfying chain conditions on normal Ω-subgroups can be decomposed into some indecomposable Ω-subgroups and up to order of the direct factors, the decomposition is unique. This theorem was first formulated for finite groups by Wedderburn in 1909 and its extension to abelian groups with operators, hence to modules, was given by O. Schmidt in 1928 (see [6], page 115).

There has long been an interest in introducing the concepts and ideas in group theory into the theory of Lie algebras. For instance, complete Lie algebras come from the concept of complete groups ([7, 8]); the study of varieties of Lie algebras has closed connection with the theory of group varieties ([10]).

The first purpose of this paper is to introduce a notion of Lie operator algebras which is an analogy of operator groups and discuss their some elementary properties. Secondly, we are interested in an analogy of Krull-Remark-Schmidt theorem for Lie operator algebras. We shall prove that there is a kind of direct decomposition for a Lie operator algebra (see Theorem 3.17). Furthermore, we

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prove that up to order of the direct factors, this decomposition is unique (see Theorem 4.13).

Definition 1.1. A Lie operator algebra is a triple $(\mathbb{g}, \Omega, \alpha)$ consisting of a Lie algebra $\mathbb{g}$, a set Ω called the operator domain and a function $\alpha : \mathbb{g} \times \Omega \to \mathbb{g}$ such that the mapping $x \mapsto \alpha(x, \omega)$ is an endomorphism of $\mathbb{g}$ for each $\omega \in \Omega$. We shall write $\omega(x)$ for $\alpha(x, \omega)$ and if α is understood, then $\mathbb{g}$ is called a Lie Ω-algebra.

Remark 1.2. Any Lie algebra can be regarded as a Lie operator algebra with empty operator domain, so a Lie operator algebra is a generalization of a Lie algebra.

Remark 1.3. A Lie operator algebra is a Lie algebra with a set of operators which act on the Lie algebra like endomorphisms. In fact, $\mathbb{g}$ is a Lie algebra if and only if $\mathbb{g}$ is a Lie $R(\Omega)$-algebra, where $R(\Omega)$ is the ring generated by endomorphisms in Ω.

This paper is organized as follows. In Section 2, we shall give several basic concepts, such as Ω-subalgebras, Ω-ideals and Ω-quotient algebras; we shall develop some basic results for the latter discussions. In Section 3, we discuss Ω-composition series, chain conditions and Ω-direct decompositions for a Lie Ω-algebra. We prove that if a Lie Ω-algebra has the descending chain condition (or ascending chain condition) on Ω-direct factors, then it has an Ω-direct decomposition. Section 4 is devoted to proving the uniqueness of this Ω-direct decomposition.

Throughout this paper we assume that all Lie algebras and Lie Ω-algebras are finite-dimensional over the field of complex numbers.

2. Some examples and properties

A subalgebra $\mathbb{h}$ of a Lie algebra $\mathbb{g}$ is said to be *fully-invariant* if $f(\mathbb{h}) \subseteq \mathbb{h}$ for each endomorphism f, be *characteristic* if $f(\mathbb{h}) \subseteq \mathbb{h}$ for each automorphism f and be *normal* if $f(\mathbb{h}) \subseteq \mathbb{h}$ for each inner automorphism f.

Definition 2.1. Let $\mathbb{g}$ be a Lie Ω-algebra and $\mathbb{h}$ be a subalgebra (ideal) of $\mathbb{g}$, then $\mathbb{h}$ is called an Ω-subalgebra (ideal) if it is Ω-admissible. That is, $\omega(x) \in \mathbb{h}$ for all $x \in \mathbb{h}$ and $\omega \in \Omega$. The symbol $\mathbb{h} \lhd^{\Omega} \mathbb{g}$ means that $\mathbb{h}$ is an Ω-deal of $\mathbb{g}$.

Let $\mathbb{n}$ be an Ω-ideal of $\mathbb{g}$, it is easy to verify that the quotient algebra $\mathbb{g}/\mathbb{n}$ becomes a Lie Ω-algebra if we define $\omega(\mathbb{n} + x) = \mathbb{n} + \omega(x)$.

Definition 2.2. A Lie Ω-algebra is said to be Ω-simple if it is not of dimension 1 and it has no proper nontrivial Ω-ideals. As usual we speak of a simple Lie algebra if Ω is empty.

Examples of Lie Ω-algebras are as follows:

Example 2.3. Let $\mathbb{g}$ be any Lie algebra and let $\Omega = \mathrm{End}\mathbb{g}$ denote the set of all endomorphisms of $\mathbb{g}$. Then $\mathbb{g}$ is a Lie Ω-algebra if we allow endomorphisms to operate on $\mathbb{g}$ in the natural way. An Ω-subalgebra of $\mathbb{g}$ is simply a fully-invariant subalgebra.

In the same way, if $\Omega = (\mathrm{Int}\mathbb{g})\mathrm{Aut}\mathbb{g}$ denotes the set of all (inner) automorphisms of $\mathbb{g}$, then here the Ω-subalgebras are just (normal) characteristic subalgebras.

Definition 2.4. A Lie homomorphism $f : \mathbb{g} \to \mathbb{h}$ is said to be an Ω-homomorphism between Lie Ω-algebras $\mathbb{g}$ and $\mathbb{h}$ if $f(\omega(x)) = \omega(f(x))$ for all $x \in \mathbb{g}$ and $\omega \in \Omega$. The set of all Ω-homomorphism from $\mathbb{g}$ to $\mathbb{h}$ is written $\mathrm{Hom}_\Omega(\mathbb{g}, \mathbb{h})$. Similarly, we can define the Ω-endomorphisms and Ω-automorphisms, and they form the sets $\mathrm{End}_\Omega\mathbb{g}$ and $\mathrm{Aut}_\Omega\mathbb{g}$ respectively.

Remark 2.5. Clearly $\mathrm{End}_\Omega\mathbb{g} \subseteq \mathrm{End}\mathbb{g}$ and $\mathrm{Aut}_\Omega\mathbb{g} \subseteq \mathrm{Aut}\mathbb{g}$. The symbol $\simeq^\Omega$ means Ω-isomorphism.

Definition 2.6. Let $\mathbb{g}$ be a Lie Ω-algebra. (An) A (inner) derivation D of $\mathbb{g}$ is called the Ω-(inner) derivation if $\omega D = D\omega$ for all $\omega \in \Omega$. We write $\mathrm{Der}_\Omega\mathbb{g}$ and $\mathrm{ad}_\Omega\mathbb{g}$ for the sets of all Ω-derivations and Ω-inner derivations of $\mathbb{g}$ respectively.

Theorem 2.7. *If $f : \mathbb{g} \to \mathbb{h}$ is an Ω-homomorphism of Lie Ω-algebras, then the mapping $f' : \mathrm{Ker}f + x \mapsto f(x)$ is an Ω-isomorphism from $\mathbb{g}/\mathrm{Ker}f$ to $\mathrm{Im}f$.*

Proof. If $x \in \mathrm{Ker}f$, then $\omega(x) \in \mathrm{Ker}f$ since $f(\omega(x)) = \omega(f(x)) = 0$. Thus $\mathrm{Ker}f$ is an Ω-ideal of $\mathbb{g}$. Now f' is well-defined since $f(t + x) = f(x)$ for all $t \in \mathrm{Ker}f$, and it is clearly an Ω-epimorphism. Also $\mathrm{Ker}f + x \in \mathrm{Ker}f'$ if and only if $x \in \mathrm{Ker}f$, that is to say, $\mathrm{Ker}f' = 0$; thus f' is an Ω-isomorphism. □

Remark 2.8. If $\mathbb{n}$ is an Ω-ideal of a Lie Ω-algebra $\mathbb{g}$, the mapping $\pi : x \mapsto \mathbb{n} + x$ is an Ω-epimorphism from $\mathbb{g}$ to $\mathbb{g}/\mathbb{n}$ with kernel $\mathbb{n}$. This π is called the canonical homomorphism.

Corollary 2.9. *Let $\mathbb{m}$ and $\mathbb{n}$ be two Ω-ideals of a Lie Ω-algebra $\mathbb{g}$ and $\mathbb{n}$ be an Ω-ideal of $\mathbb{m}$. Then $\mathbb{m}/\mathbb{n}$ is an Ω-ideal of $\mathbb{g}/\mathbb{n}$ and $(\mathbb{g}/\mathbb{n})/(\mathbb{m}/\mathbb{n}) \simeq^\Omega \mathbb{g}/\mathbb{m}$.*

Proof. Define $f : \mathbb{g}/\mathbb{n} \to \mathbb{m}/\mathbb{n}$ by $f(\mathbb{n} + x) = \mathbb{m} + x$. This is a well-defined Ω-epimorphism with kernel $\mathbb{m}/\mathbb{n}$. This result follows from Theorem 2.7. □

Suppose that $\mathbb{m}$ and $\mathbb{n}$ are two Ω-subalgebras of a Lie Ω-algebra $\mathbb{g}$. It is easy to check that the sum $\mathbb{m} + \mathbb{n}$ and intersection $\mathbb{m} \cap \mathbb{n}$ are also Ω-subalgebras of $\mathbb{g}$.

Corollary 2.10. *Let $\mathbb{m}$ be an Ω-subalgebra and $\mathbb{n}$ an Ω-ideal of a Lie Ω-algebra $\mathbb{g}$. Then $\mathbb{n}\cap\mathbb{m}$ is an Ω-ideal of $\mathbb{m}$ and $(\mathbb{n}\cap\mathbb{m})+x \mapsto \mathbb{n}+x$ is an Ω-isomorphism from $\mathbb{m}/(\mathbb{m} \cap \mathbb{n})$ to $(\mathbb{n} + \mathbb{m})/\mathbb{n}$.*

Proof. The function $x \mapsto \mathbb{n} + x$ is clearly an Ω-epimorphism from $\mathbb{m}$ to $(\mathbb{n} + \mathbb{m})/\mathbb{n}$ whose kernel is $\mathbb{m} \cap \mathbb{n}$. This result follows from Theorem 2.7. □

Lemma 2.11. *Let $\mathbb{n} \subset \mathbb{g}$ be two Lie Ω-algebras and $\mathbb{m}$ be an Ω-ideal of $\mathbb{n}$. If f is an Ω-endomorphism of $\mathbb{g}$, then $f(\mathbb{m})$ is an Ω-ideal of $f(\mathbb{n})$ and $\mathbb{m} + \mathbb{h}$ is an Ω-ideal of $\mathbb{n} + \mathbb{h}$, where $\mathbb{h}$ is an Ω-ideal of $\mathbb{g}$.*

Lemma 2.12. *Let* $\mathbb{h}, \mathbb{k}, \mathbb{g}$ *be three* Ω*-subalgebras of a Lie* Ω*-algebra and assume that* $\mathbb{k} \subseteq \mathbb{g}$*. Then* $(\mathbb{h} + \mathbb{k}) \cap \mathbb{g} = (\mathbb{h} \cap \mathbb{g}) + \mathbb{k}$.

Proof. Obviously $(\mathbb{h} \cap \mathbb{g}) + \mathbb{k} \subseteq \mathbb{h} + \mathbb{k}$ and $(\mathbb{h} \cap \mathbb{g}) + \mathbb{k} \subseteq \mathbb{g} + \mathbb{k} = \mathbb{g}$. Thus $(\mathbb{h} \cap \mathbb{g}) + \mathbb{k} \subseteq (\mathbb{h} + \mathbb{k}) \cap \mathbb{g}$. Conversely, let $x \in (\mathbb{h} + \mathbb{k}) \cap \mathbb{g}, x = h + k \in \mathbb{h} + \mathbb{k}$, where $h \in \mathbb{h}$ and $k \in \mathbb{k}$. Then $h = x - k \in \mathbb{g} + \mathbb{k} = \mathbb{g}$. Thus $h \in \mathbb{g} \cap \mathbb{h}$ and $x \in (\mathbb{h} \cap \mathbb{g}) + \mathbb{k}$. Hence $(\mathbb{h} + \mathbb{k}) \cap \mathbb{g} = (\mathbb{h} \cap \mathbb{g}) + \mathbb{k}$. □

Proposition 2.13. *Let* $\mathbb{h}_1, \mathbb{h}_2, \mathbb{k}_1, \mathbb{k}_2$ *be* Ω*-subalgebras of a Lie* Ω*-algebra* $\mathbb{g}$ *such that* $\mathbb{h}_1 \lhd^\Omega \mathbb{h}_2$ *and* $\mathbb{k}_1 \lhd^\Omega \mathbb{k}_2$*. Let* $\mathbb{c}_{ij} = \mathbb{h}_i \cap \mathbb{k}_j$*. Then* $(\mathbb{h}_1 + \mathbb{c}_{21}) \lhd^\Omega (\mathbb{h}_1 + \mathbb{c}_{22})$ *and* $(\mathbb{k}_1 + \mathbb{c}_{12}) \lhd^\Omega (\mathbb{k}_1 + \mathbb{c}_{22})$*. Moreover, the Lie* Ω*-algebras* $(\mathbb{h}_1 + \mathbb{c}_{22})/(\mathbb{h}_1 + \mathbb{c}_{21})$ *and* $(\mathbb{k}_1 + \mathbb{c}_{22})/(\mathbb{k}_1 + \mathbb{c}_{12})$ *are* Ω*-isomorphic.*

Proof. Since $\mathbb{k}_1 \lhd^\Omega \mathbb{k}_2$, we have $\mathbb{c}_{21} \lhd^\Omega \mathbb{c}_{22}$. Since also $\mathbb{h}_1 \lhd^\Omega \mathbb{h}_2$, it follows that $(\mathbb{h}_1 + \mathbb{c}_{21}) \lhd^\Omega (\mathbb{h}_1 + \mathbb{c}_{22})$ by Lemma 2.11. Similarly $(\mathbb{k}_1 + \mathbb{c}_{12}) \lhd^\Omega (\mathbb{k}_1 + \mathbb{c}_{22})$. Apply Corollary 2.10 with $\mathbb{m} = \mathbb{c}_{22}$ and $\mathbb{n} = \mathbb{h}_1 + \mathbb{c}_{21}$, noting that $\mathbb{n} + \mathbb{m} = \mathbb{h}_1 + \mathbb{c}_{22}$ and $\mathbb{n} \cap \mathbb{m} = \mathbb{c}_{12} + \mathbb{c}_{21}$ by Lemma 2.12. The conclusion is that $(\mathbb{h}_1 + \mathbb{c}_{22})/(\mathbb{h}_1 + \mathbb{c}_{21}) \simeq^\Omega \mathbb{c}_{22}/(\mathbb{c}_{12} + \mathbb{c}_{21})$. Similarly $(\mathbb{k}_1 + \mathbb{c}_{22})/(\mathbb{k}_1 + \mathbb{c}_{12}) \simeq^\Omega \mathbb{c}_{22}/(\mathbb{c}_{12} + \mathbb{c}_{21})$, thus the result follows. □

3. Ω-composition series, chain conditions and Ω-direct decompositions

Definition 3.1. Let $\mathbb{g}$ be a Lie Ω-algebra. An Ω-series (of finite length) in $\mathbb{g}$ is a finite sequence of Ω-subalgebras including $\{0\}$ and $\mathbb{g}$ such that each member of the sequence is an Ω-ideal of its successor: thus a series can be written

$$\{0\} = \mathbb{g}_0 \lhd^\Omega \mathbb{g}_1 \lhd^\Omega \cdots \lhd^\Omega \mathbb{g}_l = \mathbb{g}.$$

The $\mathbb{g}_i$ are called terms and the Ω-quotient algebras $\mathbb{g}_{i+1}/\mathbb{g}_i$ are called factors. If all the $\mathbb{g}_i$ are distinct, then integer l is called the length of this Ω-series. When Ω is empty, we shall simply speak of a series.

Definition 3.2. Let $\mathcal{S}$ and $\mathcal{T}$ be two Ω-series of a Lie Ω-algebra $\mathbb{g}$. We call $\mathcal{S}$ a refinement of $\mathcal{T}$ if every term of $\mathcal{T}$ is also a term of $\mathcal{S}$. If there is at least one term of $\mathcal{S}$ which is not a term of $\mathcal{T}$, then $\mathcal{S}$ is a proper refinement of $\mathcal{T}$.

Remark 3.3. Clearly the relation of refinement is a partial ordering of the set of all Ω-series of $\mathbb{g}$.

Definition 3.4. Two Ω-series $\mathcal{S}$ and $\mathcal{T}$ of a Lie Ω-algebra $\mathbb{g}$ are said to be Ω-isomorphic if there is a bijection from the set of factors of $\mathcal{S}$ to the set of factors of $\mathcal{T}$ such that corresponding factors are Ω-isomorphic.

We now have the fundamental result on refinements.

Proposition 3.5. *Any two* Ω*-series of a Lie* Ω*-algebra possess* Ω*-isomorphic refinements.*

Proof. Let $\{0\} = \mathbb{g}_0 \lhd^\Omega \mathbb{g}_1 \lhd^\Omega \cdots \lhd^\Omega \mathbb{g}_l = \mathbb{g}$ and $\{0\} = \mathbb{c}_0 \lhd^\Omega \mathbb{c}_1 \lhd^\Omega \cdots \lhd^\Omega \mathbb{c}_m = \mathbb{g}$ be two Ω-series of $\mathbb{g}$. Define $\mathbb{g}_{ij} = \mathbb{g}_i + (\mathbb{g}_{i+1} \cap \mathbb{c}_j)$ and $\mathbb{c}_{ij} = \mathbb{c}_j + (\mathbb{g}_i \cap \mathbb{c}_{j+1})$. Apply Proposition 2.13 with $\mathbb{h}_1 = \mathbb{g}_i, \mathbb{h}_2 = \mathbb{g}_{i+1}, \mathbb{k}_1 = \mathbb{c}_j$, and $\mathbb{k}_2 = \mathbb{c}_{j+1}$, we have

$$\mathbb{g}_{ij} \lhd^\Omega \mathbb{g}_{ij+1}, \mathbb{c}_{ij} \lhd^\Omega \mathbb{c}_{i+1j}$$

and $\mathbb{g}_{ij+1}/\mathbb{g}_{ij} \simeq^\Omega \mathbb{c}_{i+1j}/\mathbb{c}_{ij}$. Hence the series $\{\mathbb{g}_{ij} \mid i = 0, \ldots, l-1, j = 0, \ldots, m\}$ and $\{\mathbb{c}_{ij} \mid i = 0, \ldots, l, j = 0, \ldots, m-1\}$ are Ω-isomorphic refinements of $\{\mathbb{g}_i \mid i = 0, \ldots, l\}$ and $\{\mathbb{c}_j \mid j = 0, \ldots, m\}$ respectively. □

Definition 3.6. An Ω-series which has no proper refinements is called an Ω-composition series.

Remark 3.7. It is clear that we shall arrive at an Ω-composition series of a Lie Ω-algebra of finite dimension if we repeatedly refine any given series. If Ω is empty, we speak of a composition series.

The following theorem pointed out that an Ω-composition series can be recognized by the structure of its factors.

Theorem 3.8. *An Ω-series is an Ω-composition series if and only if all its factors are Ω-simple.*

Proof. If some factor $\mathbb{h}/\mathbb{k}$ of an Ω-series of a Lie Ω-algebra $\mathbb{g}$ is not Ω-simple, it possesses a nontrivial Ω-ideal $\mathbb{c}/\mathbb{k}$ where $\mathbb{k} \lhd^\Omega \mathbb{c} \lhd^\Omega \mathbb{h}$. Adjunction of $\mathbb{c}$ to the series produces a proper refinement, so the initial series is not an Ω-composition series. Conversely, if an Ω-series is not an Ω-composition series, it has a proper refinement and there exist consecutive term $\mathbb{k} \lhd^\Omega \mathbb{h}$ and an Ω-subalgebra $\mathbb{c}$ of $\mathbb{g}$ with $\mathbb{k} \lhd^\Omega \mathbb{c} \lhd^\Omega \mathbb{h}$. But $\mathbb{c}/\mathbb{k}$ is an Ω-ideal of $\mathbb{h}/\mathbb{k}$ and the latter cannot be Ω-simple. □

Corollary 3.9. *If $\mathcal{S}$ is an Ω-composition series and $\mathcal{T}$ is any Ω-series of a Lie Ω-algebra $\mathbb{g}$, then $\mathcal{T}$ has a refinement which is an Ω-composition series and is Ω-isomorphic with $\mathcal{S}$. In particular, if $\mathcal{T}$ is an Ω-composition series, it is Ω-isomorphic with $\mathcal{S}$.*

Remark 3.10. Corollary 3.9 indicated that the factors of an Ω-composition series are independent of the series and constitute a set of invariants of the Lie algebra, the Ω-composition factors of $\mathbb{g}$. Also all Ω-composition series of $\mathbb{g}$ have the same length, the Ω-composition length of $\mathbb{g}$.

We associate with each Lie Ω-algebra $\mathbb{g}$ a set $\mathcal{F}(\mathbb{g})$ of Ω-subalgebras such that if $\varphi : \mathbb{g} \to \mathbb{h}$ is an Ω-isomorphism, $\mathcal{F}(\mathbb{h}) = \{\varphi(X) | X \in \mathcal{F}(\mathbb{g})\}$. For example, $\mathcal{F}(\mathbb{g})$ might consist of all Ω-subalgebras or of all Ω-ideals of $\mathbb{g}$. Obverse that $\mathcal{F}(\mathbb{g})$ is a partially ordered set with respect to set containment, so we may apply it to the notion of a chain condition.

Definition 3.11. A Lie Ω-algebra $\mathbb{g}$ satisfies the ascending chain condition and the descending chain condition on Ω-subalgebras if the partially ordered set $\mathcal{F}(\mathbb{g})$ satisfies the corresponding chain conditions respectively.

Example 3.12. 1. Let $\mathcal{F}(\mathbb{g})$ be the set of all Ω-subalgebras of a Lie Ω-algebra $\mathbb{g}$. We obtain the ascending chain condition and descending chain condition on Ω-subalgebras, denoted by acc-Ω and dcc-Ω. When Ω is empty, we simply write acc and dcc, and speak of the ascending chain condition and descending chain condition on subalgebras.

2. Let $\mathcal{F}(\mathbb{g})$ be the set of all Ω-ideals; this is the case which concerns us here since the corresponding properties acc-Ωi and dcc-Ωi are intimately related to the question of the existence of an Ω-composition series.

Theorem 3.13. *A Lie Ω-algebra* $\mathbb{g}$ *has an Ω-composition series if and only if it satisfies* acc-Ω*i and* dcc-Ω*i.*

Proof. Suppose that $\mathbb{g}$ has an Ω-composition series of length l but that nevertheless there exists an infinite ascending chain $\mathbb{h}_1 \lhd^{\Omega} \mathbb{h}_2 \lhd^{\Omega} \cdots$ of Ω-ideal of $\mathbb{g}$. Consider the chain $\{0\} = \mathbb{h}_0 \lhd^{\Omega} \mathbb{h}_1 \lhd^{\Omega} \cdots \lhd^{\Omega} \mathbb{h}_{l+1}$; since $\mathbb{h}_i$ is an Ω-ideal of $\mathbb{g}$, it is an Ω-ideal in $\mathbb{h}_{i+1}$. Hence our chain can be made into an Ω-series of $\mathbb{g}$ by inserting terms of a suitable Ω-series between $\mathbb{h}_i$ and $\mathbb{h}_{i+1}$ and between $\mathbb{h}_l$ and $\mathbb{g}$. The length of the resulting series is at least $l+1$ but cannot exceed the composition length by Corollary 3.9, a contradiction. In a similar manner we may prove that $\mathbb{g}$ has dcc-Ωi.

Now assume that $\mathbb{g}$ has acc-Ωi and dcc-Ωi but does not have an Ω-composition series. Apply acc-Ωi to the set of proper Ω-ideals of $\mathbb{g}$, noting that dimension of $\mathbb{g}$ is not 1, and select a maximal member $\mathbb{g}_1$; then $\mathbb{g}/\mathbb{g}_1$ is Ω-simple. Now $\dim \mathbb{g}_1 \neq 1$ since $\mathbb{g}$ has no Ω composition series, and by acc-Ωi again we may choose a maximal proper Ω-ideal $\mathbb{g}_2$ of $\mathbb{g}_1$. Again $\mathbb{g}_1/\mathbb{g}_2$ is Ω-simple and $\dim \mathbb{g}_2 \neq 1$. This process cannot terminate, so there is an infinite descending chain of Ω-ideals of the form

$$\cdots \lhd^{\Omega} \mathbb{g}_2 \lhd^{\Omega} \mathbb{g}_1 \lhd^{\Omega} \mathbb{g}_0 = \mathbb{g}$$

in contradiction to dcc-Ωi. □

Definition 3.14. Let $\mathbb{g}$ be a Lie Ω-algebra. An Ω-subalgebra $\mathbb{h}$ is called an Ω-direct factor of $\mathbb{g}$ if there exists an Ω-subalgebra $\mathbb{k}$ such that $\mathbb{g} = \mathbb{h} \oplus \mathbb{k}$. In this situation, $\mathbb{k}$ is called an Ω-direct complement of $\mathbb{h}$ in $\mathbb{g}$. If there are no proper nontrivial Ω-direct factors of $\mathbb{g}$, then $\mathbb{g}$ is said to be Ω-indecomposable (or just indecomposable if $\Omega = \emptyset$).

Note that Lie Ω-simple algebra is Ω-indecomposable. We consider chain conditions on the set of direct factors.

Proposition 3.15. *For a Lie Ω-algebra* $\mathbb{g}$*, the ascending chain condition and the descending chain condition on the Ω-direct factors are equivalent properties.*

Proof. Assume that $\mathbb{g}$ is a Lie Ω-algebra satisfying the descending chain condition on the Ω-direct factors; let $\mathcal{O}$ be a nonempty set of Ω-direct factors of $\mathbb{g}$. We will show that $\mathcal{O}$ has a maximal element, so that $\mathbb{g}$ satisfies the ascending chain condition on Ω-direct factors.

Let $\mathcal{P}$ be the set of all Ω-subalgebras of $\mathbb{g}$ which are direct complement of at least one element of $\mathcal{O}$. Then $\mathcal{P}$ has a minimal element $\mathbb{n}$ and $\mathbb{g} = \mathbb{m} \oplus \mathbb{n}$ for some $\mathbb{m} \in \mathcal{O}$. If $\mathbb{m}$ is not maximal in $\mathcal{O}$, there exists $\mathbb{m}_1 \in \mathcal{O}$ such that $\mathbb{m} \subset \mathbb{m}_1$; then $\mathbb{g} = \mathbb{m}_1 \oplus \mathbb{n}_1$ for some $\mathbb{n}_1 \in \mathcal{P}$. Now $\mathbb{m}_1 = \mathbb{m}_1 \cap (\mathbb{m} \oplus \mathbb{n}) = \mathbb{m} \oplus (\mathbb{m}_1 \cap \mathbb{n})$ by Lemma 2.12, whence $\mathbb{g} = \mathbb{m}_1 \oplus \mathbb{n}_1 = \mathbb{m} \oplus \mathbb{n}_1 \oplus (\mathbb{m}_1 \cap \mathbb{n})$. Intersecting with $\mathbb{n}$ we obtain $\mathbb{n} = \mathbb{n}_2 \oplus (\mathbb{m}_1 \cap \mathbb{n})$ where $\mathbb{n}_2 = (\mathbb{m} \oplus \mathbb{n}_1) \cap \mathbb{n}$. Hence $\mathbb{g} = \mathbb{m} \oplus \mathbb{n} = (\mathbb{m} \oplus (\mathbb{m}_1 \cap \mathbb{n})) \oplus \mathbb{n}_2 = \mathbb{m}_1 \oplus \mathbb{n}_2$. It follows that $\mathbb{n}_2 \in \mathcal{P}$ and hence that $\mathbb{n}_2 = \mathbb{n}$ by minimality of $\mathbb{n}$ in $\mathcal{P}$. Therefore $\mathbb{n} \subseteq \mathbb{m} \oplus \mathbb{n}_1$ and $\mathbb{g} = \mathbb{m} \oplus \mathbb{n} = \mathbb{m} \oplus \mathbb{n}_1 = \mathbb{m}_1 \oplus \mathbb{n}_1$. Since $\mathbb{m} \subseteq \mathbb{m}_1$, we get $\mathbb{m} = \mathbb{m}_1$, a contradiction. The converse implication is proved in an analogous way. □

Definition 3.16. A Lie Ω-algebra is said to have an Ω-direct decomposition if it can be expressed as a direct sum of finitely many nontrivial Ω-indecomposable subalgebras.

The following theorem is the main result of this section.

Theorem 3.17. *If a Lie Ω-algebra $\mathbb{g}$ has the descending chain condition* (*or ascending chain condition*) *on the Ω-direct factors, then it has an Ω-direct decomposition.*

Proof. By Theorem 3.13 and Proposition 3.15, $\mathbb{g}$ has an Ω-composition series. Assume that $\mathbb{g}$ can not be expressed as a direct sum of finitely many nontrivial Ω-indecomposable subalgebras. Then $\mathbb{g}$ is certainly Ω-decomposable, so the set $\mathcal{O}$ of all proper nontrivial Ω-direct factors of $\mathbb{g}$ is not empty. Choose a minimal element $\mathbb{g}_1$ of $\mathcal{O}$ and write $\mathbb{g} = \mathbb{g}_1 \oplus \mathbb{h}_1$. Then $\mathbb{g}_1$ is Ω-indecomposable by minimality. Clearly $\mathbb{h}_1$ inherits the descending chain condition from $\mathbb{g}$ and cannot be indecomposable. Hence $\mathbb{h}_1 = \mathbb{g}_2 \oplus \mathbb{h}_2 \supset \mathbb{g}_2$ where $\mathbb{g}_2$ is Ω-indecomposable, and $\mathbb{g} = \mathbb{g}_1 \oplus \mathbb{g}_2 \oplus \mathbb{h}_2$. Repetition of this procedure leads to an infinite descending chain $\mathbb{h}_1 \supset \mathbb{h}_2 \supset \cdots$ of Ω-direct factors of $\mathbb{g}$, which cannot exist. By Proposition 3.15, the result follows. □

4. Uniqueness of Ω-direct decompositions

Definition 4.1. An Ω-endomorphism of a Lie Ω-algebra $\mathbb{g}$ is said to be normal if it commutes with all inner derivations of $\mathbb{g}$.

An Ω-endomorphism π of a Lie Ω-algebra $\mathbb{g}$ is said to be normally idempotent if it is normal and a projection, that is, $\pi = \pi^2$. It is clear that $\pi(\mathbb{g})$ is an Ω-ideal of $\mathbb{g}$.

Example 4.2. Let $\mathbb{g}$ be a Lie Ω-algebra with an Ω-direct decomposition $\mathbb{g} = A \oplus B$, and π be the projection into A with respect to this decomposition. Then π is normally idempotent Ω-endomorphism.

In fact, for any $x = x_1 + x_2, y = y_1 + y_2, x_1, y_1 \in A$ and $x_2, y_2 \in B$, we have $\pi \mathrm{ad}x(y) = [x_1, y_1] = \mathrm{ad}x\pi(y)$. Moreover, for all $\omega \in \Omega, x \in \mathbb{g}$, we have $\pi\omega(x) = \pi(\omega(x_1) + \omega(x_2)) = \omega(x_1)$ and $\omega(\pi(x)) = \omega(\pi(x_1 + x_2)) = \omega(x_1)$. So $\pi\omega = \omega\pi$ and π is an Ω-homomorphism of $\mathbb{g}$. It is clear that $\pi = \pi^2$. Hence π is normally idempotent.

The following two lemmas are well-known.

Lemma 4.3. *Let* g *be a Lie algebra. The mapping* $\mathrm{ad} : x \mapsto \mathrm{ad}x$ *defines a representation of* g*, where* $\mathrm{ad}x : \mathrm{g} \to \mathrm{g}, y \mapsto [x, y]$ *for all* $y \in \mathrm{g}$.

Lemma 4.4. *Let* g *be a Lie algebra. If* $\phi \in \mathrm{Autg}$*, then* $\phi \cdot \mathrm{ad}x \cdot \phi^{-1} = \mathrm{ad}(\phi(x))$ *for all* $x \in \mathrm{g}$.

Proposition 4.5. *Let* g *be a Lie* Ω*-algebra with* $\Omega = \mathrm{Autg}$*. If* ϕ *is an* Ω*-endomorphism of* g*. Then* ϕ *is normal if and only if* $\phi(x) - x \in C(\mathrm{g})$ *for all* $x \in \mathrm{g}$.

Proof. For all $x \in \mathrm{g}$, $\phi \in \mathrm{Autg} = \Omega$, by Lemma 4.4 we have

$$\mathrm{ad}x \in \mathrm{ad}_\Omega \mathrm{g} \Leftrightarrow \phi(\mathrm{ad}x) = (\mathrm{ad}x)\phi = (\mathrm{ad}(\phi(x)))\phi \Leftrightarrow \mathrm{ad}x = \mathrm{ad}(\phi(x)).$$

That means that $\mathrm{ad}x \in \mathrm{ad}_\Omega \mathrm{g} \Leftrightarrow \phi(x) - x \in \mathrm{Ker}(\mathrm{ad}) = C(\mathrm{g})$ by Lemma 4.3. $\square$

Lemma 4.6. *If* g *is a Lie* Ω*-algebra, then there is a bijection between the set of all finite* Ω*-direct decompositions of* g *and the set of all finite sets of normal* Ω*-endomorphisms of* g*,* $\{\pi_1, \dots, \pi_r\}$ *satisfying*

$$\begin{cases} \pi_1 + \cdots + \pi_r = 1 \\ \pi_i \pi_j = 0, (i \neq j). \end{cases} \tag{4.1}$$

Proof. Let $\mathrm{g} = \mathrm{g}_1 \oplus \cdots \oplus \mathrm{g}_r$ be an Ω-direct decomposition of g, where g_i $(1 \leq i \leq r)$ is an Ω-subalgebra of g, then each x in g is uniquely expressible in the form $x = x_1 + \cdots + x_r$ with $x_i \in \mathrm{g}_i, i = 1, \dots, r$. The endomorphism π_i defined by $\pi_i(x) = x_i$ is normal: for clearly $\pi_i^2 = \pi_i$ and

$$\begin{aligned} \pi_i(\mathrm{ad}y(x)) &= [\pi_i(y), \pi_i(x)] = [y_i, x_i] \\ &= \mathrm{ad}y_i(x_i) = \mathrm{ad}y(x_i) \\ &= \mathrm{ad}y(\pi_i(x)) \end{aligned}$$

for all $y \in \mathrm{g}$. In addition $x = x_1 + \cdots + x_r = (\pi_1 + \cdots + \pi_r)(x)$ and $\pi_i(\pi_j(x)) = 0$ if $i \neq j$. Thus the π_i satisfy (4.1).

Conversely, consider some normal Ω-endomorphisms, $\pi_1, \dots, \pi_n$ satisfying the conditions in (4.1). Then $\pi_i = \pi_i(\pi_1 + \cdots + \pi_n) = \pi_i^2$, so that π_i is a projection. Let $\mathrm{g}_i = \pi_i(\mathrm{g})$, an Ω-ideal of g. Now $\pi_1 + \cdots + \pi_n = 1$ implies that $\mathrm{g} = \mathrm{g}_1 \oplus \cdots \oplus \mathrm{g}_n$. Furthermore, if $x \in \mathrm{g}_i \cap \sum_{j \neq i} \mathrm{g}_j$, then $x = \pi_i(y)$ and $\pi_i(x) = \pi_i(y) = x$; but $\pi_i(x) = 0$ since $\pi_i(\mathrm{g}_j) = 0$ if $i \neq j$, so in fact $x = 0$. Hence $\mathrm{g} = \mathrm{g}_1 \oplus \cdots \oplus \mathrm{g}_n$. $\square$

Lemma 4.7. *Let* ϕ *be a normal* Ω*-endomorphism of a Lie* Ω*-algebra* g *and suppose that* g *satisfies the* dcc-Ωi *and* acc-Ωi*. Then there exists a positive integer* r *such that* $\mathrm{Im}\phi^r = \mathrm{Im}\phi^{r+1} = \cdots$*, and* $\mathrm{Ker}\phi^r = \mathrm{Ker}\phi^{r+1} = \cdots$*. Thus* $\mathrm{g} = \mathrm{Im}\phi^r \oplus \mathrm{Ker}\phi^r$.

Proof. Since ϕ is normal, it is clear that ϕ^i is also normal and $\mathrm{Im}\phi^i$ is an Ω-subalgebra of g. Since $\mathrm{ad}x\phi^i = \phi^i\mathrm{ad}x$ for all $x \in \mathrm{g}$, so $\mathrm{ad}x\phi^i(y) = \phi^i\mathrm{ad}x(y)$

for all $y \in \mathbb{g}$, i.e., $[\phi^i(y), x] = -\phi^i(\mathrm{ad}x(y)) \in \phi^i(\mathbb{g})$. Thus $\mathrm{Im}\phi^i$ is an Ω-ideal of $\mathbb{g}$. Similarly, $\mathrm{Ker}\phi^i$ is an Ω-ideal of $\mathbb{g}$. Clearly $\mathrm{Ker}\phi \subseteq \mathrm{Ker}\phi^2 \subseteq \cdots$ and $\mathrm{Im}\phi \supseteq \mathrm{Im}\phi^2 \supseteq \cdots$, so there is a positive integer r such that $\mathrm{Ker}\phi^r = \mathrm{Ker}\phi^{r+1} = \cdots = \mathbb{m}$ and $\mathrm{Im}\phi^r = \mathrm{Im}\phi^{r+1} = \cdots = \mathbb{n}$, say. Let $x \in \mathbb{g}$; then $\phi^r(x) \in \mathrm{Im}\phi^r = \mathrm{Im}\phi^{2r}$ and $\phi^r(x) = \phi^{2r}(y)$ for some $y \in \mathbb{g}$. Hence $x - \phi^r(y) \in \mathbb{m}$ and $x \in \mathbb{m} + \mathbb{n}$, which shows that $\mathbb{g} = \mathbb{m} + \mathbb{n}$. Next, if $x \in \mathbb{m} \cap \mathbb{n}$, then $x = \phi^r(y)$ with $y \in \mathbb{g}$. Therefore $0 = \phi^r(x) = \phi^{2r}(y)$, whence $y \in \mathrm{Ker}\phi^{2r} = \mathrm{Ker}\phi^r$ and $x = \phi^r(y) = 0$. It follows that $\mathbb{g} = \mathbb{m} \oplus \mathbb{n}$. □

Definition 4.8. An endomorphism ϕ is said to be nilpotent if $\phi^r = 0$ for some positive integer r.

Proposition 4.9. *If* $\mathbb{g}$ *is an indecomposable Lie* Ω*-algebra satisfying the* dcc-Ωi *and* acc-Ωi*, then a normal* Ω*-endomorphism of* $\mathbb{g}$ *is either nilpotent or an* Ω*-automorphism.*

Proof. Let ϕ be a normal Ω-endomorphism of $\mathbb{g}$. By Lemma 4.7, there is an $r > 0$ such that $\mathbb{g} = \mathrm{Im}\phi^r \oplus \mathrm{Ker}\phi^r$. But $\mathbb{g}$ is Ω-indecomposable, so either $\mathrm{Im}\phi^r = 0$ and $\phi^r = 0$ or $\mathrm{Im}\phi^r = \mathbb{g}$ and $\mathrm{Ker}\phi^r = 0$; in the latter case ϕ is an Ω-automorphism. □

Definition 4.10. An Ω-endomorphism ϕ of a Lie Ω-algebra $\mathbb{g}$ is said to be central if $\phi(x) - x \in C(\mathbb{g})$ for all $x \in \mathbb{g}$.

Proposition 4.11. *Let* ϕ *be an* Ω*-endomorphism of a Lie* Ω*-algebra* $\mathbb{g}$*. Then the following statements hold*:

(1) *If* ϕ *is central, then it is normal.*
(2) *If* ϕ *is surjective and normal, then* ϕ *is central.*
(3) *If* ϕ *is an* Ω*-automorphism, then* ϕ *is normal if and only if* ϕ *is central.*

Proof. (1) For all $x, y \in \mathbb{g}$, $\phi(\mathrm{ad}y(x)) - \mathrm{ad}y(\phi(x)) = [\phi(y) - y, \phi(x)] = 0$ because ϕ is central. Thus $\phi(\mathrm{ad}y) = \mathrm{ad}y(\phi)$ and ϕ is an Ω-projection. Proposition 4.5 implies that ϕ is normal.

(2) Since ϕ is surjective and normal, so $[\phi(y), \phi(x)] = \phi(\mathrm{ad}y(x)) = \mathrm{ad}y(\phi(x)) = [y, \phi(x)]$ for all $x, y \in \mathbb{g}$. That is, $[\phi(y) - y, \phi(x)] = 0$. Since $\mathbb{g} = \phi(\mathbb{g})$, this implies that $\phi(y) - y \in C(\mathbb{g})$ and ϕ is central.

(3) It is immediate from statements (1) and (2). □

Lemma 4.12. *Let* $\mathbb{g}$ *be an indecomposable Lie* Ω*-algebra satisfying the* dcc-Ωi *and* acc-Ωi*. Suppose that* $\phi_1, \ldots, \phi_k$ *are normal* Ω*-endomorphisms of* $\mathbb{g}$*. If* $\phi_1 + \cdots + \phi_k$ *is an* Ω*-automorphism, then so is at least one* ϕ_i*.*

Proof. By induction we may assume that $k = 2$ and $\alpha = \phi_1 + \phi_2$ is an Ω-automorphism. Put $\psi_i = \alpha^{-1}\phi_i$, so that $\psi_1 + \psi_2 = id$. Now α is normal since ϕ_1 and ϕ_2 are; hence ψ_1 and ψ_2 are also normal. Suppose that neither ϕ_1 nor ϕ_2 is an Ω-automorphism; then neither ψ_1 nor ψ_2 can be an Ω-automorphism. By Proposition 4.9, both ψ_1 and ψ_2 are nilpotent, so $\psi_1^r = 0 = \psi_2^r$ for some $r > 0$. Now $\psi_1 = id - \psi_2$, so $\psi_1\psi_2 = \psi_2\psi_1$. Hence $id = (\psi_1 + \psi_2)^{2r-1} =$

$\sum_{i=0}^{2r-1} C_{2r-1}^{i}\psi_1^i\psi_2^{2r-i-1}$ by the binomial theorem. Since either $i \geq r$ or $2r - i - 1 \geq r$, we have $\psi_1^i\psi_2^{2r-i-1} = 0$ for all i. Hence $id = 0$, which implies that $\mathbb{g} = \{0\}$ and $\phi_1 = id = \phi_2$, a contradiction. □

The following theorem is the main result of this section.

Theorem 4.13. *Let* $\mathbb{g}$ *be a Lie* Ω*-algebra satisfying the* dcc-Ωi *and* acc-Ωi. *If*

$$\mathbb{g} = \mathbb{h}_1 \oplus \cdots \oplus \mathbb{h}_r = \mathbb{n}_1 \oplus \cdots \oplus \mathbb{n}_s$$

are two Ω*-direct decompositions, then* $r = s$ *and there is a central* Ω*-automorphism* ϕ *of* $\mathbb{g}$ *such that, after suitable relabeling of the* $\mathbb{n}_j$*'s if necessary,* $\phi(\mathbb{h}_i) = \mathbb{n}_i$ *and* $\mathbb{g} = \mathbb{h}_1 \oplus \cdots \oplus \mathbb{h}_k \oplus \mathbb{n}_{k+1} \oplus \cdots \oplus \mathbb{n}_r$ *for* $k = 1, \ldots, r$.

Proof. Assume that for some k satisfying $1 \leq k \leq \max\{r, s\}$ there is an Ω-direct decomposition $\mathbb{g} = \mathbb{h}_1 \oplus \cdots \oplus \mathbb{h}_{k-1} \oplus \mathbb{n}_k \oplus \cdots \oplus \mathbb{n}_r$. Certainly this is true if $k = 1$. Let $\{\sigma_1, \ldots, \sigma_r\}$ be the set of projections specifying this decomposition, and let $\{\pi_1, \ldots, \pi_r\}$ and $\{\rho_1, \ldots, \rho_s\}$ be the corresponding sets of projections for the decompositions $\mathbb{g} = \mathbb{h}_1 \oplus \cdots \oplus \mathbb{h}_r$ and $\mathbb{g} = \mathbb{n}_1 \oplus \cdots \oplus \mathbb{n}_s$. If $x \in \mathbb{g}$, then $\rho_j(x) \in \mathbb{n}_j$ and $\sigma_k\rho_j(x) = 0$ if $j < k$. Hence $\sigma_k\rho_j = 0$ if $j < k$. Since $\sigma_k = \sigma_k \cdot id = \sigma_k \cdot (\rho_1 + \cdots + \rho_s)$, we obtain

$$\sigma_k\rho_k + \sigma_k\rho_{k+1} + \cdots + \sigma_k\rho_s = \sigma_k. \tag{4.2}$$

Consider the restriction of $\sigma_k\rho_j$ to $\mathbb{h}_k$, certainly a normal Ω-endomorphism of $\mathbb{h}_k$. Now $\mathbb{h}_k$ inherits the ascending chain and descending chain conditions from $\mathbb{g}$ and the restriction of σ_k to $\mathbb{h}_k$ is, of course, id. By (4.2) and Lemma 4.12, some $\sigma_k\rho_j, k \leq j \leq s$, is an Ω-automorphism on $\mathbb{h}_k$. The $\mathbb{n}_j$ can be labeled in such a way that $\sigma_k\rho_k$ is an Ω-automorphism on $\mathbb{h}_k$.

Let $\bar{\mathbb{n}}_k = \rho_k(\mathbb{h}_k) \subseteq \mathbb{n}_k$. Then $\bar{\mathbb{n}}_k \lhd \mathbb{n}_k$ since ρ_k is normal. If $\rho_k(y) = 0$ with $y \in \mathbb{h}_k$, then $\sigma_k\rho_k(y) = 0$ and $y = 0$, thus ρ_k maps $\mathbb{h}_k$ isomorphically onto $\bar{\mathbb{n}}_k$. For the same reason σ_k maps $\bar{\mathbb{n}}_k$ monomorphically into $\mathbb{h}_k$. Write $\tilde{\mathbb{n}}_k = \mathrm{Ker}\sigma_k \cap \mathbb{n}_k$; then $\mathbb{n}_k \cap \bar{\mathbb{n}}_k = \{0\}$. Also, for $x \in \mathbb{n}_k$ we have $\sigma_k(x) \in \mathbb{h}_k$ and hence $\sigma_k(x) = \sigma_k\rho_k(y)$ for some y in $\mathbb{h}_k$; thus $x - \rho_k(y) \in \tilde{\mathbb{n}}_k$, and $x \in \tilde{\mathbb{n}}_k + \bar{\mathbb{n}}_k$. Consequently $\mathbb{n}_k = \tilde{\mathbb{n}}_k \oplus \bar{\mathbb{n}}_k$. But $\mathbb{n}_k$ is Ω-indecomposable, and $\bar{\mathbb{n}}_k \simeq \mathbb{h}_k \neq \{0\}$, hence $\tilde{\mathbb{n}}_k = \{0\}$ and $\bar{\mathbb{n}}_k = \mathbb{n}_k$. It follows that ρ_k maps $\mathbb{h}_k$ isomorphically to $\mathbb{n}_k$.

Next write $\mathbb{c}_k = \mathbb{n}_1 \oplus \cdots \oplus \mathbb{n}_{k-1} \oplus \mathbb{h}_{k+1} \oplus \cdots \oplus \mathbb{h}_r$, so that $\mathbb{g} = \mathbb{c}_k \oplus \mathbb{h}_k$. The proof proceeds by showing that $\mathbb{g} = \mathbb{c}_k \oplus \mathbb{n}_k$. Firstly $\sigma_k(\mathbb{c}_k) = \{0\}$ and $\mathbb{c}_k \cap \mathbb{n}_k = \{0\}$. Next define $\phi = \rho_k\sigma_k + (\mathrm{id} - \sigma_k)$, a normal Ω-endomorphism of $\mathbb{g}$. If $x = y + z$ where $y \in \mathbb{c}_k, z \in \mathbb{h}_k$, then $\phi(x) = \phi(y) + \phi(z) = y + \rho_k(z)$ since $\sigma_k(y) = 0$ and $\sigma_k(z) = z$. Hence $\phi(x) = 0$ implies that $y = 0 = \rho_k(z)$ (because $\mathbb{c}_k \cap \mathbb{n}_k = \{0\}$); since ρ_k is monomorphic on $\mathbb{n}_k$, we conclude that $y = 0 = z$. Hence ϕ is a monomorphism. It follows form Lemma 4.7 that ϕ is an Ω-automorphism and therefore $\mathbb{g} = \phi(\mathbb{g}) \subseteq \rho_k\sigma_k(\mathbb{g}) + (\mathrm{id} - \sigma_k)(\mathbb{g}) \subseteq \mathbb{n}_k \oplus \mathbb{c}_k$ and $\mathbb{g} = \mathbb{n}_k \oplus \mathbb{c}_k$. This is just to say that $\mathbb{g} = \mathbb{n}_1 \oplus \cdots \oplus \mathbb{n}_k \oplus \mathbb{h}_{k+1} \oplus \cdots \oplus \mathbb{h}_r$, so far we have proved that there is an Ω-decomposition

$$\mathbb{g} = \mathbb{n}_1 \oplus \cdots \oplus \mathbb{n}_k \oplus \mathbb{h}_{k+1} \oplus \cdots \oplus \mathbb{h}_r$$

for $1 \leq k \leq \max\{r, s\}$, after relabeling the $\mathbb{n}'_j$s. If we put $k = \min\{r, s\}$, it follows that $r = s$. We also saw that ρ_k maps $\mathbb{h}_k$ isomorphically to $\mathbb{n}_k$. Define $\alpha = \rho_1\pi_1 + \cdots + \rho_r\pi_r$, a normal Ω-endomorphism. Now $\alpha(\mathbb{h}_i) = \rho_i\pi_i(\mathbb{h}_i) = \rho_i(\mathbb{h}_i) = \mathbb{n}_i$, so $\alpha(\mathbb{g}) = \mathbb{g}$. By Proposition 4.9, α is an Ω-automorphism and so by Proposition 4.11, it is central. □

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