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ON EXISTENCE OF WEAK SOLUTIONS OF NEUMANN PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING p-LAPLACIAN IN AN UNBOUNDED DOMAIN

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Abstract. In this paper we study the existence of non-trivial weak solutions of the Neumann problem for quasilinear elliptic equations in the form

$$-\operatorname{div}(h(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = f(x,u), \quad p \geq 2$$

in an unbounded domain $\Omega \subset \mathbb{R}^N, N \geq 3$, with sufficiently smooth bounded boundary $\partial\Omega$, where $h(x) \in L^1_{loc}(\overline{\Omega})$, $\overline{\Omega} = \Omega \cup \partial\Omega$, $h(x) \geq 1$ for all $x \in \Omega$. The proof of main results rely essentially on the arguments of variational method.

1. Introduction and preliminaries results

We are concerned with the study of a Neumann problem of the type

$$\text{(1.1)} \qquad \begin{cases} -\operatorname{div}(h(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad u(x) \to 0 \text{ as } |x| \to +\infty, \end{cases}$$

where $p \geq 2$, $\Omega \subset \mathbb{R}^N, N \geq 3$, is an unbounded domain with sufficiently smooth bounded boundary $\partial\Omega$, $\overline{\Omega} = \Omega \cup \partial\Omega$, n is the outward unit normal to $\partial\Omega$, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function which will be specified later, $h(x)$ and $b(x)$ are satisfied the following conditions:

(H) $h(x) \in L^1_{\text{loc}}(\overline{\Omega})$, $h(x) \geq 1$ for all $x \in \overline{\Omega}$.

(B) $b(x) \in L^\infty_{\text{loc}}(\overline{\Omega})$, $b(x) \geq b_0 > 0$ for all $x \in \overline{\Omega}$.

We first make some comments on the problem (1.1). In the case when Ω is a bounded domain in $\mathbb{R}^N$ or $h(x) = 1$ there were extensive studies in the last decades dealing with the Neumann problems of type (1.1). We just remember the papers [1, 2, 4, 3], [10, 12, 13, 16], where different techniques of finding

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solutions are illustrated. We also find that in the case that $h(x) \in L^1_{loc}(\Omega)$, the quasilinear elliptic equations of type (1.1), with Dirichlet boundary condition, have been studied by D. M. Duc, N. T. Vu ([7]), H. Q. Toan, N. Q. Anh, N. T. Chung (see [15, 14, 5]). The goal of this work we study the existence of weak solutions of Neumann problem for quasilinear elliptic equations with singular coefficients involving the p-Laplace operator of type (1.1) in an unbounded domain $\Omega \subset \mathbb{R}^N$ with sufficiently smooth bounded boundary $\partial\Omega$.

In order to state our main results let us introduce following some hypotheses:

(F1) $f(x,t) \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0, x \in \overline{\Omega}$.

(F2) There exist functions $\tau : \overline{\Omega} \longrightarrow \mathbb{R}$, $\tau(x) \geq 0$ for $x \in \overline{\Omega}$ and constant $r \in (p-1, \frac{N+p}{N-p})$ such that

$$|f'_z(x,z)| \leq \tau(x)|z|^{r-1} \quad \text{for a.e. } x \in \overline{\Omega},$$

$$\tau(x) \in L^\infty(\Omega) \cap L^{r_0}(\Omega), \quad r_0 = \frac{Np}{Np-(r+1)(N-p)}.$$

(F3) There exists $\mu > p$ such that

$$0 < \mu F(x,z) = \mu \int_0^z f(x,t)dt \leq z f(x,z),\ x \in \overline{\Omega},\ z \neq 0.$$

Denote by

$$C_0^\infty(\overline{\Omega}) = \{u \in C^\infty(\overline{\Omega}) : \text{supp } u \text{ compact} \subset \overline{\Omega}\}$$

and $W^{1,p}(\Omega)$ is the usual Sobolev space which can be defined as the completion of $C_0^\infty(\overline{\Omega})$ under the norm

$$||u|| = \left(\int_\Omega (|\nabla u|^p + |u|^p)dx\right)^{\frac{1}{p}}.$$

We now consider following subspace of $W^{1,p}(\Omega)$, defined by

$$H = \left\{u \in W^{1,p}(\Omega) : \int_\Omega (h(x)|\nabla u|^p + b(x)|u|^p)dx < +\infty\right\}$$

and H can be endowed with the norm

$$||u||_H = \left(\int_\Omega h(x)|\nabla u|^p + b(x)|u|^p dx\right)^{\frac{1}{p}}.$$

Applying the method as those used in [14] or [5], we can prove that:

Proposition 1.1. *H is a Banach space. The embedding continuous $H \hookrightarrow W^{1,p}(\Omega)$ holds true.*

Proof. It is clear that H is a normed space. Let $\{u_m\}$ be a Cauchy sequence in H. Then

$$\lim_{m,k\to\infty} \int_\Omega (h(x)|\nabla(u_m - u_k)|^p + b(x)|u_m - u_k|^p)dx = 0$$

and $\{||u_m||_H\}$ is bounded.

Since $||u_m - u_k||_{W^{1,p}(\Omega)} \le \overline{b}||u_m - u_k||_H$, $\overline{b}$ is a positive constant for all m, k, $\{u_m\}$ is also a Cauchy sequence in $W^{1,p}(\Omega)$ and it converges to u in $W^{1,p}(\Omega)$, i.e.,

$$\lim_{m\to+\infty}\int_\Omega (|\nabla u_m - \nabla u|^p + |u_m - u|^p)dx = 0.$$

It follows the sequence $\{\nabla u_m\}$ converges to ∇u and $\{u_m\}$ converges to u in $L^p(\Omega)$. Therefore $\{\nabla u_m(x)\}$ converges to $\nabla u(x)$ and $\{u_m(x)\}$ converges to $\{u(x)\}$ for almost everywhere $x \in \Omega$. Applying Fatou's lemma we get

$$\int_\Omega (h(x)|\nabla u|^p + b(x)|u|^p)dx \le \lim_{m\to+\infty}\inf \int_\Omega (h(x)|\nabla u_m|^p + b(x)|u_m|^p)dx < +\infty.$$

Hence $u \in H$. Applying again Fatou's lemma

$$\begin{aligned} 0 &\le \lim_{m\to+\infty}\int_\Omega (h(x)|\nabla u_m - \nabla u|^p + b(x)|u_m - u|^p)dx \\ &\le \lim_{m\to+\infty}\left[\lim_{k\to+\infty}\inf\int_\Omega (h(x)|\nabla u_m - \nabla u_k|^p + b(x)|u_m - u_k|^p)dx\right] = 0. \end{aligned}$$

Hence $\{u_m\}$ converges to u in H. Thus H is a Banach space and the continuous embedding $H \hookrightarrow W^{1,p}(\Omega)$ holds true. □

Definition 1.1. A function $u \in H$ is a weak solution of the problem (1.1) if and only if

$$\int_\Omega h(x)|\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega b(x)|u|^{p-2}u\varphi dx - \int_\Omega f(x,u)\varphi dx = 0 \tag{1.2}$$

for all $\varphi \in C_0^\infty(\overline{\Omega})$.

Remark 1.1. If $u_0 \in C_0^\infty(\overline{\Omega})$ satisfied the condition (1.2), hence u_0 is a classical solution of the problem (1.1). Indeed, since $u_0 \in C_0^\infty(\overline{\Omega})$, supp u_0 compact, hence there exists $R > 0$ large enough such that $\partial\Omega \subset B_R(0)$, supp $u_0 \subset \overline{\Omega} \cap B_R(0)$ where $B_R(0)$ is ball of radius R.

By denote $\Omega_R = \Omega \cap B_R(0)$, then from (F1) we have

$$\int_{\Omega_R} h(x)|\nabla u_0|^{p-2}\nabla u_0 \nabla\varphi dx + \int_{\Omega_R} b(x)|u_0|^{p-2}u_0\varphi dx - \int_{\Omega_R} f(x,u_0)\varphi dx = 0$$

for all $\varphi \in C_0^\infty(\overline{\Omega})$.

Applying Green's formula and remark that supp $u_0 \subset \overline{\Omega} \cap B_R(0)$ we get

$$\begin{aligned} &\int_{\Omega_R} -\mathrm{div}(h(x)|\nabla u_0|^{p-2}\nabla u_0)\varphi + b(x)|u_0|^{p-2}u_0\varphi)dx \\ &+ \int_{\partial\Omega} h(x)|\nabla u_0|^{p-2}\frac{\partial u_0}{\partial n}\varphi d\sigma - \int_{\Omega_R} f(x,u_0)\varphi dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\overline{\Omega}). \end{aligned}$$

This implies that

$$\int_{\Omega_R} (-\mathrm{div}\ (h(x)|\nabla u_0|^{p-2}\nabla u_0)\varphi + b(x)|u_0|^{p-2}u_0\varphi)dx - \int_{\Omega_R} f(x,u_0)\varphi dx = 0$$

for all $\varphi \in C_0^\infty(\Omega_R)$. From this it follows that

$$\begin{cases} -\text{div}(h(x)|\nabla u_0|^{p-2}\nabla u_0) + b(x)|u_0|^{p-2}u_0 = f(x, u_0) \text{ in } \Omega, \\ \dfrac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Thus u_0 is a classical solution of (1.1).

Our main result given by the following theorem:

Theorem 1.1. *Assuming hypotheses* (F1)-(F3) *are fulfilled then the problem* (1.1) *has at least one nontrivial weak solution in* H.

Theorem 1.1 will be proved by using a variation of the Mountain pass theorem in [6].

2. Existence of a weak solution

We define the functional $J : H \longrightarrow \mathbb{R}$ by

$$\begin{aligned} J(u) &= \frac{1}{p}\int_\Omega h(x)|\nabla u|^p dx + \frac{1}{p}\int_\Omega b(x)|u|^p dx - \int_\Omega F(x, u)dx \\ &= T(u) - P(u), \end{aligned} \tag{2.4}$$

where

$$T(u) = \frac{1}{p}\int_\Omega h(x)|\nabla u|^p dx + \frac{1}{p}\int_\Omega b(x)|u|^p dx$$

and

$$P(u) = \int_\Omega F(x, u)dx.$$

Firstly we remark that, due to the presence of $h(x) \in L^1_{loc}(\overline{\Omega})$, in general, the functional T does not belong to $C^1(H)$. This mean that we cannot apply the classical Mountain pass theorem by Ambrossetti-Rabinowitz. In order to overcome this difficulty, we shall apply a weak version of the Mountain pass theorem introduced by D. M. Duc ([6]). Now we first recall the following useful concept:

Definition 2.1. Let J be a functional from a Banach space Y into $\mathbb{R}$. We say that J is weakly continuously differentiable on Y if and only if three following conditions are satisfied:

(i) J is continuous on Y.

(ii) For any $u \in Y$ there exists a linear map $DJ(u)$ from Y into $\mathbb{R}$ such that
$$\lim_{t\to 0} \frac{J(u + t\varphi) - J(u)}{t} = \big\langle DJ(u), \varphi\big\rangle, \forall\varphi \in Y.$$

(iii) For any $\varphi \in Y$, the map $u \mapsto \big\langle DJ(u), \varphi\big\rangle$ is continuous on Y.

Proposition 2.1. *Assuming hypotheses of Theorem 1.1 are fulfilled. We assert that*

(i) *P is continuous on H. Moreover, P is weakly continuously differentiable on H and*

$$\big\langle DP(u), v\big\rangle = \int_\Omega f(x,u)v\,dx, \quad \forall u, v \in H.$$

(ii) *T is continuous on H.*

(iii) *T is weakly continuously differentiable on H and*

$$\big\langle DT(u), v\big\rangle = \int_\Omega \left(h(x)|\nabla u|^{p-2}\nabla u\nabla v + b(x)|u|^{p-2}uv\right)dx, \quad \forall u, v \in H.$$

Thus $J = T - P$ is weakly continuously differentiable on H and

$$\big\langle DJ(u), v\big\rangle = \int_\Omega \left(h(x)|\nabla u|^{p-2}\nabla u\nabla v + b(x)|u|^{p-2}uv\right)dx - \int_\Omega f(x,u)v\,dx \tag{2.5}$$

$\forall u, v \in H.$

Proof. (i) By hypotheses of Theorem 1.1, applying Theorem C1 in [11, p. 248], we have $P \in C^1(W^{1,p}(\Omega))$. Since the embedding $H \hookrightarrow W^{1,p}(\Omega)$ is continuous, we also have $P \in C^1(H)$ and then P is weakly continuously differentiable on H. Moreover,

$$\big\langle DP(u), v\big\rangle = \int_\Omega f(x,u)v\,dx \quad \forall u, v \in H.$$

(ii) Let $\{u_m\}$ be a sequence converging to u in H, i.e.,

$$\lim_{m\longrightarrow+\infty}\int_\Omega \left(h(x)|\nabla u_m - \nabla u|^p + b(x)|u_m - u|^p\right)dx = 0.$$

Then $\{||u_m||_H\}$ is bounded.

First we observe that: for some $\theta \in (0,1)$:

$$\begin{aligned}||\nabla u_m|^p - |\nabla u|^p| &= p|\nabla u_m + \theta(\nabla u_m - \nabla u)|^{p-1}|\nabla u_m - \nabla u| \\ &\le p2^{p-2}\left(|\nabla u_m|^{p-1}|\nabla u_m - \nabla u| + |\nabla u_m - \nabla u|^p\right).\end{aligned}$$

Hence by applying the Holder's inequality we get

$$\begin{aligned}&\left|\frac{1}{p}\int_\Omega h(x)|\nabla u_m|^p dx - \frac{1}{p}h(x)|\nabla u|^p dx\right| \\ &\le \frac{1}{p}\int_\Omega h(x)||\nabla u_m|^p - |\nabla u|^p|dx \\ &\le 2^{p-2}\int_\Omega h(x)|\nabla u_m|^{p-1}|\nabla u_m - \nabla u|dx + 2^{p-2}\int_\Omega h(x)|\nabla u_m - \nabla u|^p dx \\ &\le 2^{p-2}\left(\int_\Omega (h(x)^{\frac{p-1}{p}}|\nabla u_m|^{p-1})^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega (h(x)|\nabla(u_m - u)|^p)dx\right)^{\frac{1}{p}} \\ &\quad + 2^{p-2}\int_\Omega (h(x)|\nabla(u_m - u)|^p)dx \\ &\le c_1\left(||u_m||_H^{p-1}||u_m - u||_H + ||u_m - u||_H^p\right).\end{aligned} \tag{2.6}$$

Similarly, we also have

$$\left|\frac{1}{p}\int_\Omega b(x)|u_m|^p dx - \frac{1}{p}\int_\Omega b(x)|u|^p dx\right| \leq c_2\left(||u_m||_H^{p-1}||u_m-u||_H + ||u_m-u||_H^p\right). \tag{2.7}$$

Combining (2.6) and (2.7) we have

$$|T(u_m)-T(u)| \leq c_3\left(||u_m||_H^{p-1}||u_m-u||_H + ||u_m-u||_H^p\right)$$

with $c_1, c_2, c_3 > 0$. Letting $m \to +\infty$ since $||u_m - u||_H \to 0$ and $\{||u_m||_H\}$ bounded, we obtain

$$\lim_{m\to+\infty} T(u_m) = T(u).$$

Thus T is continuous on H.

(iii) For all $u, v \in H$, any $t \in (-1,1)\setminus\{0\}$ and a.e. $x \in \Omega$ we have

$$\begin{aligned}
&\left|\frac{h(x)|\nabla u + t\nabla v|^p - h(x)|\nabla u|^p}{t}\right| \\
&= p\left|\int_0^1 h(x)|\nabla u + st\nabla v|^{p-2}(\nabla u + st\nabla v)\nabla v ds\right| \\
&\leq p\int_0^1 h(x)|\nabla u + st\nabla v|^{p-1}|\nabla v| ds \leq p2^{p-2}h(x)(|\nabla u|^{p-1}|\nabla v| + |\nabla v|^p) \\
&\leq p2^{p-2}\left(h(x)^{\frac{p-1}{p}}|\nabla u|^{p-1}h(x)^{\frac{1}{p}}|\nabla v| + h(x)|\nabla v|^p\right).
\end{aligned}$$

Since $u, v \in H$, we observe that

$$\begin{aligned}
&\int_\Omega \left(h(x)^{\frac{p-1}{p}}|\nabla u|^{p-1}h(x)^{\frac{1}{p}}|\nabla v| + h(x)|\nabla v|^p\right) dx \\
&\leq \left(\int_\Omega (h(x)^{\frac{p-1}{p}}|\nabla u|^{p-1})^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_\Omega h(x)|\nabla v|^p dx\right)^{\frac{1}{p}} + c_5||v||_H^p \\
&\leq c_4||u||_H^{p-1}||v||_H + c_5||v||_H^p < +\infty,
\end{aligned}$$

where c_4, c_5 two positive constants.

Hence $G(x) = h(x)|\nabla u|^{p-1}|\nabla v| + h(x)|\nabla v|^p \in L^1(\Omega)$. Applying the Lebesgue dominated convergence theorem we get

$$\lim_{t\to 0}\int_\Omega \frac{h(x)|\nabla u + t\nabla v|^p - h(x)|\nabla u|^p}{t} dx = p\int_\Omega h(x)|\nabla u|^{p-2}\nabla u\nabla v dx.$$

Similarly we also have

$$\lim_{t\to 0}\int_\Omega \frac{b(x)|u+tv|^p - b(x)|u|^p}{t} dx = p\int_\Omega b(x)|u|^{p-2}uv dx.$$

This implies that

$$\langle DT(u), v\rangle = \lim_{t\to 0}\frac{T(u+tv)-T(u)}{t} = \int_\Omega (h(x)|\nabla u|^{p-2}\nabla u\nabla v + b(x)|u|^{p-2}uv) dx.$$

Thus T is weakly differentiable on H.

Let $v \in H$ be fixed, we now prove that the map $u \mapsto \langle DT(u), v \rangle$ is continuous on H.

Assume $u_m \to u$ in H, that is

$$\lim_{m\to+\infty} \int_\Omega (h(x)|\nabla u_m - \nabla u|^p + b(x)|u_m - u|^p)dx = 0.$$

By hypotheses (H) and (B) it follows that $\nabla u_m \to \nabla u$ and $u_m \to u$ in $L^p(\Omega)$. Applying Theorem C.2 in [11, p. 249] for function $g(x,s) = |s|^{p-2}s$, we deduce that

$$g(x, \nabla u_m) = |\nabla u_m|^{p-2}\nabla u_m \longrightarrow |\nabla u|^{p-2}\nabla u$$

and

$$g(x, u_m) = |u_m|^{p-2}u_m \longrightarrow |u|^{p-2}u$$

in $(L^{\frac{p}{p-1}}(\Omega))^N$ as $m \to +\infty$, where $(L^r(\Omega))^N = L^r(\Omega) \times L^r(\Omega) \times \cdots \times L^r(\Omega)$ (N times). Using this fact we shall proved that the map $u \to \langle DT(u), v \rangle$ is continuous on H for every v fixed in H.

Indeed for $\varphi \in C_0^\infty(\overline{\Omega})$, $\omega = \text{supp}\varphi$, we have

$$\begin{aligned}
&|\langle DT(u_m) - DT(u), \varphi \rangle| \\
&= \left| \int_\Omega \{h(x)(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u)\nabla\varphi + b(x)(|u_m|^{p-2}u_m - |u|^{p-2}u)\varphi\}dx \right| \\
&= \left| \int_\omega \{h(x)(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u)\nabla\varphi + b(x)(|u_m|^{p-2}u_m - |u|^{p-2}u)\varphi\}dx \right| \\
&\le C(\varphi)\{||g(x,\nabla u_m) - g(x,\nabla u)||_{L^{\frac{p}{p-1}}(\omega)} ||\nabla\varphi||_{L^p(\omega)} \\
&\quad + ||g(x,u_m) - g(x,u)||_{L^{\frac{p}{p-1}}(\omega)} ||\varphi||_{L^p(\omega)}\},
\end{aligned}$$

where $C(\varphi)$ is a constant positive. From this letting $m \to +\infty$ we get

$$\lim_{m\to+\infty} |\langle DT(u_m) - DT(u), \varphi \rangle| = 0.$$

Since $C_0^\infty(\overline{\Omega})$ is dense in H we deduce that for every $v \in H$ fixed

$$\lim_{m\to+\infty} |\langle DT(u_m) - DT(u), v \rangle| = 0.$$

The proof of Proposition 2.1 is complete. □

Proposition 2.2. *Suppose that sequence* $\{u_m\}$ *is weakly converging to* u *in* $W^{1,p}(\Omega)$*. Then we have*

$$T(u) \le \lim_{m\to+\infty} \inf T(u_m).$$

Proof. Since $\{u_m\}$ weakly converging in $W^{1,p}(\Omega)$ hence for all bounded $\Omega' \subset\subset \Omega$, $\{u_m\}$ is also weakly converging in $W^{1,p}(\Omega')$. By compactness of the embedding $W^{1,p}(\Omega')$ into $L^p(\Omega')$, the sequence $\{u_m\}$ converges strongly in $L^p(\Omega')$

then $\{u_m\}$ converges strongly in $L^1(\Omega')$. Applying Theorem 1.6 in [6, p. 9] or Theorem 4.5 [8, p. 129], we deduce that

$$T(u) \leq \lim_{m\to+\infty} \inf T(u_m).$$

The proof of Proposition 2.2 is complete. □

Proposition 2.3. *The functional* $J : H \longrightarrow \mathbb{R}$ *is defined by* (2.4), *i.e.,*

$$J(u) = T(u) - P(u), \quad u \in H$$

satisfies the Palais-Smale condition on H.

Proof. Let $\{u_m\}$ be a sequence in H such that

$$\lim_{m\to\infty} J(u_m) = c, \quad \lim_{m\to+\infty} ||DJ(u_m)||_{H^*} = 0.$$

First, we shall proved that $\{u_m\}$ is bounded in H. We suppose by contradiction that $\{u_m\}$ is not bounded in H. Then there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ such that $||u_{m_k}||_H \to +\infty$ as $k \to +\infty$. Observe further that

$$\begin{aligned} & J(u_{m_k}) - \frac{1}{\mu}\big\langle DJ(u_{m_k}), u_{m_k}\big\rangle \\ &= T(u_{m_k}) - \frac{1}{\mu}\big\langle DT(u_{m_k}), u_{m_k}\big\rangle + \frac{1}{\mu}\big\langle DP(u_{m_k}), u_{m_k}\big\rangle - P(u_{m_k}) \\ &\geq (\frac{1}{p} - \frac{1}{\mu})||u_{m_k}||_H^p \end{aligned}$$

yields

$$\begin{aligned} J(u_{m_k}) &\geq (\frac{1}{p} - \frac{1}{\mu})||u_{m_k}||_H^p + \frac{1}{\mu}\big\langle DJ(u_{m_k}), u_{m_k}\big\rangle \\ &\geq (\frac{1}{p} - \frac{1}{\mu})||u_{m_k}||_H^p - \frac{1}{\mu}||DJ(u_{u_{m_k}})||_{H^*}||u_{m_k}||_H \\ &\geq ||u_{m_k}||_H \left(\gamma_0||u_{m_k}||_H^{p-1} - \frac{1}{\mu}||DJ(u_{m_k})||_{H^*}\right), \end{aligned}$$

where $\gamma_0 = \frac{1}{p} - \frac{1}{\mu} > 0$.

From this letting $k \to +\infty$, since $||u_{m_k}||_H \to +\infty$, $||DJ(u_{m_k})||_{H^*} \to 0$, we deduce $J(u_{m_k}) \to +\infty$ yields a contradiction. Hence $\{u_m\}$ is bounded in H. By the continuous embedding H into $W^{1,p}(\Omega)$, $\{u_m\}$ is bounded in $W^{1,p}(\Omega)$. Therefore, there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ converging weakly to u in $W^{1,p}(\Omega)$. Since the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is continuous, the subsequence $\{u_{m_k}\}$ converges weakly to u in $L^{p^*}(\Omega)$ and $u_{m_k} \to u$ a.e. $x \in \Omega$. It follows that $\{u_{m_k}\}$ is bounded in $L^{p^*}(\Omega)$, that is there exists a constant $M > 0$ such that

$$||u_{m_k}||_{L^{p^*}(\Omega)} \leq M \quad \text{for all } k = 1, 2, \ldots.$$

We remark that by hypotheses (F2) and (F3) we get

$$0 \leq F(x,z) \leq \tau(x)|z|^{r+1} \quad \text{for } x \in \overline{\Omega},\ z \in \mathbb{R} - \{0\},$$

where $\tau(x) \in L^{r_0}(\Omega) \cap L^{\infty}(\Omega)$.

Then by Holder's inequality and remark that $\frac{1}{r_0} + \frac{r+1}{p^*} = 1$ we deduce

$$\begin{aligned} P(u_{m_k}) = \int_\Omega F(x, u_{m_k})dx &\leq \int_\Omega \tau(x)|u_{m_k}|^{r+1} \\ &\leq ||\tau(x)||_{L^{r_0}(\Omega)}||u_{m_k}||^{r+1}_{L^{p^*}(\Omega)} \\ &\leq M^{r+1}||\tau(x)||_{L^{r_0}(\Omega)}. \end{aligned}$$

By Proposition 2.2 we get

$$\begin{aligned} T(u) &\leq \lim_{k\to+\infty} \inf T(u_{m_k}) \leq \lim_{k\to+\infty}[P(u_{m_k}) + J(u_{m_k})] \\ &\leq c + ||\tau(x)||_{L^{r_0}(\Omega)} M^{r+1} < +\infty. \end{aligned}$$

Thus $u \in H$.

Since $\{u_{m_k}\}$ is weakly converges to u in $L^{p^*}(\Omega)$ and $u_{m_k} \to u$ a.e. $x \in \Omega$. Then it is clear that $|u_{m_k}|^{r-1}u_{m_k}$ is converges weakly to $|u|^{r-1}u$ in $L^{\frac{p^*}{r}}(\Omega)$. With similar arguments as those in [9], we define the map $K(u) : L^{\frac{p^*}{r}}(\Omega) \longrightarrow \mathbb{R}$ by

$$\left\langle K(u), \omega \right\rangle = \int_\Omega \tau(x)u\omega dx \quad \text{for } \omega \in L^{\frac{p^*}{r}}(\Omega).$$

We remark that $K(u)$ is linear and continuous provided that $\tau(x) \in L^{r_0}(\Omega)$, $u \in L^{p^*}(\Omega)$, $\omega \in L^{\frac{p^*}{r}}(\Omega)$ and $\frac{1}{r_0} + \frac{1}{p^*} + \frac{r}{p^*} = 1$. Hence

$$\left\langle K(u), |u_{m_k}|^{r-1}u_{m_k} \right\rangle \longrightarrow \left\langle K(u), |u|^{r-1}u \right\rangle \text{ as } k \to +\infty,$$

i.e.,

$$\lim_{k\to+\infty} \int_\Omega \tau(x)|u_{m_k}|^{r-1}u_{m_k}u dx = \int_\Omega \tau(x)|u|^{r+1}dx. \tag{2.8}$$

Similarly we also have

$$\lim_{k\to+\infty} \int_\Omega \tau(x)|u_{m_k}|^{r+1}dx = \int_\Omega \tau(x)|u|^{r+1}dx. \tag{2.9}$$

Combining (2.8), (2.9) we get

$$\lim_{k\to+\infty} \int_\Omega \tau(x)|u_{m_k}|^{r-1}u_{m_k}(u_{m_k} - u)dx = 0. \tag{2.10}$$

By (2.10), (F1), (F2) we obtain

$$\lim_{m\to+\infty} \int_\Omega f(x, u_{m_k})(u_{m_k} - u)dx = 0,$$

i.e.,

$$\lim_{k\to+\infty} \left\langle DP(u_{m_k}), u_{m_k} - u \right\rangle = 0. \tag{2.11}$$

It follows from (2.11) that

$$\lim_{k\to+\infty}\big\langle DT(u_{m_k}), u_{m_k}-u\big\rangle = \lim_{k\to+\infty}\big\langle DJ(u_{m_k}), (u_{m_k}-u)\big\rangle + \lim_{k\to+\infty}\big\langle DP(u_{m_k}), (u_{m_k}-u)\big\rangle = 0.$$

Moreover, since T is convex we have

$$T(u) - T(u_{m_k}) \geq \big\langle DT(u_{m_k}, u - u_{m_k})\big\rangle.$$

Letting $k \to +\infty$ we obtain that

$$T(u) - \lim_{k\to+\infty} T(u_{m_k}) = \lim_{k\to+\infty}[T(u) - T(u_{m_k})] \geq \lim_{k\to+\infty}\big\langle DT(u_{m_k}), u - u_{m_k}\big\rangle = 0.$$

Thus

$$T(u) \geq \lim_{k\to+\infty} T(u_{m_k}).$$

On other hand, by Proposition 2.2 we have

$$T(u) \leq \lim_{k\to+\infty}\inf T(u_{m_k}).$$

Hence, from two above inequalities, we get $T(u) = \lim_{k\to+\infty} T(u_{m_k})$.

Now, we shall prove that the subsequence $\{u_{m_k}\}$ converges strongly to u in H, i.e., $\lim_{k\to+\infty} ||u_{m_k} - u||_H = 0$.

Indeed, we suppose by contradiction that $\{u_{m_k}\}$ does not converge strongly to u in H. Then there exist a constant $\varepsilon_0 > 0$ and a subsequence $\{u_{m_{k_j}}\}$ of $\{u_{m_k}\}$ such that $||u_{m_{k_j}} - u||_H \geq \varepsilon_0$ for any $j = 1, 2, \ldots$.

By recalling the Clarkson's inequality

$$|\frac{\alpha+\beta}{2}|^p + |\frac{\alpha-\beta}{2}|^p \leq \frac{1}{2}(|\alpha|^p + |\beta|^p), \forall \alpha, \beta \in \mathbb{R}.$$

We deduce that

$$\frac{1}{2}T(u) + \frac{1}{2}T(v) - T(\frac{u+v}{2}) \geq T(\frac{u-v}{2}),\ \forall u, v \in H.$$

From this, for any $j = 1, 2, \ldots$, we have

$$\frac{1}{2}T(u_{m_{k_j}}) + \frac{1}{2}T(u) - T(\frac{u_{m_{k_j}}+u}{2}) \geq T(\frac{u_{m_{k_j}}-u}{2}).$$

Remark that

$$T(\frac{u_{m_{k_j}}-u}{2}) = \frac{1}{p2^p}||u_{m_{k_j}} - u||_H^p \geq \frac{1}{p2^p}\varepsilon_0^p.$$

We get

$$\frac{1}{2}T(u_{m_{k_j}}) + \frac{1}{2}T(u) - T(\frac{u_{m_{k_j}}+u}{2}) \geq \frac{1}{p2^p}\varepsilon_0^p. \tag{2.12}$$

Again instead of the remark that since $\{\frac{u_{m_{k_j}}+u}{2}\}$ converges weakly to u in $W^{1,p}(\Omega)$, by Proposition 2.2 we have

$$T(u) \leq \lim_{j\to+\infty} \inf T(\frac{u_{m_{k_j}}+u}{2}).$$

From (2.12), letting $j \to +\infty$ we obtain that

$$T(u) - \lim_{j\to+\infty} \inf T(\frac{u_{m_{k_j}}+u}{2}) \geq \frac{1}{p2^p}\varepsilon_0^p.$$

Hence $0 \geq \frac{1}{p2^p}\varepsilon_0^p$, which is a contradiction.

Therefore, $\{u_{m_k}\}$ converges strongly to u in H. Thus, the functional J satisfies the Palais-Smale condition on H. The proof of Proposition 2.3 is complete. □

We remark that the critical points of the functional J correspond to the weak solutions of the problem (1.1). Thus our idea is to apply a variation of the Mountain pass theorem (see [6]) in order to obtain at least one non-trivial weak solution of the problem (1.1).

In what follows, we will prove proposition which shows that the functional J has the Mountain pass geometry.

Proposition 2.4. (i) *There exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \geq \alpha > 0$ for all $u \in H$, $||u||_H = \rho$.*

(ii) *There exists $u_0 \in H$, $||u_0||_H > \rho$ and $J(u_0) < 0$.*

Proof. (i) Using (F2) and L'Hospital theorem we have

$$\lim_{z\to0}\frac{F(x,z)}{z^p} = \lim_{z\to0}\frac{f(x,z)}{pz^{p-1}} = \lim_{z\to0}\frac{f'_z(x,z)}{p(p-1)z^{p-2}} = 0.$$

Thus

$$\lim_{z\to0}\frac{F(x,z)}{z^p} = 0. \tag{2.13}$$

Using (F2) there exists A a positive constant such that

$$|f(x,z)| \leq A|z|^r.$$

We integrate again

$$0 < F(x,z) \leq \overline{A}|z|^{r+1},$$

where $\overline{A}$ is a positive constant. Then

$$0 \leq \lim_{z\to+\infty}\frac{F(x,z)}{z^{\frac{Np}{N-p}}} \leq \lim_{z\to+\infty}\frac{\overline{A}|z|^{r+1}}{z^{\frac{Np}{N-p}}} = 0$$

with $r \in (p-1, \frac{N+p}{N-p})$. Hence

$$\lim_{z\to+\infty}\frac{F(x,z)}{z^{\frac{Np}{N-p}}} = 0. \tag{2.14}$$

Using (2.13), (2.14), we obtain

$\forall\varepsilon > 0$, $\exists\delta_1 > 0$ such that $|\frac{F(x,z)}{z^p}| < \varepsilon$ for all z with $|z| < \delta_1$.

$\forall\varepsilon > 0$, $\exists\delta_2 > 0$ such that $|\frac{F(x,z)}{z^{\frac{Np}{N-p}}}| < \varepsilon$ for all z with $|z| > \delta_2$.

Thus $\forall\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$F(x,z) < \varepsilon|z|^p, \quad |z| < \delta_1 \text{ and } F(x,z) < \varepsilon|z|^{\frac{Np}{N-p}}, \quad |z| > \delta_2.$$

Using the relation $0 < F(x,z) \leq \overline{A}|z|^{r+1}$ there exists a constant $b > 0$ such that $F(x,z) \leq b$ for all $|z| \in [\delta_1, \delta_2]$. We conclude that for all $\varepsilon > 0$, there exists $b_\varepsilon > 0$ such that

$$F(x,z) \leq \varepsilon|z|^p + b_\varepsilon|z|^{\frac{Np}{N-p}}. \tag{2.15}$$

Using (2.15) we have

$$\begin{aligned} J(u) &= \frac{1}{p}||u||_H^p - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p}||u||_H^p - \varepsilon\int_\Omega |u|^p dx - b_\varepsilon\int_\Omega |u|^{\frac{Np}{N-p}}dx. \end{aligned}$$

For $p \leq q \leq \frac{Np}{N-p}$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. So the embedding $H \hookrightarrow L^q(\Omega)$ is continuous, $|u|_{L^q(\Omega)} \leq c||u||_H$. Thus we have

$$|u|_{L^p} \leq C_1||u||_H.$$

$$|u|_{L^{\frac{Np}{N-p}}} \leq C_2||u||_H.$$

Therefore

$$\begin{aligned} J(u) &\geq \frac{1}{p}||u||_H^p - \varepsilon C_1^p||u||_H^p - b_\varepsilon C_2^{\frac{Np}{N-p}}||u||_H^{\frac{Np}{N-p}} \\ &\geq ||u||_H^p\left(\frac{1}{p} - \varepsilon C_1^p - b_\varepsilon C_2^{\frac{Np}{N-p}}||u||^{\frac{Np}{N-p}-p}\right). \end{aligned}$$

Letting $\varepsilon \in (0, \frac{1}{pC_1^p})$ and $||u||_H = \rho$ small enough such that

$$\frac{1}{p} - \varepsilon C_1^p - b_\varepsilon C_2^{\frac{Np}{N-p}}||u||_H^{\frac{Np}{N-p}-p} > 0,$$

we obtain

$$J(u) \geq \left(\frac{1}{p} - \varepsilon C_1^p - b_\varepsilon C_2^{\frac{Np}{N-p}}||u||^{\frac{Np}{N-p}-p}\right)\rho^p = \alpha > 0.$$

ii) Denote $h(t) = \frac{F(x,tz)}{t^\mu}$ for all $t > 0$.

Then using (F3) we get

$$h'(t) = \frac{1}{t^{\mu+1}}[tzf(x,tz) - \mu F(x,tz)] \geq 0, \ \forall t > 0.$$

Thus we deduce for any $t \geq 1$, $F(x,tz) \geq t^{\mu} F(x,z)$. Let $w_0 \in C_0^{\infty}(\overline{\Omega})$ be such that meas $(\{x \in (\overline{\Omega}) : |w_0(x)| > 0\}) > 1$ then with $t > 1$ we get

$$\begin{aligned} J(tw_0) &= \int_{\Omega} \frac{1}{p} \left(h(x)|\nabla(tw_0)|^p + b(x)|tw_0|^p\right) dx - \int_{\Omega} F(x, tw_0)dx \\ &= \int_{\Omega} \frac{t^p}{p} \left(h(x)|\nabla w_0|^p + b(x)|w_0|^p\right) dx - \int_{\Omega} F(x, tw_0)dx \\ &\leq \frac{t^p}{p} ||w_0||_H^p - t^{\mu} \int_{\Omega} F(x, w_0)dx. \end{aligned}$$

Since $\mu > p$, the right hand-side of above inequality converges to $-\infty$ when $t \to +\infty$. Then there exists $t_0 > 0$ such that $||t_0 w_0||_H > \rho$ and $J(t_0 w_0) < 0$. Set $u_0 = t_0 w_0$, we have $J(u_0) < 0$ and $||u_0|| > \rho$.

The proof of Proposition 2.4 is complete. □

Proposition 2.5. (i) $J(0) = 0$.

(ii) *The acceptable set* $G = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = u_0\}$ *is not empty, where* u_0 *is given in Proposition 2.4.*

It is clear that: (i) follows from (F1) and the definition of J.

(ii) Let $\gamma(t) = tu_0$, then $\gamma(t) \in G$.

Proof of Theorem 1.1. By Propositions 2.1-2.5, all assumptions of the variations of the Mountain pass theorem introduced in [6] are satisfied. Therefore there exists $\hat{u} \in H$ such that

$$0 < \alpha \leq J(\hat{u}) = \inf\{\max J(\gamma([0,1])) : \gamma \in G\}$$

and $\langle DJ(\hat{u}), v\rangle = 0$ for all $v \in H$, i.e., $\hat{u}$ is a weak solution of the problem (1.1). Moreover since $J(\hat{u}) > 0 = J(0)$, $\hat{u}$ is a nontrivial weak solution of the problem (1.1). The Theorem 1.1 is completely proved. □

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