Bull. Korean Math. Soc. **48** (2011), No. 6, pp. 1157–1167
http://dx.doi.org/10.4134/BKMS.2011.48.6.1157

APPROXIMATION AND BALANCING ORDERS FOR TOTALLY INTERPOLATING BIORTHOGONAL MULTIWAVELET SYSTEMS

Youngwoo Choi and Jaewon Jung

Abstract. We consider totally interpolating biorthogonal multiwavelet systems with finite impulse response two-band multifilter banks, and study balancing order conditions of such systems. Based on FIR and interpolating properties, we show that approximation order condition is completely equivalent to balancing order condition. Consequently, a prefiltering can be avoided if a totally interpolating biorthogonal multiwavelet system satisfies suitable approximation order conditions. An example with approximation order 4 is provided to illustrate the result.

1. Introduction

The approximation order property (or vanishing moments condition) plays a crucial role in many application problems. It is well known that in the scalar wavelet case, if a scaling function has approximation order $M \geq 1$, then the associated low-pass filter/high-pass filter preserves/cancels discrete-time polynomial signals up to degree $M-1$. Such result does not hold in multiwavelet systems. Furthermore, since a discrete multiwavelet transform employs multi-input/multi-output filter banks, a prefiltering is a prerequisite in various practical applications [7, 17, 18, 19]. As an alternative approach, Lebrun and Vetterli introduced the notion of *balancing order* for orthogonal multiwavelet systems [11, 12]. As a general rule if an orthogonal multiwavelet system is balanced of order M, then it has an approximation order at least M but the converse is not true in general.

This paper is mainly concerned with relationships between approximation order and balancing order conditions of compactly supported totally interpolating biorthogonal multiwavelet systems with FIR property and multiplicity 2. We follow the definition of the balancing order generalized to biorthogonal setting [1] and apply it to interpolating biorthogonal multiwavelet systems with FIR property. It turns out that the concept of approximation order is

Received April 27, 2010; Revised November 6, 2010.
2010 *Mathematics Subject Classification.* Primary 42C40; Secondary 65T60.
Key words and phrases. multiwavelets, interpolating, balancing order.

completely equivalent to that of balancing order in this setting. Several examples of balanced interpolating orthogonal multiwavelet systems [10, 15] are known. It was shown that any compactly supported interpolating orthogonal multiscaling function providing approximation order M is balanced of order M by using Plonka factorization [9]. Our result is based on a different approach and can be regarded as a biorthogonal analogue. Consequently, as is mentioned in [1, 11, 18, 19, 21], a prefiltering can be avoided if a totally interpolating biorthogonal multiwavelet system satisfies suitable approximation order conditions.

This paper is organized as follows: basic notations and elementary facts on biorthogonal multiwavelet systems with the interpolating property are introduced in Section 2. In Section 3, we introduce the definition of the balancing order for biorthogonal multiwavelet systems. Based on a simple characterization of balancing order condition in terms of the dilation coefficients the main result is stated and proved in the same section. Finally, to illustrate our result, an example with approximation order 4 is provided.

2. Preliminaries

We consider a *multiresolution analysis* (*MRA*) *of multiplicity* 2 that is a nested sequence $\{V_n\}$ of closed linear subspaces in $L^2(\mathbb{R})$ [3, 6, 16]. A vector function $\mathbf{\Phi} = (\phi_1, \phi_2)^T$ is called a *multiscaling function* if

$$\mathbf{\Phi}(t) = 2\sum_{\ell\in\mathbb{Z}} \mathbf{P}_\ell \mathbf{\Phi}(2t-\ell) \tag{1}$$

for some 2×2 real matrices $\mathbf{P}_\ell$. The Fourier transform is defined by

$$\hat{\mathbf{\Phi}} = \left(\hat{\phi_1}, \hat{\phi_2}\right)^T,$$

where $\hat{\phi_j}(\omega) := \int_{-\infty}^{\infty} \phi_j(t) e^{-i\omega t} dt$ with $i = \sqrt{-1}$ and $j = 1, 2$. Taking the Fourier transform on (1) yields

$$\hat{\mathbf{\Phi}}(\omega) = \mathbf{P}\left(\frac{\omega}{2}\right)\hat{\mathbf{\Phi}}\left(\frac{\omega}{2}\right),$$

where the *two-scale matrix symbol* (*or the refinement mask*) $\mathbf{P}(\omega)$ corresponding to $\mathbf{\Phi}$ is given by $\mathbf{P}(\omega) := \sum_{\ell\in\mathbb{Z}} \mathbf{P}_\ell e^{-i\omega\ell}$. A vector function $\mathbf{f} \in L^2(\mathbb{R})^2$ is said to be L^2-*stable* if there are constants $0 < A \le B < \infty$ such that

$$A\sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{b}_k \ \le\ \left\| \sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{f}(\cdot - k) \right\|_{L^2}^2 \ \le\ B\sum_{k=-\infty}^{\infty} \mathbf{b}_k^* \mathbf{b}_k$$

holds for any vector sequence $\{\mathbf{b}_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})^2$, where $*$ stands for the complex conjugate transpose [14]. A pair of L^2-stable multiscaling functions $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ associated with matrix refinement equations

$$\mathbf{\Phi}(t) \;=\; 2\sum_{\ell\in\mathbb{Z}} \mathbf{P}_\ell \mathbf{\Phi}(2t-\ell),$$

$$\tilde{\mathbf{\Phi}}(t) = 2\sum_{\ell\in\mathbb{Z}} \tilde{\mathbf{P}}_\ell \tilde{\mathbf{\Phi}}(2t-\ell),$$

are said to be *biorthogonal* if

$$\left\langle \mathbf{\Phi}(\cdot), \tilde{\mathbf{\Phi}}(\cdot - k)\right\rangle = \delta_{k,0}\mathbf{I}_2, \tag{2}$$

where $\langle \mathbf{f}, \mathbf{g}\rangle := \int_{-\infty}^{\infty} \mathbf{f}(t)\,\mathbf{g}^T(t)\,dt$ and $\delta_{k,\ell}$ denotes the *Kronecker δ-symbol* for $k, \ell \in \mathbb{Z}$ [5]. Here and in the sequel, $\mathbf{I}_n$ and $\mathbf{0}_{m\times n}$ denote the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively. Let $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ be *multiwavelets* associated with $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, respectively. They are given by

$$\begin{aligned}\mathbf{\Psi}(t) &= 2\sum_{\ell\in\mathbb{Z}} \mathbf{Q}_\ell \mathbf{\Phi}(2t-\ell),\\ \tilde{\mathbf{\Psi}}(t) &= 2\sum_{\ell\in\mathbb{Z}} \tilde{\mathbf{Q}}_\ell \tilde{\mathbf{\Phi}}(2t-\ell),\end{aligned}$$

where $\mathbf{Q}_\ell$ and $\tilde{\mathbf{Q}}_\ell$ are 2×2 real matrices. The pair multiwavelets $\{\mathbf{\Psi}, \tilde{\mathbf{\Psi}}\}$ is said to be *biorthogonal* if

$$\left\langle \mathbf{\Psi}(\cdot), \tilde{\mathbf{\Psi}}(\cdot - k)\right\rangle = \delta_{k,0}\mathbf{I}_2, \tag{3}$$

$$\left\langle \mathbf{\Phi}(\cdot), \tilde{\mathbf{\Psi}}(\cdot - k)\right\rangle = \left\langle \tilde{\mathbf{\Phi}}(\cdot), \mathbf{\Psi}(\cdot - k)\right\rangle = \mathbf{0}_{2\times 2} \tag{4}$$

for all $k \in \mathbb{Z}$. By taking Fourier transform, one can rewrite (2), (3), and (4) as

$$\mathbf{P}(\omega)\tilde{\mathbf{P}}^*(\omega) + \mathbf{P}(\omega+\pi)\tilde{\mathbf{P}}^*(\omega+\pi) = \mathbf{I}_2, \tag{5}$$

$$\mathbf{Q}(\omega)\tilde{\mathbf{Q}}^*(\omega) + \mathbf{Q}(\omega+\pi)\tilde{\mathbf{Q}}^*(\omega+\pi) = \mathbf{I}_2, \tag{6}$$

$$\mathbf{P}(\omega)\tilde{\mathbf{Q}}^*(\omega) + \mathbf{P}(\omega+\pi)\tilde{\mathbf{Q}}^*(\omega+\pi) = \mathbf{0}_{2\times 2}, \tag{7}$$

$$\mathbf{Q}(\omega)\tilde{\mathbf{P}}^*(\omega) + \mathbf{Q}(\omega+\pi)\tilde{\mathbf{P}}^*(\omega+\pi) = \mathbf{0}_{2\times 2}. \tag{8}$$

We say that a multiscaling function $\mathbf{\Phi}$ *provides approximation order* $M \geq 1$ if there exist vectors $\mathbf{y}_\ell^m \in \mathbb{R}^2$ such that

$$\sum_{\ell\in\mathbb{Z}} (\mathbf{y}_\ell^m)^T \mathbf{\Phi}(t-\ell) = t^m \tag{9}$$

for all $t \in \mathbb{R}$ and $m = 0, \ldots, M-1$. A multiscaling function $\mathbf{\Phi}$ is said to be *interpolating* if it satisfies the condition

$$\left[\mathbf{\Phi}(n), \mathbf{\Phi}\left(n+\frac{1}{2}\right)\right] = \sqrt{2}\delta_{n,0}\mathbf{I}_2$$

for $n \in \mathbb{Z}$. If each function in a biorthogonal multiwavelet system $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}}\}$ has interpolating property, then this system is said to be *totally interpolating* [21].

In what follows, we consider totally interpolating biorthogonal multiwavelet systems $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}}\}$ which are FIR two-band multifilter banks. We further assume that both $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ are L^2-stable and compactly supported. For a sequence $\mathbf{A}_k$ of 2×2 real matrices, we let $\mathbf{A}_k^{j,\ell} := [\mathbf{A}_k]_{j,\ell}$ for $j,\ell = 1,2$ and $k\in\mathbb{Z}$. Recall the following lemma [21].

Lemma 2.1. *If $\{\mathbf{\Phi},\tilde{\mathbf{\Phi}},\mathbf{\Psi},\tilde{\mathbf{\Psi}}\}$ is a totally interpolating biorthogonal multiwavelet system, then*

$$\mathbf{P}(\omega) = \begin{pmatrix} \frac{1}{2} & p_1(\omega) \\ \frac{1}{2}e^{-i\omega} & p_2(\omega)\end{pmatrix} \quad \textit{and} \quad \tilde{\mathbf{P}}(\omega) = \begin{pmatrix} \frac{1}{2} & \tilde{p}_1(\omega) \\ \frac{1}{2}e^{-i\omega} & \tilde{p}_2(\omega)\end{pmatrix}, \tag{10}$$

$$\mathbf{Q}(\omega) = \begin{pmatrix} \frac{1}{2} & q_1(\omega) \\ \frac{1}{2}e^{-i\omega} & q_2(\omega)\end{pmatrix} \quad \textit{and} \quad \tilde{\mathbf{Q}}(\omega) = \begin{pmatrix} \frac{1}{2} & \tilde{q}_1(\omega) \\ \frac{1}{2}e^{-i\omega} & \tilde{q}_2(\omega)\end{pmatrix}, \tag{11}$$

$$\textit{where} \begin{cases} p_j(\omega) := \sum_{k\in\mathbb{Z}} \mathbf{P}_k^{j,2} e^{-i\omega k}, & \tilde{p}_j(\omega) := \sum_{k\in\mathbb{Z}} \tilde{\mathbf{P}}_k^{j,2} e^{-i\omega k}, \\ q_j(\omega) := \sum_{k\in\mathbb{Z}} \mathbf{Q}_k^{j,2} e^{-i\omega k}, & \tilde{q}_j(\omega) := \sum_{k\in\mathbb{Z}} \tilde{\mathbf{Q}}_k^{j,2} e^{-i\omega k}, \end{cases} \textit{for } j=1,2.$$

For convenience, we let $c_k^j := \mathbf{P}_k^{j,2}$ for $j=1,2$ and $k\in\mathbb{Z}$. The following theorem for FIR property was shown in [21, 2].

Theorem 2.2. *Let $\{\mathbf{\Phi},\tilde{\mathbf{\Phi}}\}$ be a biorthogonal pair of interpolating multiscaling functions with two-scale matrix symbols $\mathbf{P}(\omega)$ and $\tilde{\mathbf{P}}(\omega)$ as in* (10)*. If both $p_1(\omega)$ and $p_2(\omega)$ are FIR filters, then $\tilde{\mathbf{P}}(\omega)$ is an FIR filter if and only if*

$$p_1(\omega)\,p_2(\omega+\pi) - p_1(\omega+\pi)\,p_2(\omega) = Ce^{im'\omega}, \tag{12}$$

that is,

$$\sum_{k\in\mathbb{Z}} c_{2k+1}^1 c_{2\ell-2k}^2 - \sum_{k\in\mathbb{Z}} c_{2k}^1 c_{2\ell-2k+1}^2 = \frac{C}{2}\delta_{2\ell+1,-m'} \tag{13}$$

for some real constant $C\neq 0$ and some odd integer m'.

Based on biorthogonality and FIR conditions, we showed the following [2]:

Theorem 2.3. *Let $\mathbf{\Phi}(t)$ and $\tilde{\mathbf{\Phi}}(t)$ be L^2-stable interpolating biorthogonal multiscaling functions. Assume that $\mathbf{P}(\omega)$ and $\tilde{\mathbf{P}}(\omega)$ are both FIR filters. Then both $\mathbf{\Phi}(t)$ and $\tilde{\mathbf{\Phi}}(t)$ provide approximation order M if and only if*

$$\sum_{k\in\mathbb{Z}} k^n c_{2k}^1 = \frac{(-1)^n\left(1-2C\,(2m'+1)^n\right)}{2^{2n+2}}, \tag{14}$$

$$\sum_{k\in\mathbb{Z}} k^n c_{2k+1}^1 = \frac{(-1)^n\left(3^n+2C\,(2m'+3)^n\right)}{2^{2n+2}}, \tag{15}$$

$$\sum_{k\in\mathbb{Z}} k^n c_{2k}^2 = \frac{1+2C\,(1-2m')^n}{2^{2n+2}}, \tag{16}$$

$$\sum_{k\in\mathbb{Z}} k^n c_{2k+1}^2 = \frac{(-1)^n\left(1-2C\,(2m'+1)^n\right)}{2^{2n+2}} \tag{17}$$

for $n=0,\ldots,M-1$.

3. Balancing order

Define the block Toeplitz matrices $\mathcal{L} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ and $\mathcal{H} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ corresponding to the low-pass analysis and high-pass analysis of $\mathbf{\Phi}$ and $\mathbf{\Psi}$, i.e.,

$$\mathcal{L} = \begin{pmatrix} \cdots & & & & & & & & & \\ & \mathbf{P_{-1}} & \mathbf{P_0} & \mathbf{P_1} & \mathbf{P_2} & \mathbf{P_3} & \cdots & & & \\ & & & \mathbf{P_{-1}} & \mathbf{P_0} & \mathbf{P_1} & \mathbf{P_2} & \mathbf{P_3} & \cdots & \\ & & & & & & & & & \cdots \end{pmatrix},$$

$$\mathcal{H} = \begin{pmatrix} \cdots & & & & & & & & & \\ & \mathbf{Q_{-1}} & \mathbf{Q_0} & \mathbf{Q_1} & \mathbf{Q_2} & \mathbf{Q_3} & \cdots & & & \\ & & & \mathbf{Q_{-1}} & \mathbf{Q_0} & \mathbf{Q_1} & \mathbf{Q_2} & \mathbf{Q_3} & \cdots & \\ & & & & & & & & & \cdots \end{pmatrix}.$$

Similarly, define $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{H}}$ corresponding to the low-pass analysis and high-pass analysis of $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{\Psi}}$. Lebrun and Vetterli [11, 12] introduced the balancing order condition for orthogonal multiwavelets. We follow the definition by [1] generalized to biorthogonal setting. Recall that the biorthogonality provides

$$2\begin{pmatrix} \mathcal{L}^T & \mathcal{H}^T \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}} \\ \tilde{\mathcal{H}} \end{pmatrix} = \mathbf{I} \quad \text{and} \quad 2\begin{pmatrix} \mathcal{L} \\ \mathcal{H} \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}}^T & \tilde{\mathcal{H}}^T \end{pmatrix} = \mathbf{I},$$

i.e.,

$$\begin{aligned} 2\mathcal{L}^T\tilde{\mathcal{L}} + 2\mathcal{H}^T\tilde{\mathcal{H}} = \mathbf{I} \qquad &\text{and} \qquad 2\mathcal{L}\tilde{\mathcal{L}}^T = \mathbf{I}, \\ 2\mathcal{L}\tilde{\mathcal{H}}^T = \mathbf{0}, \quad 2\mathcal{H}\tilde{\mathcal{L}}^T = \mathbf{0}, \qquad &\text{and} \qquad 2\mathcal{H}\tilde{\mathcal{H}}^T = \mathbf{I}, \end{aligned}$$

where $\mathbf{I}$ is the identity block series and $\mathbf{0}$ is the zero block series, respectively. For $k = 0, \ldots, M-1$, we let $\mathbf{u}_k := [\ldots, (-2)^k, (-1)^k, 0^k, 1^k, 2^k, \ldots]^T$. In view of the biorthogonality property, one can see that $2\tilde{\mathcal{L}}^T\mathbf{u}_k = 2^{-k}\mathbf{u}_k$ implies $\mathcal{L}\mathbf{u}_k = 2^k\mathbf{u}_k$. Unlike the orthogonal case, balancing property of $\mathbf{\Phi}$ or $\tilde{\mathbf{\Phi}}$ does not lead to that of $\mathbf{\Psi}$ or $\tilde{\mathbf{\Psi}}$. In other words, $2\tilde{\mathcal{L}}^T\mathbf{u}_k = 2^{-k}\mathbf{u}_k$ or $\mathcal{L}\mathbf{u}_k = 2^k\mathbf{u}_k$ does not imply $\mathcal{H}\mathbf{u}_k = \mathbf{0}$ or $\tilde{\mathcal{H}}\mathbf{u}_k = \mathbf{0}$. This motivates the following definition of balancing order for a pair $\{\mathbf{\Phi}, \mathbf{\Psi}\}$.

Definition. A pair $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ is said to be *balanced of order* M or to *have balancing order* M if for $k = 0, \ldots, M-1$ the signal $\mathbf{u}_k$ is preserved by low-pass branch $\mathcal{L}$ and cancelled by the high-pass branch $\mathcal{H}$, i.e., for $k = 0, \ldots, M-1$,

$$\mathcal{L}\mathbf{u}_k = 2^k\mathbf{u}_k \quad \text{and} \quad \mathcal{H}\mathbf{u}_k = \mathbf{0}.$$

From the definition of balancing order and the biorthogonality, we observe that $\mathbf{u}_k^T\left(2\mathcal{L}^T\tilde{\mathcal{L}} + 2\mathcal{H}^T\tilde{\mathcal{H}}\right) = \mathbf{u}_k^T$ provides $2\tilde{\mathcal{L}}^T\mathbf{u}_k = 2^{-k}\mathbf{u}_k$. Therefore, balancing of $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ is equivalent to that of $\{\tilde{\mathbf{\Phi}}, \mathbf{\Psi}\}$. A simple calculation shows the following:

Proposition 3.1. *A pair* $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ *is balanced of order* M *if and only if there exists at least one real number* b *such that*

$$\sum_{j\in\mathbb{Z}} \mathbf{P}_j \begin{pmatrix} (2j)^k \\ (2j+1)^k \end{pmatrix} = 2^k \begin{pmatrix} b^k \\ (b+1)^k \end{pmatrix}, \tag{18}$$

$$\sum_{j\in\mathbb{Z}} \mathbf{Q}_j \begin{pmatrix} (2j)^k \\ (2j+1)^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

for $k = 0, \ldots, M-1$.

In general, balancing condition of order M implies approximation property of order M, but the converse is not true in general. Therefore, it is interesting to ask under what circumstances balancing of order M (discrete time property) is equivalent to approximation of order M (continuous time property). We prove the equivalence for totally interpolating biorthogonal multiwavelet systems.

Under the totally interpolating condition, balancing order condition can be rephrased in terms of two scale coefficients.

Theorem 3.2. *Let* $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Psi}, \tilde{\mathbf{\Psi}}\}$ *be a totally interpolating biorthogonal multiwavelet system. If* $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ *is balanced of order* $M > 1$, *then the number* b *must be zero in* (18). *Furthermore,* (18) *can be written as*

$$\sum_{j\in\mathbb{Z}} (2j+1)^k c_j^1 = \frac{1}{2}\delta_{k,0}, \tag{20}$$

$$\sum_{j\in\mathbb{Z}} (2j+1)^k c_j^2 = 2^{k-1} \tag{21}$$

for $k = 0, \ldots, M-1$.

Proof. Condition (18) is equivalent to

$$\frac{1}{2}\delta_{k,0} + \sum_{j\in\mathbb{Z}} c_j^1 (2j+1)^k = 2^k b^k, \tag{22}$$

$$\frac{1}{2}2^k + \sum_{j\in\mathbb{Z}} c_j^2 (2j+1)^k = 2^k (b+1)^k \tag{23}$$

for $k = 0, \ldots, M-1$. Using (22) and $q_1(\omega) = -p_1(\omega)$ from [21], we see

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \left[\mathbf{Q}_j^{1,1}(2j)^k + \mathbf{Q}_j^{1,2}(2j+1)^k\right] &= \frac{1}{2}\delta_{k,0} - \sum_{j\in\mathbb{Z}} c_j^1 (2j+1)^k \\ &= \frac{1}{2}\delta_{k,0} - \left(2^k b^k - \frac{1}{2}\delta_{k,0}\right) \\ &= \delta_{k,0} - 2^k b^k \end{aligned}$$

for $k = 0, \ldots, M-1$. Thus, condition (19) forces $b = 0$ when $M > 1$. One can easily check that (22) and (23) are equivalent to (20) and (21), respectively. $\square$

One can easily check (20) and (21) for the examples provided in [21].

We are now ready to state and prove our main result.

Theorem 3.3. *Let $\{\mathbf{\Phi}, \tilde{\mathbf{\Phi}}\}$ be a pair of biorthogonal interpolating multiscaling functions with FIR property. Both $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ provide approximation order M if and only if both $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ and $\{\tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}\}$ are balanced of order M.*

Proof. It suffices to prove the necessity. Suppose that both $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ have approximation order M and $\mathbf{P}(\omega)$ and $\tilde{\mathbf{P}}(\omega)$ are both FIR filters in (10). If we substitute (14) and (15) into (20), the binomial theorem provides the following:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} (2j+1)^k c_j^1 &= \sum_{j\in\mathbb{Z}} \left((4j+1)^k c_{2j}^1 + (4j+3)^k c_{2j+1}^1\right) \\
&= \sum_{j\in\mathbb{Z}} \left(\sum_{\ell=0}^{k} \binom{k}{\ell} (4j)^\ell c_{2j}^1 + \sum_{\ell=0}^{k} \binom{k}{\ell} (4j)^\ell 3^{k-\ell} c_{2j+1}^1 \right) \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} 4^\ell \left(\frac{1-2C(2m'+1)^\ell}{4\cdot 4^\ell} \right) (-1)^\ell \\
&\quad + \sum_{\ell=0}^{k} \binom{k}{\ell} 4^\ell 3^{k-\ell} \left(\frac{3^\ell + 2C(2m'+3)^\ell}{4\cdot 4^\ell} \right) (-1)^\ell \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{1}{4} \left((-1)^\ell - 2C(-2m'-1)^\ell\right) \\
&\quad + \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{1}{4} \left(3^k(-1)^\ell + 2C(-2m'-3)^\ell 3^{k-\ell}\right) \\
&= \frac{1}{4}\delta_{k,0} - \frac{C}{2}(-2m')^k + \frac{1}{4}\delta_{k,0} + \frac{C}{2}(-2m')^k \\
&= \frac{1}{2}\delta_{k,0}
\end{aligned}$$

for $k = 0, \ldots, M-1$. Substituting (16) and (17) into (21) leads to

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} (2j+1)^k c_j^2 &= \sum_{\ell=0}^{k} \binom{k}{\ell} 4^\ell \left(\frac{1+2C(1-2m')^\ell}{4\cdot 4^\ell} \right) \\
&\quad + \sum_{\ell=0}^{k} \binom{k}{\ell} 4^\ell 3^{k-\ell} \left(\frac{1-2C(2m'+1)^\ell}{4\cdot 4^\ell} \right) (-1)^\ell \\
&= 2^{k-2} + \frac{C}{2}(2-2m')^k + 2^{k-2} - \frac{C}{2}(2-2m')^k \\
&= 2^{k-1}.
\end{aligned}$$

Next, we will show $I := \sum_{j\in\mathbb{Z}} (2j+1)^k \tilde{c}_j^1 = \frac{1}{2}\delta_{k,0}$ and $II := \sum_{j\in\mathbb{Z}} (2j+1)^k \tilde{c}_j^2 = 2^{k-1}$ for $k = 0,\ldots,M-1$. Recall the following formulas [2]:

$$\tilde{c}_j^1 = \frac{1}{2C}(-1)^{m'+j} c^2_{-m'-j} \quad \text{and} \quad \tilde{c}_j^2 = -\frac{1}{2C}(-1)^{m'+j} c^1_{-m'-j}.$$

Hence, (16) and (17) imply for $\ell = 0,\ldots,M-1$

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \left(\frac{-m'-2j-1}{2}\right)^\ell \tilde{c}^1_{2j+1} &= \frac{1}{2C}\left(\frac{1+2C(1-2m')^\ell}{4\cdot 4^\ell}\right), \\
\sum_{j\in\mathbb{Z}} \left(\frac{-m'-2j-1}{2}\right)^\ell \tilde{c}^1_{2j} &= -\frac{1}{2C}\left(\frac{(-1)^\ell\left(1-2C(2m'+1)^\ell\right)}{4\cdot 4^\ell}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
I &= \sum_{j\in\mathbb{Z}} \left((4j+1)^k \tilde{c}^1_{2j} + (4j+3)^k \tilde{c}^1_{2j+1}\right) \\
&= (-4)^k \sum_{j\in\mathbb{Z}}\sum_{\ell=0}^{k} \binom{k}{\ell}\left(\frac{m'}{2}+\frac{1}{4}\right)^{k-\ell}\left(\frac{-m'-2j-1}{2}\right)^\ell \tilde{c}^1_{2j} \\
&\quad +(-4)^k \sum_{j\in\mathbb{Z}}\sum_{\ell=0}^{k} \binom{k}{\ell}\left(\frac{m'}{2}-\frac{1}{4}\right)^{k-\ell}\left(\frac{-m'-2j-1}{2}\right)^\ell \tilde{c}^1_{2j+1} \\
&= (-4)^k \frac{0^k}{2} \\
&= \frac{1}{2}\delta_{k,0}
\end{aligned}$$

for $k = 0,\ldots,M-1$. Similarly, from (14) and (15) we obtain

$$\begin{aligned}
II &= \frac{(-4)^k}{2C}\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{m'}{2}+\frac{1}{4}\right)^{k-\ell}\left(\frac{(-1)^\ell\left(3^\ell+2C(2m'+3)^\ell\right)}{4\cdot 4^\ell}\right) \\
&\quad -\frac{(-4)^k}{2C}\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{m'}{2}-\frac{1}{4}\right)^{k-\ell}\left(\frac{(-1)^\ell\left(1-2C(2m'+1)^\ell\right)}{4\cdot 4^\ell}\right) \\
&= (-4)^k \frac{1}{2}\left(-\frac{1}{2}\right)^k \\
&= 2^{k-1}.
\end{aligned}$$

Since $\mathbf{Q}_j^{n,2} = -\mathbf{P}_j^{n,2}$ for $n = 1,2$ and $j \in \mathbb{Z}$, we get

$$\sum_{j\in\mathbb{Z}} (2j+1)^k \mathbf{Q}_j^{1,2} = -\frac{1}{2}\delta_{k,0} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} (2j+1)^k \mathbf{Q}_j^{2,2} = -2^{k-1}$$

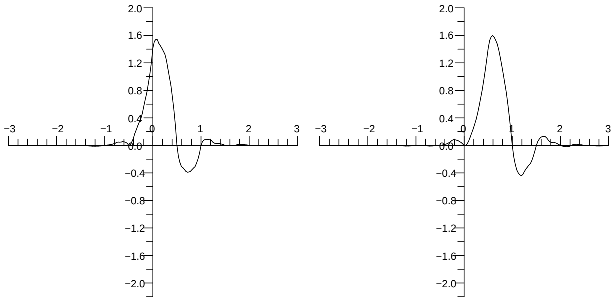

FIGURE 1. ϕ_1 and ϕ_2

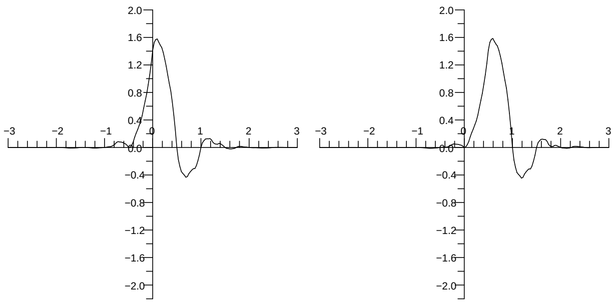

FIGURE 2. $\tilde{\phi}_1$ and $\tilde{\phi}_2$

for $k = 0, \ldots, M-1$. The interpolating property provides that $\mathbf{Q}_j^{1,1} = \frac{1}{2}\delta_{j,0}$ and $\mathbf{Q}_j^{2,1} = \frac{1}{2}\delta_{j,1}$ for $j \in \mathbb{Z}$. Therefore, we obtain

$$\sum_{j\in\mathbb{Z}} \mathbf{Q}_j \begin{pmatrix} (2j)^k \\ (2j+1)^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., it suffices to use (18) for the balancing order of $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ in our system. Hence $\{\mathbf{\Phi}, \mathbf{\Psi}\}$ is balanced of order M. Similarly, $\{\tilde{\mathbf{\Phi}}, \tilde{\mathbf{\Psi}}\}$ is also balanced of order M. □

Example. Based on a method introduced in [2], one can construct a biorthogonal multiwavelet system with approximation order 4:

$$\begin{aligned} p_1(\omega) &= -\frac{1}{256}e^{3i\omega} + \frac{1}{64}e^{2i\omega} + \frac{35}{256}e^{i\omega} \\ &\quad + \frac{15}{32} - \frac{35}{256}e^{-i\omega} + \frac{1}{64}e^{-2i\omega} + \frac{1}{256}e^{-3i\omega}, \\ p_2(\omega) &= \frac{7}{131072}e^{5i\omega} - \frac{7}{32768}e^{4i\omega} - \frac{181}{65536}e^{3i\omega} - \frac{93}{32768}e^{2i\omega} + \frac{3433}{131072}e^{i\omega} \end{aligned}$$

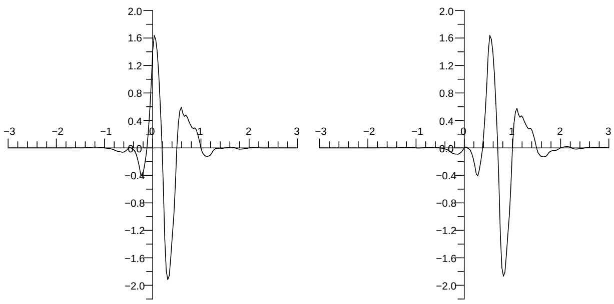

FIGURE 3. ψ_1 and ψ_2

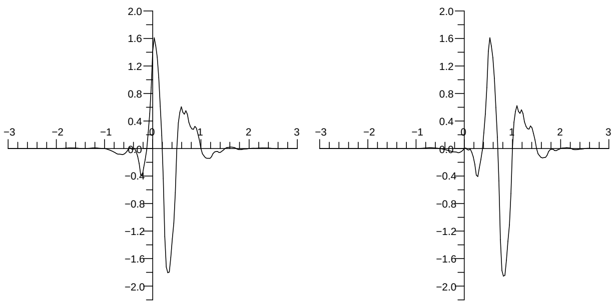

FIGURE 4. $\tilde{\psi}_1$ and $\tilde{\psi}_2$

$$+\frac{2205}{16384}+\frac{14845}{32768}e^{-i\omega}-\frac{2205}{16384}e^{-2i\omega}+\frac{3433}{131072}e^{-3i\omega}$$
$$+\frac{93}{32768}e^{-4i\omega}-\frac{181}{65536}e^{-5i\omega}+\frac{7}{32768}e^{-6i\omega}+\frac{7}{131072}e^{-7i\omega},$$

with $m' = -1$ and $C = -\frac{1}{2}$. And the associated filters are given by

$$\tilde{p}_1(\omega) = -e^{-i\omega}p_2(-\omega+\pi),\ \tilde{p}_2(\omega) = e^{-i\omega}p_1(-\omega+\pi),$$

$$q_1(\omega) = -p_1(\omega),\ q_2(\omega) = -p_2(\omega),\ \tilde{q}_1(\omega) = -\tilde{p}_1(\omega),\ \tilde{q}_2(\omega) = -\tilde{p}_2(\omega).$$

The multiscaling functions and wavelets are shown in Figures 1-4. According to Theorem 3.3, the system has desired balancing order 4.

References

[1] S. Bacchelli, M. Cotronei, and D. Lazzaro, *An algebraic construction of k-balanced multiwavelets via the lifting scheme*, Numer. Algorithms **23** (2000), no. 4, 329–356.

[2] Y. Choi and J. Jung, *Totally interpolating biorthogonal multiwavelet systems with approximation order*, Preprint.

[3] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[4] G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Construction of orthogonal wavelets using fractal interpolation functions*, SIAM J. Math. Anal. **27** (1996), no. 4, 1158–1192.
[5] G. B. Folland, *Fourier Analysis and Its Applications*, Brooks/Cole Publishing Company, A Division of Wadsworth, Inc. 1992.
[6] D. P. Hardin and J. A. Marasovich, *Biorthogonal Multiwavelets on* $[-1, 1]$, Appl. Comput. Harmon. Anal. **7** (1999), no. 1, 34–53.
[7] S. Hongli, C. Yuanli, and Q. Zulian, *On Design of Multiwavelet Prefilters*, Appl. Math. Comput. **172** (2006), no. 2, 1175–1187.
[8] Q. Jiang, *On the Regularity of Matrix Refinable Functions*, SIAM J. Math. Anal. **29** (1998), no. 5, 1157–1176.
[9] K. Koch, *Interpolating scaling vectors*, Int. J. Wavelets Multiresolut. Inf. Process. **3** (2005), no. 3, 389–416.
[10] ______, *Multivariate orthogonal interpolating scaling vectors*, Appl. Comput. Harmon. Anal. **22** (2007), no. 2, 198–216.
[11] J. Lebrun and M. Vetterli, *Balanced multiwavelets theory and design*, IEEE Trans. Signal Process. **46** (1998), no. 4, 1119–1125.
[12] ______, *High order balanced multiwavelets: theory, factorization, and design*, IEEE Trans. Signal Process. **49** (2001), no. 9, 1918–1930.
[13] P. R. Massopust, D. K. Rush, and P. J. Fleet, *On the support properties of scaling vectors*, Appl. Comput. Harmon. Anal. **3** (1996), no. 3, 229–238.
[14] G. Plonka and V. Strela, *From wavelets to multiwavelets*, Mathematical Methods for Curves and Surface II(M. Dahlen, T. Lyche, and L. L. Schumaker, Eds. Vanderbilt Univ. Press, Nashville), (1998), 375–399.
[15] I. W. Selesnick, *Interpolating multiwavelet bases and the sampling theorem*, IEEE Trans. Signal Process. **47** (1999), no. 6, 1615–1621.
[16] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, MA, 1995.
[17] W. Sweldens and R. Piessens, *Wavelet sampling techniques*, Proc. Statistical Computing Section, 20–29, 1993.
[18] X. G. Xia, J. S. Geronimo, D. P. Hardin, and B. W. Suter, *Design of prefilters for discrete multiwavelet transforms*, IEEE Trans. Signal Process. **44** (1996), 25–35.
[19] ______, *Why and how prefiltering for discrete multiwavelet transforms*, Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Proces. Vol. 1, 1578–1581, 1996.
[20] S. Yang and H. Wang, *High-order balanced multiwavelets with dilation factor a*, Appl. Math. Comput. **181** (2006), no. 1, 362–369.
[21] J. K. Zhang, T. N. Davidson, Z. Q. Luo, and K. M. Wong, *Design of interpolating biorthogonal multiwavelet systems with compact support*, Appl. Comput. Harmon. Anal. **11** (2001), no. 3, 420–438.

Youngwoo Choi
Department of Mathematics
Ajou University
Suwon 443-749, Korea
E-mail address: youngwoo@ajou.ac.kr

Jaewon Jung
Department of Mathematics
Ajou University
Suwon 443-749, Korea
E-mail address: jaewon95@ajou.ac.kr