

**ON THE RATES OF THE ALMOST SURE CONVERGENCE  
 FOR SELF-NORMALIZED LAW OF  
 THE ITERATED LOGARITHM**

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ABSTRACT. Let  $\{X_i, i \geq 1\}$  be a sequence of i.i.d. nondegenerate random variables which is in the domain of attraction of the normal law with mean zero and possibly infinite variance. Denote  $S_n = \sum_{i=1}^n X_i$ ,  $M_n = \max_{1 \leq i \leq n} |S_i|$  and  $V_n^2 = \sum_{i=1}^n X_i^2$ . Then for  $d > -1$ , we showed that under some regularity conditions,

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n)\} \\ &= \frac{2}{\sqrt{\pi}(1+d)} \Gamma(d+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+2}} \text{ a.s.} \end{aligned}$$

holds in this paper, where  $I\{\cdot\}$  denotes the indicator function.

**1. Introduction and main results**

Let  $\{X_i, i \geq 1\}$  be a sequence of i.i.d. random variables with common distribution function, and set  $S_n = \sum_{i=1}^n X_i$  and  $V_n^2 = \sum_{i=1}^n X_i^2$  for  $n \geq 1$ ,  $\log x = \ln(x \vee e)$  and  $\log \log x = \log(\log x)$ . Hsu and Robbins [9] and Erdős [6] established the well-known complete convergence,

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n| \geq \varepsilon n) < \infty, \quad \varepsilon > 0$$

if and only if  $\mathbb{E}X = 0$  and  $\mathbb{E}X^2 < \infty$ . Baum and Katz [1] extended this result and proved that for  $1 \leq p < 2$  and  $r \geq p$ ,

$$(1.1) \quad \sum_{n=1}^{\infty} n^{r-2} \mathbb{P}(|S_n| \geq \varepsilon n^{1/p}) < \infty, \quad \varepsilon > 0$$

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holds if and only if  $EX = 0$  and  $E|X|^p < \infty$ .

Many authors considered various extensions of the results of Hsu-Robbins-Erdős and Baum-Katz. Some of them studied the precise convergence rates of the infinite sums as  $\varepsilon \rightarrow 0$  (c.f. Heyde [8], Chen [2] and Spătaru [13]). But, this kind of results do not hold for  $p = 2$ . However, by replacing  $n^{1/p}$  by  $\sqrt{n \log \log n}$ , Gut and Spătaru [7] established the following results called the precise asymptotics of the law of the iterated logarithm.

**Theorem A.** *Suppose that  $EX = 0$ ,  $EX^2 = \sigma^2$  and  $EX^2(\log \log |X|)^{1+\delta} < \infty$  for some  $\delta > 0$ , and let  $a_n = O(\sqrt{n}/(\log \log n)^\gamma)$  for some  $\gamma > 1/2$ . Then*

$$(1.2) \quad \lim_{\varepsilon \searrow 1} \sqrt{\varepsilon^2 - 1} \sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \varepsilon \sigma \sqrt{2n \log \log n} + a_n) = 1.$$

In the paper of Gut and Spătaru [7], the proof of Theorem A relies heavily on the Berry-Esseen inequality. Of late, by using a different method, that is, the strong approximation method (cf. Csörgő and Révész [3]), Zhang [14] provided the sufficient and necessary conditions of precise asymptotics of the law of the iterated logarithm not only for  $S_n$ , but also for  $M_n$ , here and in what follows, we denote  $M_n = \max_{1 \leq i \leq n} |S_i|$ .

On the other hand, it is well-known that the so-called self-normalized limit theorems put a totally new countenance upon classical limit theorems. The main advantage of self-normalization is that the limit theorems in probability and statistics, compared with the classical versions, may still hold under less or even without any moment condition (cf. Shao [12]) if the normalizing constants in the classical versions are replaced by an appropriate sequence of random variables. For a survey on recent developments in this area, the reader is referred to Lai and Shao [10] or de la Peña, Lai and Shao [5] for details.

Naturally, Pang, Zhang and Wang [11] studied the precise asymptotics in the self-normalized law of the iterated logarithm and below are their main results.

**Theorem B.** *Let  $\{X_i, i \geq 1\}$  be a sequence of i.i.d. nondegenerate random variables which is in the domain of attraction of the normal law with mean zero. Suppose  $l(x) := EX_1^2 I\{|X_1| \leq x\} \leq c_1 \exp(c_2(\log x)^\beta)$  for some  $c_1 > 0$ ,  $c_2 > 0$  and  $0 \leq \beta < 1$ ,  $\alpha_n = O(1/\log \log n)$ . Then, for  $d > -1$ , we have*

$$(1.3) \quad \begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P(M_n \geq \sqrt{2V_n^2 \log \log n}(\varepsilon + \alpha_n)) \\ &= \frac{2}{\sqrt{\pi}(1+d)} \Gamma(d+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+2}} \end{aligned}$$

and

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} P(|S_n| \geq \sqrt{2V_n^2 \log \log n}(\varepsilon + \alpha_n))$$

$$(1.4) \quad = \frac{1}{\sqrt{\pi}(1+d)}\Gamma(d+3/2),$$

where  $\Gamma(\cdot)$  is the gamma function.

**Theorem C.** Let  $\{X_i, i \geq 1\}$  be a sequence of i.i.d. nondegenerate, symmetric random variables which is in the domain of attraction of the normal law with mean zero. Suppose  $l(x) \leq c_1 \exp(c_2(\log x)^\beta)$  for some  $c_1 > 0, c_2 > 0$  and  $0 \leq \beta < 1$ . Moreover, assume that for  $a > -1$  and  $b > -1/2, \alpha_n(\varepsilon)$  is a nonnegative function of  $\varepsilon$  such that

$$(1.5) \quad \alpha_n(\varepsilon) \log \log n \rightarrow \tau \text{ as } n \rightarrow \infty \text{ and } \varepsilon \searrow \sqrt{1+a}.$$

Then

$$(1.6) \quad \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} P(|S_n| \geq \sqrt{2V_n^2 \log \log n}(\varepsilon + \alpha_n(\varepsilon))) \\ = \sqrt{\frac{1}{\pi(1+a)}} \exp(-2\tau\sqrt{1+a})\Gamma(b+1/2),$$

where,  $\tau$  is a finite constant.

The purpose of the present paper is devoted to study the almost sure convergence versions of Theorem B and Theorem C. In what follows, we denote  $I\{\cdot\}$  the indicator function and  $C$  the positive constant whose value can differ from line to line, and  $C_1$  and  $C_2$  stand for two universal constants respectively. Now, we give our main results.

**Theorem 1.1.** Suppose the assumptions of Theorem B are satisfied, and in addition, if there exists a  $\delta_0 > 1/2$  such that

$$nP(|X_1| > \eta_n) \leq C_1(\log \log n)^{-(2d+2+\delta_0)}$$

for some  $d > -1$ , where  $\eta_n$  is defined in (2.1) and (2.2) below. Then we have

$$(1.7) \quad \lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n)\} \\ = \frac{2}{\sqrt{\pi}(1+d)}\Gamma(d+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+2}} \text{ a.s.}$$

and

$$(1.8) \quad \lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} I\{|S_n|/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n)\} \\ = \frac{1}{\sqrt{\pi}(1+d)}\Gamma(d+3/2) \text{ a.s.}$$

**Theorem 1.2** Under the assumptions of Theorem C with  $a > -1, b > -1/2$  replaced by  $-1 < a < -1/2, b > -1/2$  or  $a = -1/2, -1/2 < b < -1/4$ , and in

addition, if there exists a  $\delta'_0 > 0$  such that

$$nP(|X_1| > \eta_n) \leq C_2(\log n)^{-(2a+2)}(\log \log n)^{-(2b+1+\delta'_0)}$$

for above  $a$  and  $b$ . Then we have

$$(1.9) \quad \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} I\{|S_n|/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n(\varepsilon))\} \\ = \sqrt{\frac{1}{\pi(1+a)}} \exp(-2\tau\sqrt{1+a})\Gamma(b+1/2) \text{ a.s.,}$$

where,  $\tau$  is the same finite constant as that in Theorem C.

### 2. Proofs

One of the ideas of proofs is based on the truncated random variables. That is, let

$$(2.1) \quad l(x) = E(X_1^2 I\{|X_1| \leq x\}), \quad b = \inf\{x \geq 1 : l(x) > 0\},$$

and

$$(2.2) \quad \eta_j = \inf\{s : s \geq b + 1, \frac{l(s)}{s^2} \leq \frac{(\log \log j)^2}{j}\} \text{ for } j = 1, 2, 3, \dots$$

Then it is easy to see  $\eta_n^2(\log \log n)^2 \approx nl(\eta_n)$ . For each  $n$  and  $1 \leq i \leq n$ , we denote

$$(2.3) \quad \bar{X}_{ni} = X_i I\{|X_i| \leq \eta_n\}, \quad \bar{S}_n = \sum_{i=1}^n \bar{X}_{ni}, \quad \bar{M}_n = \max_{1 \leq i \leq n} |\bar{S}_i|, \quad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_{ni}^2.$$

Furthermore, the following lemma is taken from Csörgő, Szyszkowicz and Wang [4] and will be used in the proofs.

**Lemma 2.1.** *Let  $X$  be a random variable, and denote  $l(x) = EX^2 I\{|X| \leq x\}$ . The following statements are equivalent:*

- a)  $X$  is in the domain of attraction of the normal law,
- b)  $x^2 P(|X| > x) = o(l(x))$ ,
- c)  $x E(|X| I\{|X| > x\}) = o(l(x))$ ,
- d)  $E(|X|^n I\{|X| \leq x\}) = o(x^{n-2} l(x))$  for  $n > 2$ .

Next, we give the following lemma which is also useful for the proofs of theorems and is stated as follows.

**Lemma 2.2.** *Let  $d > -1$  and  $\{X_i, i \geq 1\}$  be a sequence of i.i.d. nondegenerate random variables which is in the domain of attraction of the normal law with mean zero. Suppose  $f(\cdot)$  is a real function such that  $\sup_{x \in R} |f(x)| \leq C$*

and  $\sup_{x \in \mathbb{R}} |f'(x)| \leq C$ , and in addition, if there exists a  $\delta_0 > 0$  such that  $nP(|X_1| > \eta_n) \leq C(\log \log n)^{-(2d+2+\delta_0)}$ . Then for  $l \geq m \geq 1$ , we have

$$(2.4) \quad \begin{cases} \text{Var}\left(\sum_{n=m}^l \frac{(\log \log n)^d}{n \log n} f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n(\varepsilon + \alpha_n)}}\right)\right) \leq C \frac{(\log \log m)^{2d+1/2}}{(\varepsilon + \alpha_m) \log m}, \\ \text{Var}\left(\sum_{n=m}^l \frac{(\log \log n)^d}{n \log n} f\left(\frac{\bar{S}_n}{\sqrt{2nl(\eta_n) \log \log n(\varepsilon + \alpha_n)}}\right)\right) \leq C \frac{(\log \log m)^{2d+1/2}}{(\varepsilon + \alpha_m) \log m}, \\ \text{Var}\left(\sum_{n=m}^l \frac{(\log \log n)^d}{n \log n} f\left(\frac{\bar{V}_n^2}{nl(\eta_n)}\right)\right) \leq C \frac{(\log \log m)^{2d}}{\log m}, \\ \text{Var}\left(\sum_{n=m}^l \frac{(\log \log n)^d}{n \log n} \sum_{i=1}^n I\{|X_i| \geq \eta_n\}\right) \leq \frac{C}{(\log \log m)^{\delta_0}}. \end{cases}$$

*Proof.* Denote  $\bar{M}_{ij} = \max_{i \leq k < j} |\bar{S}_k - \bar{S}_i|$ . Obviously,  $\bar{M}_{ij}$  is independent of  $\bar{M}_i$ . Using  $\mathbf{E}X = 0$ ,  $\eta_n^2(\log \log n)^2 \approx nl(\eta_n)$ , Hölder inequality and Lemma 2.1, for  $i < j$  we have

$$(2.5) \quad \begin{aligned} & \text{Cov}\left(f\left(\frac{\bar{M}_i}{\sqrt{2il(\eta_i) \log \log i(\varepsilon + \alpha_i)}}\right), f\left(\frac{\bar{M}_j}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}}\right)\right) \\ &= \text{Cov}\left(f\left(\frac{\bar{M}_i}{\sqrt{2il(\eta_i) \log \log i(\varepsilon + \alpha_i)}}\right), \right. \\ & \quad \left. f\left(\frac{\bar{M}_j}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}}\right) - f\left(\frac{\bar{M}_{ij}}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}}\right)\right) \\ &\leq C \mathbf{E}\left|f\left(\frac{\bar{M}_j}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}}\right) - f\left(\frac{\bar{M}_{ij}}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}}\right)\right| \\ &\leq C \frac{\mathbf{E}\bar{M}_i}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}} \\ &\leq C \frac{\mathbf{E} \max_{1 \leq k \leq i} |\bar{S}_k - \mathbf{E}\bar{S}_k| + i\mathbf{E}|X_1|I\{|X_1| > \eta_i\}}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}} \\ &\leq C \frac{\sqrt{il(\eta_i)} + o(il(\eta_i)/\eta_i)}{\sqrt{2jl(\eta_j) \log \log j(\varepsilon + \alpha_j)}} \\ &\leq C \frac{\sqrt{i}}{\sqrt{j \log \log j(\varepsilon + \alpha_j)}} + o\left(\frac{\sqrt{i} \log \log i}{\sqrt{j \log \log j(\varepsilon + \alpha_j)}}\right), \end{aligned}$$

which yields

$$\begin{aligned} & \text{Var}\left(\sum_{n=m}^l \frac{(\log \log n)^d}{n \log n} f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n(\varepsilon + \alpha_n)}}\right)\right) \\ &\leq C \sum_{n=m}^l \frac{(\log \log n)^{2d}}{n^2(\log n)^2} + 2 \sum_{i=m}^{l-1} \sum_{j=i+1}^l \frac{(\log \log i)^d}{i \log i} \frac{(\log \log j)^d}{j \log j} \\ & \quad \cdot \left(\frac{C\sqrt{i}}{\sqrt{j \log \log j(\varepsilon + \alpha_j)}} + o\left(\frac{\sqrt{i} \log \log i}{\sqrt{j \log \log j(\varepsilon + \alpha_j)}}\right)\right) \end{aligned}$$

$$\begin{aligned}
&\leq C \frac{(\log \log m)^{2d}}{m(\log m)^2} + C \frac{(\log \log m)^{2d-1/2}}{(\varepsilon + \alpha_m) \log m} + o\left(\frac{(\log \log m)^{2d+1/2}}{(\varepsilon + \alpha_m) \log m}\right) \\
(2.6) \quad &\leq C \frac{(\log \log m)^{2d+1/2}}{(\varepsilon + \alpha_m) \log m}.
\end{aligned}$$

The first part of (2.4) is proved. Similarly, the second and the third parts of (2.4) can be proved by the same arguments, and we omit the details here. Consider the fourth part of (2.4) now. In view of the assumption  $n\mathbb{P}(|X_1| \geq \eta_n) \leq C_1(\log \log n)^{-(2d+2+\delta_0)}$  for some  $\delta_0 > 0$ , we have

$$\begin{aligned}
(2.7) \quad &\text{Var}\left(\sum_{n=m}^l \frac{(\log \log n)^d}{n \log n} \sum_{i=1}^n I\{|X_i| \geq \eta_n\}\right) \\
&\leq C \sum_{n=m}^l \frac{(\log \log n)^{2d}}{n^2(\log n)^2} \cdot n\mathbb{P}(|X_1| \geq \eta_n) \\
&\quad + 2 \sum_{j=m+1}^l \sum_{i=m}^{j-1} \frac{(\log \log i)^d}{i \log i} \frac{(\log \log j)^d}{j \log j} \text{Cov}\left(\sum_{k=1}^i I\{|X_k| \geq \eta_i\}, \sum_{k=1}^j I\{|X_k| \geq \eta_j\}\right) \\
&\leq C \sum_{n=m}^l \frac{(\log \log n)^{2d}}{n^2(\log n)^2} \cdot (\log \log n)^{-(2d+2+\delta_0)} \\
&\quad + 2 \sum_{j=m+1}^l \sum_{i=m}^{j-1} \frac{(\log \log i)^d}{i \log i} \frac{(\log \log j)^d}{j \log j} \cdot i\mathbb{P}(|X_1| \geq \eta_i) \\
&\leq C \frac{(\log \log m)^{-2-\delta_0}}{m(\log m)^2} + C \sum_{j=m+1}^l \sum_{i=m}^{j-1} \frac{(\log \log i)^d}{i \log i} \frac{(\log \log j)^d}{j \log j} \cdot (\log \log i)^{-(2d+2+\delta_0)} \\
&\leq \frac{C}{(\log \log m)^{\delta_0}}. \quad \square
\end{aligned}$$

*Proofs of Theorem 1.1 and Theorem 1.2.* We only show (1.7), since (1.8) and (1.9) can be proved by the same way. We first give an upper and a lower bound for the indicator function of self-normalized partial sums. For any given  $0 < \varepsilon < 1$ , it is clear that

$$\begin{aligned}
(2.8) \quad I\left\{\frac{M_k}{V_k} \geq x\right\} &\leq I\left\{\frac{\bar{M}_k}{\sqrt{(1-\varepsilon)kl(\eta_k)}} \geq x\right\} \\
&\quad + I\left\{\bigcup_{i=1}^k \{|X_i| > \eta_k\}\right\} + I\{\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\}
\end{aligned}$$

and

$$I\left\{\frac{M_k}{V_k} \geq x\right\} \geq I\left\{\frac{\bar{M}_k}{\sqrt{(1+\varepsilon)kl(\eta_k)}} \geq x\right\}$$

$$(2.9) \quad -I\left\{\bigcup_{i=1}^k \{|X_i| > \eta_k\}\right\} - I\{\bar{V}_k^2 > (1 + \varepsilon)kl(\eta_k)\}.$$

Denote  $B(\varepsilon) = \exp(\exp(1/(\varepsilon^2 M)))$  for any  $M > 0$ . Fix  $M > 4$  and  $0 < \delta_1 < 1/2$ , and let  $f(\cdot)$  be a real function, such that  $I\{|x| \geq \sqrt{1 - \delta_1}\} \leq f(x) \leq I\{|x| \geq 1 - \delta_1\}$ ,  $\sup_x |f'(x)| < \infty$ . Define  $\varepsilon_k = 1/k$ ,  $k \geq 4M + 1$ . Then, it follows from Lemma 2.2 and noting that  $\alpha_n = O(1/\log \log n) = O(M/k^2) = o(\varepsilon_k)$  for  $n > B(\varepsilon_k)$ ,

$$(2.10) \quad \begin{aligned} & \text{Var}\left(\sum_{n>B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n(\varepsilon_k + \alpha_n)}}\right)\right) \\ & \leq C \frac{(\log \log B(\varepsilon_k))^{2d+1/2}}{\varepsilon_k \log B(\varepsilon_k)} \\ & \leq CM^{-(2d+1/2)} k^{4d+2} e^{-4k}, \end{aligned}$$

which together with Borel-Cantelli lemma yields

$$(2.11) \quad \begin{aligned} & \sum_{n>B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \left(f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n(\varepsilon_k + \alpha_n)}}\right) \right. \\ & \left. - \mathbf{E}f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n(\varepsilon_k + \alpha_n)}}\right)\right) \rightarrow 0 \text{ a.s.} \end{aligned}$$

Similarly, one has

$$(2.12) \quad \sum_{n>B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \left(f\left(\frac{\bar{V}_n^2}{nl(\eta_n)}\right) - \mathbf{E}f\left(\frac{\bar{V}_n^2}{nl(\eta_n)}\right)\right) \rightarrow 0 \text{ a.s.}$$

By means of the similar arguments and using the assumption  $\delta_0 > 1/2$ , one also has

$$(2.13) \quad \sum_{n>B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \left(\sum_{i=1}^n I\{|X_i| \geq \eta_n\} - n\mathbf{P}(|X_1| \geq \eta_n)\right) \rightarrow 0 \text{ a.s.},$$

since

$$(2.14) \quad \text{Var}\left(\sum_{n>B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \sum_{i=1}^n I\{|X_i| \geq \eta_n\}\right) \leq \frac{CM^{\delta_0}}{k^{2\delta_0}}.$$

Using the inequality (2.8), we have

$$(2.15) \quad \begin{aligned} & \lim_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n>B(\varepsilon)} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n(\varepsilon + \alpha_n)}\} \\ & \leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n>B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n(\varepsilon_k + \alpha_n)}\} \end{aligned}$$

$$\begin{aligned} &\leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \left( I\{\bar{M}_n / \sqrt{(1 - \delta_1)nl(\eta_n)} \geq \sqrt{2 \log \log n}(\varepsilon_k + \alpha_n)\} \right. \\ &\quad \left. + I\left\{ \bigcup_{i=1}^n \{|X_i| \geq \eta_n\} \right\} + I\{\bar{V}_n^2 / (nl(\eta_n)) < (1 - \delta_1)\} \right) \\ &=: \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} (I_1 + I_2 + I_3). \end{aligned}$$

We are going on to treat with the three terms above respectively. Together (2.11) with Propositions 3.1 and 3.2 in Pang, Zhang and Wang [11] yields

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} I_1 \\ &\leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n}(\varepsilon_k + \alpha_n)}\right) \\ &\leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \mathbf{E}f\left(\frac{\bar{M}_n}{\sqrt{2nl(\eta_n) \log \log n}(\varepsilon_k + \alpha_n)}\right) \\ &\leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \\ &\quad \cdot \mathbf{P}\left(\frac{\bar{M}_n}{\sqrt{nl(\eta_n)}} \geq (1 - \delta_1)\sqrt{2 \log \log n}(\varepsilon_k + \alpha_n)\right) \\ (2.16) &\leq (1 - \delta_1)^{-2(d+1)} \frac{2}{\sqrt{\pi}(1 + d)} \Gamma(d + 3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^{2d+2}} \text{ a.s.} \end{aligned}$$

For the second term of (2.15), in view of (2.13) we have

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} I_2 \\ &\leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} \sum_{i=1}^n I\{|X_i| \geq \eta_n\} \\ (2.17) &\leq \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{\log n} \mathbf{P}(|X_1| \geq \sqrt{1 - \delta_1}\eta_n) = 0 \text{ a.s.,} \end{aligned}$$

where in the equation above, we have used the similar arguments for showing the right hand of (3.10) in Pang, Zhang and Wang [11]. Similarly, by using the same arguments appeared above, (2.12) and the inequality (3.13) in Pang,



Zhang and Wang [11], we have

$$(2.18) \quad \limsup_{k \rightarrow \infty} \varepsilon_{k-1}^{2(d+1)} \sum_{n > B(\varepsilon_k)} \frac{(\log \log n)^d}{n \log n} I_3 = 0 \text{ a.s.}$$

On the other hand,

$$\begin{aligned} & \limsup_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n \leq B(\varepsilon)} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n)\} \\ & \leq \limsup_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n \leq B(\varepsilon)} \frac{(\log \log n)^d}{n \log n} \\ & \leq C \limsup_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \int_{e^\varepsilon}^{B(\varepsilon)} \frac{(\log \log x)^d}{x \log x} dx \\ & \leq \frac{C}{1+d} \limsup_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} (\log \log B(\varepsilon))^{d+1} \\ (2.19) & = \frac{C}{1+d} \cdot \frac{1}{M^{d+1}}. \end{aligned}$$

Now, it follows from (2.15)-(2.19) and by the arbitrariness of  $\delta_1$  and  $M$ , we have

$$\begin{aligned} & \limsup_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n)\} \\ (2.20) & \leq \frac{2}{\sqrt{\pi}(1+d)} \Gamma(d+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+2}} \text{ a.s.} \end{aligned}$$

The similar arguments also yield

$$\begin{aligned} & \liminf_{\varepsilon \searrow 0} \varepsilon^{2(d+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^d}{n \log n} I\{M_n/V_n \geq \sqrt{2 \log \log n}(\varepsilon + \alpha_n)\} \\ (2.21) & \geq \frac{2}{\sqrt{\pi}(1+d)} \Gamma(d+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+2}} \text{ a.s.,} \end{aligned}$$

we omit the details here. The proofs are now completed. □

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