SOME INVARIANT SUBSPACES
FOR SUBSCALAR OPERATORS

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ABSTRACT. In this note, we prove that every subscalar operator with finite spectrum is algebraic. In particular, a quasi-nilpotent subscalar operator is nilpotent. We also prove that every subscalar operator with property (δ) on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace.

1. Introduction

In 1990, Eschmeier and Prunaru [12] proved that if $T \in L(X)$ has property (β) whose spectrum $\sigma(T)$ is thick, then T has a non-trivial invariant closed linear subspace. In this note, we will investigate the quasi-nilpotent part and invariant subspaces of subscalar operators on a Banach space. The relevant definitions and background material will be collected in Section 1. Section 2 contains the local spectral properties of subscalar operators. In this section, we show that if $\bigcap_{\lambda \in \mathbb{C}}(T-\lambda)^p X = \{0\}$ for some p, then $N((T-\lambda)^p) = X_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Moreover, if $T \in L(X)$ such that $\bigcap_{\lambda \in \mathbb{C}}(T-\lambda)^p X = \{0\}$ for some p, then T is algebraic if and only if the spectrum $\sigma(T)$ is finite. We also prove that every subscalar operator with finite spectrum is algebraic. In particular, a quasi-nilpotent subscalar operator is nilpotent. Finally, we show that every subscalar operator with property (δ) on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace. This results are exemplified in the case of generalized scalar operators, subnormal operators, k-quasihyponormal operators, M-hyponormal operators, ω-hyponormal operators, log-hyponormal operators and hyponormal operators.

We first recall some basic notions and results from local spectra theory. Let X be a complex Banach space and $L(X)$ denotes the Banach algebra of all bounded linear operators of X itself, equipped with the usual operator norm. For $T \in L(X)$, TX and $N(T)$ will denote the range and kernel of T, respectively. As usual, for $T \in L(X)$ and let $\sigma(T)$ and $\rho(T)$ denote the *spectrum*

and *resolvent set* of T, respectively and let $Lat(T)$ stand for the collection of all T-invariant closed linear subspaces of X.

An operator $T \in L(X)$ is called *decomposable* if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there are T-invariant closed linear subspaces Y and Z of X such that

$$Y + Z = X, \quad \sigma(T|Y) \subseteq U \quad \text{and} \quad \sigma(T|Z) \subseteq V.$$

An operator $T \in L(X)$ is said to have *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(T - \lambda)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, it follows that also $f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U. Obviously, property (β) implies that T has the *single-valued extension property* (SVEP), which means that for every open $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$.

An operator $T \in L(X)$ is said to have the *decompositions property* (δ) if for every open covering $\{U, V\}$ of $\mathbb{C}$ and for every $x \in X$ there exist a pair of elements $u, v \in X$ and a pair of analytic functions $f : \mathbb{C} \setminus \overline{U} \longrightarrow X$ and $g : \mathbb{C} \setminus \overline{V} \longrightarrow X$ such that $x = u + v$,

$$u = (T - \lambda)f(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \overline{U},$$

$$v = (T - \lambda)g(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \overline{V}.$$

It has been observed in [3] that an operator $T \in L(X)$ is decomposable if and only if it has both properties (β) and (δ). In [2], Albrecht and Eschmeier proved that the property (β) and (δ) are dual to each other in the sense that an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* on the dual space X^* satisfies (δ) and that the corresponding statement remains valid if both properties are interchanged.

Given an arbitrary operator $T \in L(X)$, let $\sigma_T(x) \subseteq \mathbb{C}$ denote the *local spectrum* of T at the point $x \in X$, i.e., the complement of the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to X$ such that $(T - \mu)f(\mu) = x$ holds for all $\mu \in U$. For every closed subset F of $\mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *analytic spectral subspace* of T. It is well known that T has the SVEP if and only if $X_T(\phi) = \{0\}$, and this is the case if and only if $X_T(\phi)$ is closed, see [18].

If $F \subseteq \mathbb{C}$, then the *algebraic spectral subspace* $E_T(F)$ is the largest subspace of X on which all the restrictions of $T - \lambda$, $\lambda \in \mathbb{C} \setminus F$, are surjective. Thus $E_T(F)$ is the largest linear subspace of Y for which

$$(T - \lambda)Y = Y \quad \text{for} \quad \text{all} \quad \lambda \in \mathbb{C} \setminus F.$$

Obviously, $E_T(F)$ is the largest linear subspace Y for which $(T - \lambda)Y = Y$ for all $\lambda \in \mathbb{C} \setminus F$. Pták and Vrbová proved in [24] that if T is a normal operator on a complex Hilbert space H, then the ranges of the spectral projections can

be represented in the form

$$H_T(F) = \Theta(F)H = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)H$$

for all closed sets $F \subseteq \mathbb{C}$, where Θ denotes the spectral measure associated with T.

An important generalization of normal operators to the setting of Banach spaces is the class of generalized scalar operators.

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. An operator $T \in L(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$. Every linear operator on a finite dimensional space as well as every spectral operator of finite type are generalized scalar operators.

2. Main results

An operator $T \in L(X)$ is said to be *subscalar*, if T is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces. Obviously, hyponormal operators and isometries are subscalar. It is well known [18] that if there exist an operator $S \in L(X)$ and a constant $c > 0$ for which

$$\|(S - \bar{\lambda})x\| \leq c\|(T - \lambda)x\|$$

for all $\lambda \in \mathbb{C}$ and $x \in X$, then T is subscalar. In particular, if $T \in L(H)$ on a complex Hilbert space H is M-hyponormal, i.e., there exists a constant $M > 0$ such that

$$\|(T^* - \bar{\lambda})x\| \leq M\|(T - \lambda)x\| \text{ for all } x \in H \text{ and } \lambda \in \mathbb{C},$$

where T^* denotes the Hilbert space adjoint of T, then T is subscalar. The class of subscalar operators is strictly larger than the class of generalized scalar operators. For instance, the unilateral right shift T on the space $\ell^2(\mathbb{N})$ is hyponormal. But T is not generalized scalar.

Theorem 2.1 ([23])**.** *For every subscalar operator $T \in L(X)$ on a Banach space X, there exists an integer $p \in \mathbb{N}$ such that*

$$X_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)^p X \text{ for all closed sets } F \subseteq \mathbb{C}.$$

Given an operator $T \in L(X)$, the *quasi-nilpotent part* of T is the set

$$H_0(T) := \{x \in X : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is clear that $H_0(T)$ is a linear subspace of X and in fact hyperinvariant under T, generally not closed. It is clear that T is quasi-nilpotent if and only

if $H_0(T) = X$. Moreover, if T is invertible, then $H_0(T) = \{0\}$. It is well known that

$$N(T^n) \subseteq N^\infty(T) \subseteq H_0(T) \subseteq X_T(\{0\})$$

holds for every positive integer $n \in \mathbb{N}$ and where $N^\infty(T) := \bigcup_{n=1}^\infty N(T^n)$ is the *generalized kernel* of T.

It is clear that the restriction of an operator T with property (β) to a closed invariant subspace certainly inherits this property. We have the following.

Lemma 2.2. *Assume that the operator $T \in L(X)$ is similar to the restriction of $S \in L(Y)$ to an invariant subspace. If S has SVEP, then T has SVEP. Moreover, if S has property (β), then T has property (β).*

Theorem 2.3. *If $\bigcap_{\lambda \in \mathbb{C}}(T - \lambda)^p X = \{0\}$ for some p, then $N((T - \lambda)^p) = X_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$.*

Proof. By Proposition 1.2.16(d) of [18], $N((T - \lambda)^p) \subseteq X_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Now suppose that $\bigcap_{\lambda \in \mathbb{C}}(T - \lambda)^p X = \{0\}$ for some p. We want to show that

$$(T - \lambda)^p X_T(\{\lambda\}) = 0.$$

By the hypothesis of the theorem in suffices to show that $(T - \lambda)^p X_T(\{\lambda\}) \subseteq (T - \mu)^p X$ for all $\mu \in \mathbb{C}$. Clearly, this is true if $\mu = \lambda$. Assume $\mu \neq \lambda$. By Proposition 1.2.16(b) of [18], $X_T(\{\lambda\}) = (T - \mu)^p X_T(\{\lambda\})$. Thus we have

$$(T - \lambda)^p X_T(\{\lambda\}) \subseteq X_T(\{\lambda\}) = (T - \mu)^p X_T(\{\lambda\}) \subseteq (T - \mu)^p X.$$

This completes the proof. $\square$

An operator $T \in L(X)$ on a complex Banach space X is said to be *algebraic* if $p(T) = 0$ for some non-zero complex polynomial p.

We may now extend Proposition 1.5.10 [18] of generalized scalar operators to subscalar operators.

Theorem 2.4. *If $T \in L(X)$ such that $\bigcap_{\lambda \in \mathbb{C}}(T - \lambda)^p X = \{0\}$ for some p, then T is algebraic if and only if the spectrum $\sigma(T)$ is finite.*

Proof. If T is algebraic, then $\sigma(T)$ is finite. Now assume that $T \in L(X)$ such that $\bigcap_{\lambda \in \mathbb{C}}(T - \lambda)^p X = \{0\}$ for some p, and $\sigma(T)$ is finite. Using Proposition 1.2.16(b) of [18] we know that $X_T(\phi) = (T - \lambda)^p X_T(\phi) \subseteq (T - \lambda)^p X$ for all $\lambda \in \mathbb{C}$. By part (f) of Proposition 1.2.16 of [18], we conclude that T has SVEP. Then by Proposition 1.2.16(g) of [18], $\sigma(T) = \{\lambda_1, \ldots, \lambda_r\}$ implies

$$X = X_T(\sigma(T)) = X_T(\{\lambda_1\}) + \cdots + X_T(\{\lambda_1\}).$$

It follows from Theorem 2.3 that $X = N((T - \lambda_1)^p) + \cdots + N((T - \lambda_r)^p)$. Hence $(T - \lambda_1)^p(T - \lambda_2)^p \cdots (T - \lambda_r)^p = 0$ on X. This shows that T is algebraic. $\square$

Note that the above result applies to all subscalar operators.

Corollary 2.5. *A subscalar operator with finite spectrum is algebraic. In particular, a quasi-nilpotent subscalar operator is nilpotent.*

It is well known from Theorem 1.5.13 [17] that all doubly power-bounded operators, i.e., T is invertible and $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$, and, in particular, all surjective isometries are generalized scalar. Thus every generalized scalar operator with finite spectrum is algebraic.

M. Putinar proved that every hyponormal operators are subscalar of order 2. Also, Benhida and Zerouali proved that if T is p-hyponormal, log-hyponormal or ω-hyponormal on a complex Hilbert space H, then T is subscalar. Moreover, E. Ko proved that every k-quasihyponormal operators are subscalar, see more details [4], [13] and [25]. Thus every hyponormal operator (k-quasihyponormal, ω-hyponormal) with finite spectrum is algebraic. In particular, a quasi-nilpotent hyponormal operator (k-quasihyponormal, ω-hyponormal) is nilpotent.

Given an operator $T \in L(X)$ on a complex Banach space X and a linear subspace M of X, we say that M is an *invariant subspace* of T if $TM \subseteq M$. Obviously $\{0\}$ and X are invariant subspaces and M invariant implies $\overline{M}$ invariant.

In [12], Eschmeier and Prunaru established that $Lat(T)$ is non-trivial provided that $\sigma(T)$ is thick, and that $Lat(T)$ is rich in the sense that it contains the lattice of all closed subspaces of some infinite-dimensional Banach space provided that the essential spectrum $\sigma_e(T)$ is thick. In [25], Putinar proved that all hyponormal operators have property (β). The preceding result subsumes, in particular, Brown's celebrated invariant subspace theorem for hyponormal operators with thick spectrum. In 1984, C. J. Read proved [26] that there exist quasi-nilpotent operators, and hence decomposable, on Banach spaces without non-trivial closed invariant subspaces. It is clear that the condition of thick spectrum cannot be dropped in general; see more details [18].

We shall also need the following elementary lemma which may be known; but we include a short proof, since we were not able to find a suitable reference.

Lemma 2.6. *Every decomposable operator whose spectrum contains at least two points has a non-trivial hyperinvariant closed linear subspace.*

Proof. Let $\lambda \in \sigma(T)$. Then, by Proposition 1.2.20 [18], the space $X_T(\{\lambda\})$ is a closed hyperinvariant subspace of T and $\sigma(T|X_T(\{\lambda\})) \subseteq \{\lambda\}$. Let U be an arbitrary open neighborhood of λ in $\mathbb{C}$. We choose another open set $V \subseteq \mathbb{C}$ such that $\lambda \notin V$ and $\{U, V\}$ is an open covering of $\mathbb{C}$. Since T is decomposable,

$$\sigma(T|X_T(\{\lambda\})) \subseteq U, \ \sigma(T|X_T(V)) \subseteq V, \ \text{and} \ X = X_T(\{\lambda\}) + X_T(V).$$

If $X_T(\{\lambda\}) = \{0\}$, then $\sigma(T) = \sigma(T|X_T(V)) \subseteq V$, which contradicts $\lambda \notin V$. If $X_T(\{\lambda\}) = X$, then $\sigma(T) = \sigma(T|X_T(\{\lambda\})) \subseteq \{\lambda\}$, which contradicts that $\sigma(T)$ contains at least two points. This contradiction shows that $X_T(\{\lambda\})$ is a non-trivial hyperinvariant closed linear subspace. This completes the proof. □

In light of Read's construction of a quasi-nilpotent, and hence decomposable, operator on a Banach space without non-trivial invariant subspaces, see [26].

We can now prove the main result of this section.

Theorem 2.7. *Every subscalar operator with property (δ) on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

Proof. Let $T \in L(X)$ be a subscalar operator with property (δ) on a Banach space X of dimension greater than 1. At first, we show that if $\sigma(T)$ contains at least two points, then T has a non-trivial hyperinvariant closed linear subspace. Since every subscalar operator has property (β), T is decomposable. It follows from Lemma 2.6 that T has a non-trivial hyperinvariant closed linear subspace. It remains to consider the case of subscalar operator $T \in L(X)$ such that X is at least two-dimensional and $\sigma(T)$ is a singleton. Then it follows from Theorem 2.4 that $T = \lambda I + N$ for some $\lambda \in \mathbb{C}$ and some nilpotent operator $N \in L(X)$. Let $p \in \mathbb{N}$ be the smallest integer for which $N^p = 0$, and choose an $x \in X$ for which $N^{p-1}x \neq 0$. The linear subspace generated by $N^{p-1}x$ is a one-dimensional T-invariant linear subspace of X. This completes the proof. $\square$

   The previous result extends [18, Proposition 1.5.11].
   The above result applies to all generalized scalar operators.

Corollary 2.8. *Every generalized scalar operator on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

   The above result applies to all k-quasihyponormal (isometries, M-hyponormal, ω-hyponormal, log-hyponormal or hyponormal) with property (δ). Thus every isometry with property (δ) on a Banach space of dimension greater than 1 has a non-trivial invariant closed linear subspace. Moreover, every M-hyponormal operator with property (δ) on a Hilbert space of dimension greater than 1 has a non-trivial invariant closed linear subspace. In particular, every hyponormal operator with property (δ) on a Hilbert space of dimension greater than 1 has a non-trivial invariant closed linear subspace.

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