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ON THE LOCAL COHOMOLOGY OF MINIMAX MODULES

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, and M a minimax R-module. We prove that the local cohomology modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax; that is, $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is minimax for all i and j in the following cases: (a) $\dim R/\mathfrak{a} = 1$; (b) $\mathrm{cd}(\mathfrak{a}) = 1$, where cd is the cohomological dimension of $\mathfrak{a}$ in R; (c) $\dim R \leq 2$. In these cases we also prove that the Bass numbers and the Betti numbers of $H^j_{\mathfrak{a}}(M)$ are finite.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. For an R-module M, the j-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H^j_{\mathfrak{a}}(M) = \varinjlim_n \mathrm{Ext}^j_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [6] or [9] for more details about local cohomology. Hartshorne [10] defined an R-module M to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all i and asked:

For which rings R and ideals $\mathfrak{a}$ are the modules $H^j_{\mathfrak{a}}(M)$ $\mathfrak{a}$-cofinite for all j and all finitely generated modules M?

With respect to this question, Hartshorne [10] showed that if R is a complete regular local ring and $\mathfrak{a}$ is either a principal ideal or a prime ideal such that $\dim R/\mathfrak{a} = 1$, then $H^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all j and for any finitely generated R-module M. Kawasaki [11] showed that, in general, for a finitely generated R-module M if $\mathfrak{a}$ is principal, then the local cohomology modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite. Delfino and Marley [7] proved that if $\mathfrak{a}$ is an ideal of an arbitrary local ring R with $\dim R/\mathfrak{a} = 1$ and M a finitely generated R-module, then the modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite (see also [14]). Recently, Bahmanpour and Naghipour [3] extended the results of Delfino and Marley [7] to arbitrary commutative Noetherian rings. Melkersson [13] showed that if $\dim R \leq 2$ and M

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a finitely generated R-module, then the modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite. In [4] Belshoff, Slattery and Wickham studied the conditions under which the local cohomology modules $H^j_{\mathfrak{a}}(M)$ are almost cofinite. In fact, they showed that over a complete Gorenstein local domain R, if $\mathfrak{a}$ is an ideal of R with $\dim R/\mathfrak{a} = 1$ and M is a Matlis reflexive module, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is Matlis reflexive for all i and j. Also, they obtained a couple of interesting results about the modules $H^j_{\mathfrak{a}}(M)$ and their Bass numbers (see also [5]). Kashyarmanesh and Khosh-Ahang [12] proved that if R is a complete local ring and M is a Matlis reflexive module, then $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is Matlis reflexive for all i and j in the following cases: (a) $\dim R/\mathfrak{a} = 1$; (b) $\mathfrak{a}$ is a principal ideal; (c) $\dim R \leq 2$. Azami, Naghipour and Vakili [1] defined an R-module M is $\mathfrak{a}$-*cominimax*, as generalization of $\mathfrak{a}$-cofinite, if $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is minimax for all i. In this paper, we eliminate the complete local hypothesis and improve the results of Kashyarmanesh and Khosh-Ahang [12]. In fact, if R is an arbitrary commutative Noetherian ring and M is a minimax module, then the modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax whenever one of the above cases of the result of Kashyarmanesh and Khosh-Ahang holds.

2. The results

Zöschinger [16] introduced the interesting class of minimax modules and he has in [16] and [17] given many equivalent conditions for a module to be minimax. The R-module M is said to be a *minimax module* if there is a finitely generated submodule N of M such that M/N is Artinian. It was shown by Zink [15] and by Enochs [8] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

Remark 2.1. Let M be an R-module.

(a) In view of [16] and [17], the class of minimax modules includes all finitely generated and all Artinian modules.
(b) Let $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ be a short exact sequence of R-modules and R-homomorphisms. Then M is minimax if and only if N and L are minimax, see [2].
(c) Every $\mathfrak{a}$-cofinite module is $\mathfrak{a}$-cominimax.

Lemma 2.2. *Let M be a minimax R-module. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ and $\mathrm{Tor}^R_i(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is minimax for all i.*

Proof. This is immediate by Remark 2.1. □

Lemma 2.3. *Suppose that for any finitely generated R-module N and for each $j \geq 0$, the local cohomology modules $H^j_{\mathfrak{a}}(N)$ are $\mathfrak{a}$-cofinite. Then for any minimax R-module M and for each $j \geq 0$, the local cohomology modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax. In fact, for each $j \geq 1$, the local cohomology modules $H^j_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite.*

Proof. Since M is minimax, there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow A \longrightarrow 0$$

with N a finitely generated module and A an Artinian module. This induces the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow A \longrightarrow H^1_{\mathfrak{a}}(N) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and for all $j \geq 2$, we have $H^j_{\mathfrak{a}}(M) \cong H^j_{\mathfrak{a}}(N)$. Hence, for all $j \geq 2$, $H^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. Let L denote the kernel of the map $H^1_{\mathfrak{a}}(N) \longrightarrow H^1_{\mathfrak{a}}(M)$. Then L is Artinian and so $(0 :_L \mathfrak{a})$ is finite length, since A is Artinian and $(0 :_{H^1_{\mathfrak{a}}(N)} \mathfrak{a})$ is finitely generated. Thus, by [13, Proposition 4.1] L is $\mathfrak{a}$-cofinite. From the exact sequence

$$0 \longrightarrow L \longrightarrow H^1_{\mathfrak{a}}(N) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

we obtain the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(N)) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(M)) \to \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, L) \to \cdots .$$

By our hypothesis, $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(N))$ is finitely generated for all $i \geq 0$ and so $H^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. Hence $H^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $j \geq 1$. □

Theorem 2.4. *Let M be a minimax R-module and suppose one of the following cases holds*:

(a) $\dim R/\mathfrak{a} = 1$;
(b) $\operatorname{cd}(\mathfrak{a}) = 1$;
(c) $\dim R \leq 2$.

Then $H^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax. In fact, $H^j_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $j \geq 1$.

Proof. This follows by Lemma 2.3, [3, Corollary 2.7], [11, Lemma 2] and [13, Theorem 7.10]. □

Theorem 2.5. *Let M be a minimax R-module and suppose one of the following cases holds*:

(a) $\dim R/\mathfrak{a} = 1$;
(b) $\operatorname{cd}(\mathfrak{a}) = 1$;
(c) $\dim R \leq 2$.

Then $\operatorname{Tor}^R_i(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is minimax for all j and i. In fact, $\operatorname{Tor}^R_i(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all i when $j \geq 1$.

Proof. This follows by Theorem 2.4, Lemma 2.2 and [13, Theorem 2.1]. □

Corollary 2.6. *Let M be a minimax R-module and suppose one of the following cases holds*:

(a) $\dim R/\mathfrak{a} = 1$;
(b) $\operatorname{cd}(\mathfrak{a}) = 1$;
(c) $\dim R \leq 2$.

Then the Bass numbers and the Betti numbers of $H^j_{\mathfrak{a}}(M)$ are finite for all j.

Proof. This follows by Theorems 2.4 and 2.5. □

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