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Determination of Initial Conditions for Tetrahedral Satellite Formation

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This paper presents an algorithm that can provide initial conditions for formation flying at the beginning of a region of interest to maximize scientific mission goals in the case of a tetrahedral satellite formation. The performance measure is to maximize the quality factor that affects scientific measurement performance. Several path constraints and periodicity conditions at the beginning of the region of interest are identified. The optimization problem is solved numerically using a direct transcription method. Our numerical results indicate that there exist an optimal configuration and states of a tetrahedral satellite formation. Furthermore, the initial states and algorithm presented here may be used for reconfiguration maneuvers and fuel balancing problems.

Keywords: tetrahedral satellite formation, quality factor, direct transcription method

1. INTRODUCTION

Satellite formation flying is defined as a set of more than one satellite whose states are coupled through a common control law (Scharf et al. 2004). Formation flying missions can be divided into two main categories, based on the motion of the chief satellite, which is a reference point to describe a relative motion. First, for the case of a circular reference orbit, formation flying can be applicable to Earth observation, including space interferometry, and synthetic apertures. The second main category is for the case of an eccentric reference orbit, which has several advantages in scientific missions, such as the magnetospheric multiscale (MMS) mission (Curtis 2000, Curtis et al. 2003) and the cluster II mission (Dow et al. 2004). To meet the scientific mission goals, four-spacecraft formations are commonly used to take measurements in three dimensions at intersatellite distances on the order of several kilometers. The work presented in this paper was motivated by the need for an algorithm that can provide initial conditions for formation flying at the beginning of the region of interest to maximize scientific mission goals for the case of a tetrahedron formation with relatively high eccentricities, such as an MMS mission. Recently, several papers have attempted to solve tetrahedral formation optimization problems similar to the one considered in this paper by focusing on various ways to calculate trajectories and determine initial conditions at the beginning of a region of interest. Huntington & Rao (2008) designed a minimum-fuel trajectory to transfer four spacecraft from a degraded tetrahedral formation into a tetrahedron that satisfied certain geometric quality constraints for a portion of its orbit. Guzman (2003) developed a general optimization procedure for tetrahedron formation control to calculate the maneuver sequence that minimizes the fuel needed to transfer the formation from its current configuration to the target configuration. Chavez Clemente & Atkins (2005) developed hierarchical optimization schemes. Zanon & Campbell (2005) presented a search space reduction technique to solve the coupled guidance and control problem. Hughes (2008) found optimal orbit states for MMS and laser interferometer space antenna mission (LISA Project Office 2000) by using a direct transcription method. To take into account the actual size of the tetrahedron, Hughes (2008) proposed a polynomial function and adopted a performance metric that evalu-

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ates both the size and the shape of a tetrahedron. Except for Hughes (2008), these works assume an active control to reconfigure the formation. Unlike the works cited above, we do not consider reconfiguration maneuver between different phases, but coast phase with a periodicity condition; however, no maneuvers are allowed in the coast phase, and several constraints are applied only in this region. Also, a quality factor is adopted as a cost function, and relative states are considered to make it easier to determine the configuration of the tetrahedron in the region of interest.

Optimal initial conditions determined in this work make it possible to estimate the configuration of tetrahedron formation and final states at the end of the region of interest phase, which can be used for reconfiguration maneuver and fuel balancing.

2. RELATIVE DYNAMICS IN AN ELLIPTICAL REFERENCE ORBIT

For tetrahedron formations, the satellites are moving in an elliptical orbit. Hence, we should use the Tschauner-Hempel equations, which were first derived by Tschauner & Hempel (1965), as follows.

$$\begin{bmatrix}\ddot{x}\\\ddot{y}\\\ddot{z}\end{bmatrix}=-2\begin{bmatrix}0&-\dot{\theta}&0\\\dot{\theta}&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{bmatrix}-\begin{bmatrix}-\dot{\theta}^2&0&0\\0&-\dot{\theta}^2&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$
$$-\begin{bmatrix}0&-\ddot{\theta}&0\\\ddot{\theta}&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}+\frac{\rho(\theta)^3}{\Gamma^4}\begin{bmatrix}2x\\-y\\-z\end{bmatrix} \tag{1}$$

where $x(t)$, $y(t)$, $z(t)$ are the relative-motion coordinates in the radial, along-track, and out-of-plane directions defined in a rotating coordinate system attached to the chief on the reference orbit as shown in Fig. 1, while the upper dot represents the differentiation with respect to time(t). In addition, θ and e refer to the true anomaly and the eccentricity of the chief satellite, respectively. $\rho(\theta) \equiv 1+e\cos\theta$ and $\Gamma \equiv L^{3/2}/GM$ are defined, where $L = R^2\dot{\theta}$ is the magnitude of the orbital angular momentum of the Chief satellite, G is the universal gravitational constant, and M is the mass of Earth.

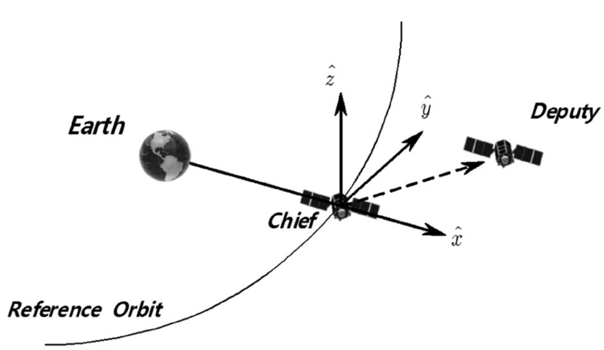

Fig. 1. Relative coordinate system.

Changing the independent variable from time (t) to the true anomaly (θ), Eq. (1) can be rewritten as:

$$\begin{bmatrix}x\\x'\\y\\y'\end{bmatrix}'=\begin{bmatrix}0&1&0&0\\1+2/\rho&-2\rho'/\rho&2\rho'/\rho&2\\0&0&0&1\\-2\rho'/\rho&-2&1-1/\rho&-2\rho'/\rho\end{bmatrix}\begin{bmatrix}x\\x'\\y\\y'\end{bmatrix} \tag{2a}$$

$$\begin{bmatrix}z\\z'\end{bmatrix}'=\begin{bmatrix}0&1\\-1/\rho&-2\rho'/\rho\end{bmatrix}\begin{bmatrix}z\\z'\end{bmatrix} \tag{2b}$$

where primer ($'$) represents differentiation with respect to the true anomaly, and state vector $[x(\theta), y(\theta), z(\theta)]^T$ are now described by the true anomaly that is easily calculated from time using Kepler's equation. The in-plane ($x(\theta)$ and $y(\theta)$) motion and the out-of-plane ($z(\theta)$) motion are decoupled, so we can deal with the problems separately. Now, for brevity, let's consider the following transformation:

$$[\tilde{x},\tilde{y},\tilde{z}]^{\mathbf{T}}=\omega^{1/2}[x,y,z]^{\mathbf{T}} \tag{3}$$

where $\omega\equiv\dot{\theta}$, the orbital rate of the Chief satellite. With the same procedure as that derived by Humi (1993), Eqs. (2a) and (2b) become very simple:

$$\begin{bmatrix}\tilde{x}\\\tilde{x}'\\\tilde{y}\\\tilde{y}'\end{bmatrix}'=\begin{bmatrix}0&1&0&0\\3/\rho&0&0&2\\0&0&0&1\\0&-2&0&0\end{bmatrix}\begin{bmatrix}\tilde{x}\\\tilde{x}'\\\tilde{y}\\\tilde{y}'\end{bmatrix} \tag{4a}$$

$$\begin{bmatrix}\tilde{z}\\\tilde{z}'\end{bmatrix}'=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}\tilde{z}\\\tilde{z}'\end{bmatrix} \tag{4b}$$

It is important to note that the actual position (x, y, z) and velocity $(\dot{x},\dot{y},\dot{z})$ are related to the pseudo-position $(\tilde{x},\tilde{y},\tilde{z})$ and pseudo-velocity $(\tilde{x}',\tilde{y}',\tilde{z}')$ as follows:

$$\begin{bmatrix}x(\theta)\\y(\theta)\\z(\theta)\end{bmatrix}=\frac{\Gamma}{\rho(\theta)}\begin{bmatrix}\tilde{x}(\theta)\\\tilde{y}(\theta)\\\tilde{z}(\theta)\end{bmatrix} \tag{5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x(\theta)\\y(\theta)\\z(\theta)\end{bmatrix}=\frac{e\sin\theta}{\Gamma}\begin{bmatrix}\tilde{x}(\theta)\\\tilde{y}(\theta)\\\tilde{z}(\theta)\end{bmatrix}+\frac{\rho(\theta)}{\Gamma}\begin{bmatrix}\tilde{x}'(\theta)\\\tilde{y}'(\theta)\\\tilde{z}'(\theta)\end{bmatrix} \tag{5b}$$

or

$$\begin{bmatrix} \tilde{x}(\theta) \\ \tilde{y}(\theta) \\ \tilde{z}(\theta) \end{bmatrix} = \frac{\rho(\theta)}{\Gamma} \begin{bmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{bmatrix} \tag{6a}$$

$$\begin{aligned} \begin{bmatrix} \tilde{x}'(\theta) \\ \tilde{y}'(\theta) \\ \tilde{z}'(\theta) \end{bmatrix} &= -\frac{e\sin\theta}{\Gamma} \begin{bmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{bmatrix} + \frac{\Gamma}{\rho(\theta)} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{bmatrix} \\ &= -\frac{e\sin\theta}{\Gamma} \begin{bmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{bmatrix} + \frac{\rho(\theta)}{\Gamma} \begin{bmatrix} x'(\theta) \\ y'(\theta) \\ z'(\theta) \end{bmatrix} \end{aligned} \tag{6b}$$

3. PROBLEM FORMULATION

In this paper, we assumed the MMS-type mission to determine the initial conditions of tetrahedral formation. For MMS, to achieve a fundamental advancement in our understanding of the Earth's magnetosphere and its dynamic interaction with the solar wind, a four-spacecraft formation has to be a near regular tetrahedron formation, within a region defined by ±20 deg in true anomaly about the orbit apogee. Also, in the region of interest, the absence of maneuvers benefits the gathering of scientific data by limiting the orbital disturbances. Consequently, the thrust is zero in the region of interest. Although no maneuvers are allowed in the region of interest phase, several constraints on the formation are applied in the region of interest. To determine the quality of the formation, we adopted a performance metric that evaluates the shape of a tetrahedron consisting of the actual volume and the average side length. In addition to minimizing the cost function, periodic conditions at the beginning of the region of interest were adopted as boundary conditions to form optimal conditions in the next measurement phase in the region of interest.

The problem can be formulated to determine state variables ($x_i, x_i', y_i, y_i', z_i, z_i'$) (i = 1, 2, 3, 4) in the region of interest to minimize the cost function given by Eq. (7).

$$J = -Q_{gm} \tag{7}$$

where, Q_{gm} is a quality factor defined by Eq. (8).

The boundary conditions are given by Eqs. (13 and 14) and Eq. (16). Dynamic constraints are set as Eq. (2). For MMS mission, the region of interest is fixed ($160^\circ < \theta < 200^\circ$) near the apogee.

3.1 Quality Factor

As the formation shifts in the region of interest, it must maintain a certain geometry to take useful scientific measurements. In particular, the two important aspects to the formation geometry are the shape and size of the formation. The most common metrics, typically called quality factors, are the Glassmeier parameter (Robert et al. 1998) defined by Eq. (8), which compares the volume and surface area of a tetrahedron with that of a regular tetrahedron of the same average leg length.

$$Q_{gm} = \frac{V_a}{V^*} + \frac{S_a}{S^*} + 1 \tag{8}$$

where V_a is the actual volume of the tetrahedron, S_a is the actual surface area of the tetrahedron, and V^* and S^* are the volume and surface area, respectively, of a regular tetrahedron whose average side length is L^*. It is noted that the largest possible value of each ratio in Eq. (8) is unity. Consequently, the maximum value of Q_{gm} is three.

If we assume one of the spacecraft is the reference, then there are three relative position vectors that describe the relative geometry of the spacecraft, and we will define them as $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$. The vector $\vec{s}_4$, $\vec{s}_5$, and $\vec{s}_6$ depicted in Fig. 2 are used to compute the surface area in Eq. (10). Knowing these quantities, we can calculate the volume of the tetrahedron using

$$V_a = \frac{1}{6}\left|\vec{s}_1 \cdot (\vec{s}_2 \times \vec{s}_3)\right| \tag{9}$$

The surface area of the actual tetrahedron is computed as follows. First, let

$$S_1 = \frac{1}{2}\left|\vec{s}_1 \times \vec{s}_2\right|,\ S_2 = \frac{1}{2}\left|\vec{s}_1 \times \vec{s}_3\right|,\ S_3 = \frac{1}{2}\left|\vec{s}_3 \times \vec{s}_2\right|,\ S_4 = \frac{1}{2}\left|\vec{s}_5 \times \vec{s}_3\right| \tag{10}$$

Then, the surface area of the actual tetrahedron is given as

$$S_a = \sum_{i=1}^{4} S_i \tag{11}$$

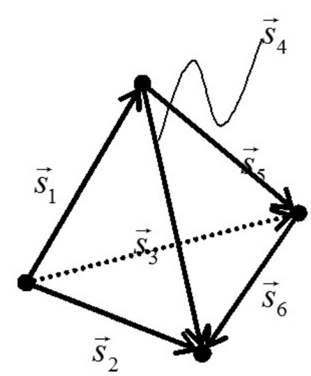

Fig. 2. Relative vectors in the tetrahedron formation.

The average side length L^* can be calculated as follows:

$$L^* = \frac{1}{6}\left(s_1 + s_2 + s_3 + s_4 + s_5 + s_6\right) \quad (12)$$

where s_i (i = 1, 2, 3, 4, 5, 6) is six unique side lengths between each spacecraft.

3.2 Constraints

The ideal geometry for this mission is a regular tetrahedron with 10 km spacing between each satellite, but investigators have noted that there is considerable flexibility here (Hughes 2004). In particular, the acceptable value of Q_{gm} is greater than $Q_{\min}$= 2.7 in the region of interest. Hence, the following path constraint on Q_{gm} is enforced in the region of interest.

$$Q_{gm} \geq Q_{\min} \quad (13)$$

Note that the Glassmeier metric is insensitive to the size of the tetrahedron, meaning an additional constraint is required to constrain the size of the tetrahedron. Although an average interspacecraft separation of 10 km is considered ideal, acceptable science return is still possible for average separations ranging from 4 to 18 km (Hughes 2004). Hence, the following path constraint is enforced during the region of interest that bounds the average side length L^* ($L_{\min}$= 4 km, $L_{\max}$= 18 km).

$$L_{\min} \leq L^* \leq L_{\max} \quad (14)$$

In addition to the aforementioned path constraints, we must satisfy periodicity conditions to have the spacecraft returning to the initial relative states. In a previous study, Sengupta & Vadali (2007) found the necessary conditions on the initial states that produce periodic solutions at an arbitrary true anomaly as follows.

$$\tilde{y}_0' = -\frac{(2+3e\cos\theta+e^2)}{(1+e\cos\theta)^2}\tilde{x}_0 - \frac{e\sin\theta}{(1+e\cos\theta)}\tilde{x}_0' \quad (15)$$

Also, Eq. (15) can be rearranged using the actual position (x, y, z) and velocity (x', y', z') from Eq. (5) as follows.

$$y_0' = -\frac{2+e\cos\theta}{1+e\cos\theta}x_0 - \frac{e\sin\theta}{1+e\cos\theta}x_0' + \frac{e\sin\theta}{1+e\cos\theta}y_0 \quad (16)$$

4. RESULTS

The numerical solutions are obtained by a general purpose optimization code: the Sparse Optimal Control Software (SOCS) (Betts & Huffman 2005) using the numerical values for the parameters shown in Table 1 and the numerical values for the initial guesses of each satellite as follows.

$$\begin{aligned} X_1 &= [0, \sqrt{3}/2\sqrt{2}, 0] \\ X_2 &= [0, -\sqrt{3}/6\sqrt{2}, -\sqrt{3}/3] \\ X_3 &= [-1/2, -\sqrt{3}/6\sqrt{2}, \sqrt{3}/6] \\ X_4 &= [1/2, -\sqrt{3}/6\sqrt{2}, \sqrt{3}/6] \end{aligned} \quad (17)$$

where X_i (i = 1, 2, 3, 4) are relative states arranged in a regular tetrahedron which is characterized by position bases that place four satellites unit distance apart.

SOCS solves optimal control problems using a direct transcription method by which the dynamic system (Eq. 4) is converted into a problem with a finite set of variables. The discretization scheme used in this paper is the second order trapezoidal method. This method produces a distinct set of nonlinear programming variables. Also, SOCS utilizes the mesh refinement algorithm to improve the accuracy of the discretization. The solutions are then determined using sequential quadratic programming.

Recall that the primary mission goal for the case of MMS we consider is to maximize the quality factor given by Eq. (8), in the region between 160 and 200 deg true anomaly. Fig. 3 shows the quality factor during one

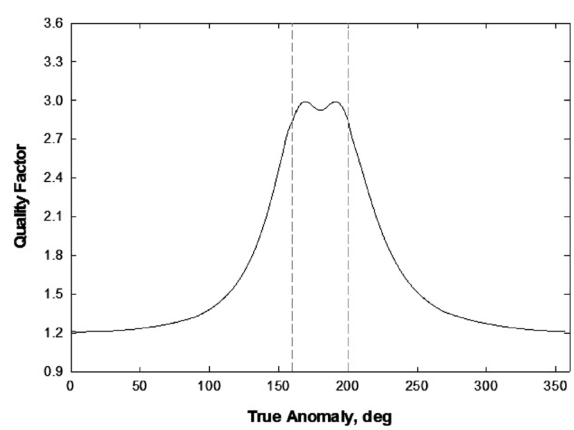

Fig. 3. Quality factor in orbit; dotted line represents the region of interest.

Table 1. Parameters used in the optimization and simulation.

Parameter	Value
Initial true anomaly, θ_i	160°
Final true anomaly, θ_i	200°
Earth's radius, R_e	6,378.14 km
Semimajor axis, a	6.6 R_e
Eccentricity, e	0.818
Q_{min}	2.7
L_{min}	4 km
L_{max}	18 km

orbital period. We can see that the quality factor is near 3 during the region of interest.

Fig. 4 shows the configuration of tetrahedron formation in the region of interest. During the coast phase in the region of interest, the orientation and the size of tetrahedron changes consistently.

Fig. 5 shows the evolution of the six individual sides and the average side length during one orbital period. Through an inspection of the plot in Fig. 5, we can see that the average side length is in the range between 4 and 18, which is acceptable in the scientific measurements.

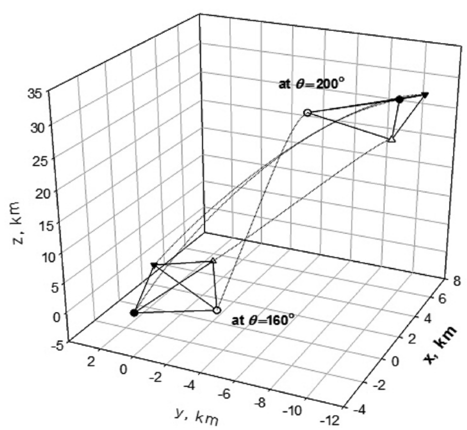

Fig. 4. Configuration of tetrahedron formation in the region of interest.

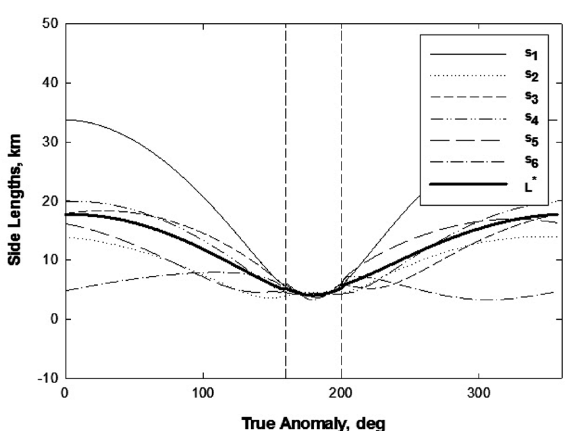

Fig. 5. Side length evolution in orbit; dotted line represents the region of interest.

Table 2. Optimal initial states for MMS (at $\theta = 160°$).

	Sat#1	Sat#2	Sat#3	Sat#4
x, km	-2.268	-1.525	0.531	3.262
x´, km/θ	14.321	6.259	8.906	-0.109
y, km	1.191	-3.344	1.993	0.160
y´, km/θ	-3.804	-3.498	-11.188	-17.039
z, km	-1.384	0.571	1.760	-0.947
z´, km/θ	40.158	43.307	45.003	40.768

MMS: magnetospheric multiscale.

Table 2 provides optimal initial relative states for MMS at the beginning of the region of interest. Initial states listed below enable scientific measurements repeatedly during the same phase without thruster maneuvering and maximize the quality factor.

5. CONCLUSIONS

In this paper, we present an algorithm that can provide initial conditions for formation flying at the beginning of a region of interest to maximize scientific mission goals for the case of a tetrahedron formation with relatively high eccentricities. Also, we found that there exist an optimal configuration and states of a tetrahedral satellite formation. As the periodicity condition is adopted, the relative position of satellites in the tetrahedral formation can be maintained during every scientific measurement phase without any thruster maneuvering. Because it is highly nonlinear and dependent on upwards of 20 variables, it is doubtful that a truly optimal solution can be found. However, finding a near-optimal solution is entirely plausible, and the evolution of quality factor we calculated shows sufficiently good performance. Hence, the optimal initial conditions determined in this work can be used to design an overall concept for reconfiguration maneuvers between different phases.

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