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New Stability Criteria for Linear Systems with Interval Time-varying State Delays

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Abstract – In the present paper, the problem of stability analysis for linear systems with interval time-varying delays is considered. By introducing a new Lyapunov-Krasovskii functional, new stability criteria are derived in terms of linear matrix inequalities (LMIs). Two numerical examples are given to show the superiority of the proposed method.

Keywords: Linear systems, Stability, Time-varying delays, Lyapunov method

1. Introduction

Stability analysis is a prerequisite and essential element of application and implementation processes in electrical and electronic fields [1]–[4]. Because time delays occur in many systems, including networked control systems, chemical processes, transportation systems, and neural networks, the issue of stability analysis for time-delay systems has received considerable attention during the last decade. For details, see [5]–[16] and the references therein.

In the field of stability analysis for time-delay systems, a major concern is the enlargement of the feasible region of stability criteria to obtain maximum delay bounds of time delays for guaranteeing system stability in a given condition. Therefore, choosing the Lyapunov-Krasovskii functional and estimating an upper bound of its time derivative play key roles to improve the feasible region of stability criteria. The descriptor system approach, Park's inequality, and free-weight matrix techniques are mainly the utilized methods in the field of delay-dependent stability analysis [7]. Based on the method of [13], the delay partition method [15] was utilized such that the upper bound of the derivative of the Lyapunov-Krasovskii functional could be estimated more tightly without increasing the decision variables. In [12], some triple-integral terms in the Lyapunov-Krasovskii functional were proposed to reduce conservatism of the stability criteria. Based on the triple-integral terms in the Lyapunov-Krasovskii functional, new delay-range-dependent stability criteria for linear systems with interval time-varying delays were reported in [14]. However, this method has much room for further improvement.

Motivated by the above discussions, we revisit the problem of stability analysis for linear systems with interval time-varying delays treated in [13]–[15]. Unlike the proposed Lyapunov-Krasovskii functional in [14], new Lyapunov-Krasovskii functionals such as

$$\left(\tau_2-\tau_1\right)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds,,$$
$$\left(\tau_2-\tau_1\right)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}x^T(u)Z_4x(u)duds,,$$

and

$$(\left(\tau_2-\tau_1\right)^2/2)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)R_2\dot{x}(u)duds,$$

where Z_2, Z_4 and R_2 are positive definite matrices, are proposed. With these novel considerations, a new delay-dependent stability criterion for linear systems with interval time-varying delays is proposed in Theorem 1. In Theorem 2, by taking a new integral form of states

$$\frac{1}{\tau_1}\int_{t-\tau_1}^{t}x(s)ds,$$
$$\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}x(s)ds,$$

and

$$\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}x(s)ds,$$

augmented vectors, a further improved stability criterion will be proposed. Through two numerical examples treated in [13]–[15], the improvement of our results will be demonstrated.

In the present presentation, the following notations will be used: R^n denotes the n-dimensional Euclidean space and $R^{m\times n}$ is the set of all $m\times n$ real matrix; $*$ denotes the

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symmetric part and $X > 0 (X \geq 0)$ means that X is a real symmetric positive definitive matrix (positive semi-definite); I denotes the identity matrix with appropriate dimensions; $\|\bullet\|$ refers to the induced matrix 2-norm; $diag\{\cdots\}$ denotes the block diagonal matrix; $C_{n,\tau} = C([-\tau, 0], R^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n, with the topology of uniform convergence; and $X_{[f(t)]} \in R^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$.

2. Problem statement and preliminaries

Consider the following linear systems with interval time-varying delays:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau(t)) \\ x(s) &= \phi(s),\ s \in [-\tau_2, 0] \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, A and A_d are the known constant matrices with appropriate dimensions, and $\phi(s) \in C_{n,\tau_2}$ is a vector-valued initial function. The delay $\tau(t)$ is a time-varying continuous function that satisfies

$$\tau_1 \leq \tau(t) \leq \tau_2,\ \dot{\tau}(t) \leq \mu, \tag{2}$$

where $\tau_2 > \tau_1 > 0$, and μ are constant values.

The objective of the present paper is to develop delay-range-dependent stability criteria for System (1). Before deriving our main results, the following facts and lemmata will be needed.

Fact 1: (Schur complement). Given constant matrices $\Sigma_1, \Sigma_2, \Sigma_3$, where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \text{ or } \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Lemma 1. For a positive matrix M, the following inequality holds:

$$-(\alpha-\beta)\int_\beta^\alpha \dot{x}^T(s) M \dot{x}(s) ds \leq \begin{bmatrix} x(\alpha) \\ x(\beta) \end{bmatrix}^T \begin{bmatrix} -M & M \\ * & M \end{bmatrix} \begin{bmatrix} x(\alpha) \\ x(\beta) \end{bmatrix}. \tag{3}$$

Proof. According to Lemma 1 in [16],

$$-(\alpha-\beta)\int_\beta^\alpha \dot{x}^T(s) M \dot{x}(s) ds \leq -\left(\int_\beta^\alpha \dot{x}(s) ds\right)^T M \left(\int_\beta^\alpha \dot{x}(s) ds\right). \tag{4}$$

Inequality (3) can thus be obtained. ■

Lemma 2. For a positive matrix M, the following inequality holds:

$$\begin{aligned} &-\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha \int_s^\alpha x^T(u) M x(u) du ds \\ &\leq -\left(\int_\beta^\alpha \int_s^\alpha x(u) du ds\right)^T M \left(\int_\beta^\alpha \int_s^\alpha x(u) du ds\right). \end{aligned} \tag{5}$$

Proof. From Lemma 1, the following inequality holds:

$$-(\alpha-s)\int_s^\alpha x^T(s) M x(s) ds \leq -\left(\int_s^\alpha x(s) ds\right)^T M \left(\int_s^\alpha x(s) ds\right). \tag{6}$$

Using Fact 1, Inequality (6) is equivalent to

$$\begin{bmatrix} -\int_s^\alpha x^T(u) M x(u) du & \left(\int_s^\alpha x(u) du\right)^T \\ * & -(\alpha-s) M^{-1} \end{bmatrix} \leq 0. \tag{7}$$

By integrating inequality (7) from β to α, we have

$$\begin{bmatrix} -\int_\beta^\alpha \int_s^\alpha x^T(u) M x(u) du ds & \left(\int_\beta^\alpha \int_s^\alpha x(u) du ds\right)^T \\ * & -\int_\beta^\alpha (\alpha-s) M^{-1} ds \end{bmatrix} \leq 0. \tag{8}$$

From Fact 1, Inequality (8) is equivalent to inequality (5). This completes the proof of Lemma 2. ■

Lemma 3 [18]. Let $\varsigma \in R^n, \Phi = \Phi^T \in R^{m\times n}$ such that $rank(B) < n$. The following statements are equivalent:

(1) $\varsigma^T \Phi \varsigma < 0,\ B\varsigma = 0,\ \varsigma \neq 0,$
(2) $\left(B^\perp\right)^T \Phi B^\perp < 0$, where $B^\perp$ is a right orthogonal complement of B.

3. Main results

In this section, we propose the improved stability criteria for System (1) with interval time-varying delays. For the convenience of readers, we have attempted to use similar notations of [14]. Here, $e_i (i = 1, ..., 9) \in R^{9n\times n}$ are defined as block entry matrices. (e.g., $e_3^T = [0\ 0\ I\ 0\ 0\ 0\ 0\ 0\ 0]$.) Some vectors and matrices are defined as

$$\begin{aligned} \varsigma(t) = col\Big\{&x(t),\ x(t-\tau(t)),\ x(t-\tau_1),\ x(t-\tau_2),\ \dot{x}(t-\tau_1), \\ &\dot{x}(t-\tau_2),\ \int_{t-\tau_1}^t x(s) ds,\ \int_{t-\tau(t)}^{t-\tau_1} x(s) ds,\ \int_{t-\tau_2}^{t-\tau(t)} x(s)\Big\}, \end{aligned}$$

$$A_c = [A\ A_d\ 0\ 0\ 0\ 0\ 0\ 0\ 0], \quad \Gamma = [e_1\ e_3\ e_4\ e_7\ e_8 + e_9],$$

$$\Upsilon = [A_c^T\ e_5\ e_6\ \ e_1 - e_3\ \ e_3 - e_4],$$

$$\Lambda = diag\{Q_1 + \tau_1^2 Z_3, -(1-\mu) Q_3, -Q_1 + Q_2 + Q_3 + (\tau_2 - \tau_1)^2 Z_4,$$

$$-Q_2, -Q_4 + Q_5 + (\tau_2 - \tau_1)^2 Z_2 + ((\tau_2 - \tau_1)^4/4)R_2, -Q_5, -Z_3, 0, 0\},$$

$$Y = Q_4 + \tau_1^2 Z_1 + (\tau_1^4/4)R_1. \quad (9)$$

We now have the following theorem:

Theorem 1. For a given scalar $\tau_2 > \tau_1 > 0$, and μ, System (1) is asymptotically stable for $\tau_1 \le \tau(t) \le \tau_2$, and $\dot{\tau}(t) \le \mu$, if there exist some positive definite matrices such as $\mathrm{P} = [P_{ij}]_{5\times 5}, Q_i (i = 1,...,5)$, $Z_i (i = 1,...,4)$, and $R_i (i = 1,2)$ that satisfy the following two LMIs:

$$\Xi_1 < 0, \quad (10)$$

$$\Xi_2 < 0, \quad (11)$$

where

$$\begin{aligned}\Xi_1 &= \Gamma \mathrm{P} \Upsilon^T + \Upsilon \mathrm{P} \Gamma^T + \Lambda + A_c^T Y A_c - (e_1 - e_3) Z_1 (e_1^T - e_3^T) \\ &\quad - 2(e_2 - e_4) Z_2 (e_2^T - e_4^T) - (e_3 - e_2) Z_2 (e_3^T - e_2^T) \\ &\quad - e_8 Z_4 e_8^T - 2 e_9 Z_4 e_9^T - (\tau_1 e_1 - e_7) R_1 (\tau_1 e_1^T - e_7^T) \\ &\quad - ((\tau_2 - \tau_1) e_3 - e_8 - e_9) R_2 ((\tau_2 - \tau_1) e_3^T - e_8^T - e_9^T),\end{aligned}$$

and

$$\begin{aligned}\Xi_2 &= \Gamma \mathrm{P} \Upsilon^T + \Upsilon \mathrm{P} \Gamma^T + \Lambda + A_c^T Y A_c - (e_1 - e_3) Z_1 (e_1^T - e_3^T) \\ &\quad - (e_2 - e_4) Z_2 (e_2^T - e_4^T) - 2(e_3 - e_2) Z_2 (e_3^T - e_2^T) \\ &\quad - 2 e_8 Z_4 e_8^T - e_9 Z_4 e_9^T - (\tau_1 e_1 - e_7) R_1 (\tau_1 e_1^T - e_7^T) \\ &\quad - ((\tau_2 - \tau_1) e_3 - e_8 - e_9) R_2 ((\tau_2 - \tau_1) e_3^T - e_8^T - e_9^T).\end{aligned}$$

Proof. For positive definite matrices $\mathrm{P} = [P_{ij}]_{5\times 5}$, $Q_i (i = 1,...,5)$, $Z_i (i = 1,...,4)$, and $R_i (i = 1,2)$, let us consider the following Lyapunov-Krasovskii functional candidate:

$$V(t) = \sum_{i=1}^{6} V_i(t) \quad (12)$$

where

$$V_1(t) = \begin{bmatrix} x(t) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ \int_{t-\tau_1}^{t} x(s)ds \\ \int_{t-\tau_2}^{t-\tau_1} x(s)ds \end{bmatrix}^T \mathrm{P} \begin{bmatrix} x(t) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ \int_{t-\tau_1}^{t} x(s)ds \\ \int_{t-\tau_2}^{t-\tau_1} x(s)ds \end{bmatrix},$$

$$\begin{aligned}V_2(t) &= \int_{t-\tau_1}^{t} x^T(s) Q_1 x(s) ds + \int_{t-\tau_2}^{t-\tau_1} x^T(s) Q_2 x(s) ds \\ &\quad + \int_{t-\tau_1}^{t} \dot{x}^T(s) Q_4 \dot{x}(s) ds + \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s) Q_5 \dot{x}(s) ds,\end{aligned}$$

$$V_3(t) = \int_{t-\tau(t)}^{t-\tau_1} x^T(s) Q_3 x(s) ds,$$

$$\begin{aligned}V_4(t) &= \tau_1 \int_{t-\tau_1}^{t} \int_{s}^{t} \dot{x}^T(u) Z_1 \dot{x}(u) du ds \\ &\quad + (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \int_{s}^{t-\tau_1} \dot{x}^T(u) Z_2 \dot{x}(u) du ds, \\ V_5(t) &= \tau_1 \int_{t-\tau_1}^{t} \int_{s}^{t} x^T(u) Z_3 x(u) du ds \\ &\quad + (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \int_{s}^{t-\tau_1} x^T(u) Z_4 x(u) du ds, \\ V_6(t) &= (\tau_1^2/2) \int_{t-\tau_1}^{t} \int_{s}^{t} \int_{u}^{t} \dot{x}^T(v) R_1 \dot{x}(v) dv du ds \\ &\quad + ((\tau_2 - \tau_1)^2/2) \int_{t-\tau_2}^{t-\tau_1} \int_{s}^{t-\tau_1} \int_{u}^{t-\tau_1} \dot{x}^T(v) R_2 \dot{x}(v) dv du ds.\end{aligned} \quad (13)$$

Note that the differences between the proposed Lyapunov-Krasovskii functional and the one in [14] are as follows:

$$(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \int_{s}^{t-\tau_1} \dot{x}^T(u) Z_2 \dot{x}(u) du ds,$$

$$(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \int_{s}^{t-\tau_1} x^T(u) Z_4 x(u) du ds,$$

and

$$((\tau_2 - \tau_1)^2/2) \int_{t-\tau_2}^{t-\tau_1} \int_{s}^{t-\tau_1} \int_{u}^{t-\tau_1} \dot{x}^T(v) R_1 \dot{x}(v) dv du ds$$

of $V_i(t) (i = 4,5,6)$, respectively.

The time derivative of $V_1(t)$ can be represented as

$$\dot{V}_1(t) = 2\varsigma^T(t) \Gamma \mathrm{P} \Upsilon^T \varsigma(t). \quad (14)$$

By calculating $\dot{V}_2(t) + \dot{V}_3(t)$, we have

$$\begin{aligned}&\dot{V}_2(t) + \dot{V}_3(t) \\ &\le x^T(t) Q_1 x(t) - x^T(t-\tau_1) Q_1 x(t-\tau_1) + x^T(t-\tau_1) Q_2 x(t-\tau_1) \\ &\quad - x^T(t-\tau_2) Q_2 x(t-\tau_2) + \dot{x}^T(t) Q_4 \dot{x}(t) - \dot{x}^T(t-\tau_1) Q_4 \dot{x}^T(t-\tau_1) \\ &\quad + \dot{x}^T(t-\tau_1) Q_5 \dot{x}(t-\tau_1) - \dot{x}^T(t-\tau_2) Q_5 \dot{x}(t-\tau_2) \\ &\quad + x^T(t-\tau_1) Q_3 x(t-\tau_1) - (1-\mu) x^T(t-\tau(t)) Q_3 x(t-\tau(t)).\end{aligned} \quad (15)$$

Calculating $\dot{V}_4(t)$ leads to

$$\begin{aligned}\dot{V}_4(t) &= \tau_1^2 \dot{x}^T(t) Z_1 \dot{x}(t) - \tau_1 \int_{t-\tau_1}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds \\ &\quad + (\tau_2 - \tau_1)^2 \dot{x}^T(t-\tau_1) Z_2 \dot{x}(t-\tau_1) \\ &\quad - (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s) Z_2 \dot{x}(s) ds.\end{aligned} \quad (16)$$

Here, by Lemma 1, one can obtain

$$-\tau_1 \int_{t-\tau_1}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds \le \begin{bmatrix} x(t) \\ x(t-\tau_1) \end{bmatrix}^T \begin{bmatrix} -Z_1 & Z_1 \\ * & -Z_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_1) \end{bmatrix}, \quad (17)$$

$$
\begin{aligned}
&-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&=-(\tau_2-\tau_1)\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&=-(\tau_2-\tau(t))\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau(t)-\tau_1)\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau_2-\tau(t))\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau(t)-\tau_1)\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds \qquad (18)\\
&\le-(\tau_2-\tau_1)^{-1}(\tau_2-\tau(t))(\tau(t)-\tau_1)\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau(t)-\tau_1)\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau_2-\tau(t))\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\quad-(\tau_2-\tau_1)^{-1}(\tau(t)-\tau_1)(\tau_2-\tau(t))\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds\\
&\le(-(\tau_2-\tau_1)^{-1}(\tau_2-\tau(t))-1)\times\\
&\begin{bmatrix}x(t-\tau_1)\\x(t-\tau(t))\end{bmatrix}^T\begin{bmatrix}-Z_2&Z_2\\ *&-Z_2\end{bmatrix}\begin{bmatrix}x(t-\tau_1)\\x(t-\tau(t))\end{bmatrix}\\
&\quad+(-1-(\tau_2-\tau_1)^{-1}(\tau(t)-\tau_1))\times\\
&\begin{bmatrix}x(t-\tau(t))\\x(t-\tau_2)\end{bmatrix}^T\begin{bmatrix}-Z_2&Z_2\\ *&-Z_2\end{bmatrix}\begin{bmatrix}x(t-\tau(t))\\x(t-\tau_2)\end{bmatrix}.
\end{aligned}
$$

Therefore, $\dot{V}_4(t)$ can be estimated as

$$
\begin{aligned}
&\dot{V}_4(t)\\
&\le\tau_1^2\dot{x}^T(t)Z_1\dot{x}(t)+\begin{bmatrix}x(t)\\x(t-\tau_1)\end{bmatrix}^T\begin{bmatrix}-Z_1&Z_1\\ *&-Z_1\end{bmatrix}\begin{bmatrix}x(t)\\x(t-\tau_1)\end{bmatrix}\\
&+(\tau_2-\tau_1)^2\dot{x}^T(t-\tau_1)Z_2\dot{x}(t-\tau_1) \qquad (19)\\
&\quad+(-(\tau_2-\tau_1)^{-1}(\tau_2-\tau(t))-1)\times\\
&\begin{bmatrix}x(t-\tau_1)\\x(t-\tau(t))\end{bmatrix}^T\begin{bmatrix}-Z_2&Z_2\\ *&-Z_2\end{bmatrix}\begin{bmatrix}x(t-\tau_1)\\x(t-\tau(t))\end{bmatrix}\\
&\quad+(-1-(\tau_2-\tau_1)^{-1}(\tau(t)-\tau_1))\times\\
&\begin{bmatrix}x(t-\tau(t))\\x(t-\tau_2)\end{bmatrix}^T\begin{bmatrix}-Z_2&Z_2\\ *&-Z_2\end{bmatrix}\begin{bmatrix}x(t-\tau(t))\\x(t-\tau_2)\end{bmatrix}.
\end{aligned}
$$

Using a similar method shown in (17) and (18), an upper bound of $\dot{V}_5(t)$ can be obtained as

$$
\begin{aligned}
&\dot{V}_5(t)\\
&\le\tau_1^2x^T(t)Z_3x(t)-\left(\int_{t-\tau_1}^{t}x(s)ds\right)^TZ_3\left(\int_{t-\tau_1}^{t}x(s)ds\right)\\
&+(\tau_2-\tau_1)^2x^T(t-\tau_1)Z_4x(t-\tau_1)\\
&\quad+(-(\tau_2-\tau_1)^{-1}(\tau_2-\tau(t))-1)\times\\
&\left(\int_{t-\tau(t)}^{t-\tau_1}x(s)ds\right)^TZ_4\left(\int_{t-\tau(t)}^{t-\tau_1}x(s)ds\right)\\
&\quad+(-1-(\tau_2-\tau_1)^{-1}(\tau(t)-\tau_1))\times\\
&\left(\int_{t-\tau_2}^{t-\tau(t)}x(s)ds\right)^TZ_4\left(\int_{t-\tau_2}^{t-\tau(t)}x(s)ds\right). \qquad (20)
\end{aligned}
$$

By utilizing Lemma 2, an upper bound of $\dot{V}_6(t)$ can be estimated as

$$
\begin{aligned}
&\dot{V}_6(t)=(\tau_1^4/4)\dot{x}^T(t)R_1\dot{x}(t)-(\tau_1^2/2)\int_{t-\tau_1}^{t}\int_s^t\dot{x}^T(u)R_1\dot{x}(u)duds\\
&\qquad+((\tau_2-\tau_1)^4/4)\dot{x}^T(t-\tau_1)R_2\dot{x}(t-\tau_2)\\
&\qquad-((\tau_2-\tau_1)^2/2)\int_{t-\tau_2}^{t-\tau_1}\int_s^{t-\tau_1}\dot{x}^T(u)R_2\dot{x}(u)duds\\
&\le(\tau_1^4/4)\dot{x}^T(t)R_1\dot{x}(t) \qquad (21)\\
&-\left(\tau_1x(t)-\int_{t-\tau_1}^{t}x(s)ds\right)^TR_1\left(\tau_1x(t)-\int_{t-\tau_1}^{t}x(s)ds\right)\\
&+((\tau_2-\tau_1)^4/4)\dot{x}^T(t-\tau_1)R_2\dot{x}(t-\tau_2)\\
&-\left((\tau_2-\tau_1)x(t-\tau_1)-\int_{t-\tau(t)}^{t-\tau_1}x(s)ds-\int_{t-\tau_2}^{t-\tau(t)}x(s)ds\right)^TR_2\times\\
&\left((\tau_2-\tau_1)x(t-\tau_1)-\int_{t-\tau(t)}^{t-\tau_1}x(s)ds-\int_{t-\tau_2}^{t-\tau(t)}x(s)ds\right).
\end{aligned}
$$

From (12)–(21), an upper bound of $\dot{V}(t)$ can be

$$V(t)\le\varsigma^T(t)\Xi_{[\tau(t)]}\varsigma(t) \qquad (22)$$

where

$$
\begin{aligned}
&\Xi_{[\tau(t)]}=\Gamma\mathrm{P}\Upsilon^T+\Upsilon\mathrm{P}\Gamma^T+\Lambda+A_c^TYA_c-(e_1-e_3)Z_1(e_1^T-e_3^T)\\
&+(-(\tau_2-\tau_1)^{-1}(\tau_2-\tau(t))-1)(e_3-e_2)Z_2(e_3^T-e_2^T)\\
&+(-1-(\tau_2-\tau_1)^{-1}(\tau(t)-\tau_1))(e_2-e_4)Z_2(e_2^T-e_4^T)\\
&+(-(\tau_2-\tau_1)^{-1}(\tau_2-\tau(t))-1)e_8Z_4e_8^T \qquad (23)\\
&+(-1-(\tau_2-\tau_1)^{-1}(\tau(t)-\tau_1))e_9Z_4e_9^T\\
&+(\tau_1e_1-e_7)R_1(\tau_1e_1^T-e_7^T)\\
&-((\tau_2-\tau_1)e_3-e_8-e_9)R_2((\tau_2-\tau_1)e_3^T-e_8^T-e_9^T).
\end{aligned}
$$

Because the elements of $\Xi_{[\tau(t)]}$ are affinely dependent on $\tau(t)$, if $\Xi_{[\tau(t)=\tau_2]}=\Xi_1<0$ and $\Xi_{[\tau(t)=\tau_1]}=\Xi_2<0$ hold, then $\Xi_{[\tau(t)]}$ for $\tau_1\le\tau(t)\le\tau_2$ is satisfied, which means that System (1) is asymptotically stable. This completes our proof. ■

Remark 1. In the proposed Lyapunov-Krasovskii functional of Theorem 1 [14], the terms

$$(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta,$$

$$(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}x^T(s)Z_4x(s)dsd\theta,$$

and

$$((\tau_2^2-\tau_1^2)/2)\int_{-\tau_2}^{-\tau_1}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)R_2\dot{x}(s)dsd\lambda d\theta$$

are proposed. These three integral terms can be represented as

$$(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t}\dot{x}^T(u)Z_2\dot{x}(u)duds,$$
$$(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t}x^T(u)Z_4x(u)duds,$$

and

$$((\tau_2^2-\tau_1^2)/2)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t}\int_{u}^{t}\dot{x}^T(v)R_2\dot{x}(v)dvduds,$$

respectively. In double integral terms, we have $t-\tau_2\le s\le t-\tau_1$ and $s\le u\le t$. In triple integral terms, we can confirm $t-\tau_2\le s\le t-\tau_1$, $s\le u\le t$, and $u\le v\le t$. Because parameter s has the integral interval from $t-\tau_2$ to $t-\tau_1$, the maximum values of u and v are effectively changed to $t-\tau_1$ instead of t. Motivated by this idea, the author proposes the following modified Lyapunov-Krasovskii functionals:

$$(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds,$$
$$(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}x^T(u)Z_4x(u)duds,$$

and

$$((\tau_2^2-\tau_1^2)/2)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\int_{u}^{t-\tau_1}\dot{x}^T(v)R_2\dot{x}(v)dvduds.$$

To the best of the author's knowledge, this form of Lyapunov-Krasovskii functional in the field of stability analysis for systems with interval time-varying delays has not yet been proposed. Through numerical examples shown in [14], the improvement of feasible region of Theorem 1 will be shown n

Remark 2. Assume that one parameter $\theta(t)$ belongs to $\theta_1\le\theta(t)\le\theta_2$ and the elements of matrix $G_{[\theta(t)]}$ are affinely dependent on $\theta(t)$. If $G_{[\theta(t)=\theta_1]}<0$ and $G_{[\theta(t)=\theta_2]}<0$ hold, then $G_{[\theta(t)]}<0$ for $\theta_1\le\theta(t)\le\theta_2$ is satisfied. In [9], this fact is applied to delay-dependent stability analysis for systems with time-varying delays for the first time. Therefore, if a stability condition for systems with interval time-varying delays $\tau(t)$ is affinely dependent on interval time-varying delays $\tau(t)$ with $\tau_1\le\tau(t)\le\tau_2$, then, by checking the feasibility region of stability condition at vertex set of $\tau(t)$, one can obtain maximum delay bounds of the concerned systems.

Remark 3. In the proposed Theorem 1, the integral forms of state, such as

$$\int_{t-\tau_1}^{t}x(s)ds,$$
$$\int_{t-\tau(t)}^{t-\tau_1}x(s)ds,$$

and

$$\int_{t-\tau_2}^{t-\tau(t)}x(s)ds,$$

are utilized. If these integral terms are changed as

$$\frac{1}{\tau_1}\int_{t-\tau_1}^{t}x(s)ds,$$
$$\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}x(s)ds,$$

and

$$\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}x(s)ds,$$

whether the feasible region of Theorem 1 can be improved with the same Lyapunov-Krasovskii functional remains unclear. To answer this question, we propose a new stability condition as shown in Theorem 2.

For the simple representation of Theorem 2, $\tilde{e}_i(i=1,\ldots,10)\in R^{10n\times n}$ are defined as block entry matrices. (For example, $\tilde{e}_3=[0\ 0\ I\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$). Some vectors and matrices are defined as

$$\begin{aligned}\zeta(t)&=col\{x(t),\,x(t-\tau(t)),\,x(t-\tau_1),\,x(t-\tau_2),\dot{x}(t),\dot{x}(t-\tau_1),\\&\quad\dot{x}(t-\tau_2),\,\frac{1}{\tau_1}\int_{t-\tau_1}^{t}x(s)ds,\,\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}x(s)ds,\\&\quad\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}x(s)\Big\},\\ \tilde{\Gamma}&=[\tilde{e}_1\ \tilde{e}_3\ \tilde{e}_4\ \tilde{e}_8\ \tilde{e}_9\ \tilde{e}_{10}],\ \tilde{\Upsilon}=[\tilde{e}_5\ \tilde{e}_6\ \tilde{e}_7\ \tilde{e}_1-\tilde{e}_3\ \tilde{e}_3-\tilde{e}_4],\\ \Psi_{[\tau(t)]}&=\begin{bmatrix}I&0&0&0&0\\0&I&0&0&0\\0&0&I&0&0\\0&0&0&\tau_1I&0\\0&0&0&0&(\tau(t)-\tau_1)I\\0&0&0&0&(\tau_2-\tau(t))I\end{bmatrix},\\ \Pi&=[A\ A_d\ 0\ 0\ -I\ 0\ 0\ 0\ 0\ 0].\end{aligned}\tag{24}$$

Theorem 2. For a given scalar $\tau_2>\tau_1>0$, and μ, System (1) is asymptotically stable for $\tau_1\le\tau(t)\le\tau_2$, and $\dot{\tau}(t)\le\mu$, if there exist positive definite matrices such as $P=[P_{ij}]_{5\times5}, Q_i(i=1,\ldots,5),\ \ Z_i(i=1,\ldots,4)$, and $R_i(i=1,2)$, which satisfy the following two LMIs:

$$(\Pi^{\perp})^T \tilde{\Xi}_{[\tau(t)=\tau_1]}\Pi^{\perp} < 0, \quad \text{and} \tag{25}$$

$$(\Pi^{\perp})^T \tilde{\Xi}_{[\tau(t)=\tau_2]}\Pi^{\perp} < 0, \tag{26}$$

where

$$\begin{aligned}\tilde{\Xi}_{[\tau(t)]} &= \tilde{\Gamma}\Psi_{[\tau(t)]}\mathrm{P}\tilde{\Upsilon}^T + \tilde{\Upsilon}\mathrm{P}\Psi_{[\tau(t)]}^T\tilde{\Gamma}^T + \tilde{e}_1 Q_1 \tilde{e}_1^T - \tilde{e}_3 Q_1 \tilde{e}_3^T + \tilde{e}_3 Q_2 \tilde{e}_3^T \\ &\quad - \tilde{e}_4 Q_2 \tilde{e}_4^T + \tilde{e}_5 Q_4 \tilde{e}_5^T - \tilde{e}_6 Q_4 \tilde{e}_6^T + \tilde{e}_6 Q_5 \tilde{e}_6^T - \tilde{e}_7 Q_5 \tilde{e}_7^T + \tilde{e}_3 Q_3 \tilde{e}_3^T \\ &\quad - (1-\mu)\tilde{e}_2 Q_3 \tilde{e}_2^T + \tau_1^2 \tilde{e}_1 Z_3 \tilde{e}_1^T - \tau_1^2 \tilde{e}_8 Z_3 \tilde{e}_8^T + (\tau_2-\tau_1)^2 \tilde{e}_3 Z_4 \tilde{e}_3^T \\ &\quad - (\tau_2-\tau_1)(\tau(t)-\tau_1)\tilde{e}_9 Z_4 \tilde{e}_9^T - (\tau_2-\tau_1)(\tau_2-\tau(t))\tilde{e}_{10} Z_4 \tilde{e}_{10}^T \\ &\quad + (\tau_1^4/4)\tilde{e}_5 R_1 \tilde{e}_5^T - \tau_1^2(\tilde{e}_1-\tilde{e}_8)R_1(\tilde{e}_1^T-\tilde{e}_8^T) \\ &\quad + ((\tau_2-\tau_2)^4/4)\tilde{e}_6 R_2 \tilde{e}_6^T - (\tau_2-\tau_1)^2(\tilde{e}_3-\tilde{e}_9)R_2(\tilde{e}_3^T-\tilde{e}_9^T) \\ &\quad - (\tau_2-\tau_1)^2(\tilde{e}_2-\tilde{e}_{10})R_2(\tilde{e}_2^T-\tilde{e}_{10}^T). \end{aligned} \tag{27}$$

Proof. Consider the same Lyapunov-Krasovskii functional in (12).

Taking the time-derivative of $\dot{V}_1(t)$, we have

$$\begin{aligned}\dot{V}_1(t) &= 2\begin{bmatrix} x(t) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ \int_{t-\tau_1}^{t} x(s)ds \\ \int_{t-\tau_2}^{t-\tau_1} x(s)ds \end{bmatrix}^T \mathrm{P}\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-\tau_1) \\ \dot{x}(t-\tau_2) \\ \int_{t-\tau_1}^{t} x(s)ds \\ \int_{t-\tau_2}^{t-\tau_1} x(s)ds \end{bmatrix} \\
&= 2\begin{bmatrix} x(t) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ \left(\frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right)\tau_1 \\ \left(\left(\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds\right)(\tau(t)-\tau_1) + \left(\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds\right)(\tau_2-\tau(t))\right) \end{bmatrix}^T \\
&\quad \times \mathrm{P}\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-\tau_1) \\ \dot{x}(t-\tau_2) \\ x(t)-x(t-\tau_1) \\ x(t-\tau_1)-x(t-\tau_2) \end{bmatrix} \\
&= 2\begin{bmatrix} x(t) \\ x(t-\tau_1) \\ x(t-\tau_2) \\ \frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds \\ \frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds \\ \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & \tau_1 I & 0 \\ 0 & 0 & 0 & 0 & (\tau(t)-\tau_1)I \\ 0 & 0 & 0 & 0 & (\tau_2-\tau(t))I \end{bmatrix} \\
&\quad \times \mathrm{P}\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-\tau_1) \\ \dot{x}(t-\tau_2) \\ x(t)-x(t-\tau_1) \\ x(t-\tau_1)-x(t-\tau_2) \end{bmatrix} \\
&= 2\tilde{\varsigma}^T(t)\tilde{\Gamma}\Psi_{[\tau(t)]}\mathrm{P}\tilde{\Upsilon}^T\tilde{\varsigma}(t). \end{aligned} \tag{28}$$

Calculating $\dot{V}_5(t)$ leads to

$$\begin{aligned} &\dot{V}_5(t) \\ &= \tau_1^2 x^T(t)Z_3 x(t) - \tau_1\int_{t-\tau_1}^{t} x^T(s)Z_3 x(s)ds \\ &\quad + (\tau_2-\tau_1)^2 x^T(t-\tau_1)Z_4 x(t-\tau_1) \\ &\quad - (\tau_2-\tau_1)\int_{t-\tau(t)}^{t-\tau_1} x^T(s)Z_4 x(s)ds \\ &\quad - (\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau(t)} x^T(s)Z_4 x(s)ds \\
&\le \tau_1^2 x^T(t)Z_3 x(t) - \left(\frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right)^T (\tau_1^2 Z_3)\left(\frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right) \\ &\quad + (\tau_2-\tau_1)^2 x^T(t-\tau_1)Z_4 x(t-\tau_1) \\ &\quad - \left(\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds\right)^T ((\tau_2-\tau_1)(\tau(t)-\tau_1))Z_4 \times \\ &\quad \left(\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds\right) \\ &\quad - \left(\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds\right)^T ((\tau_2-\tau_1)(\tau_2-\tau(t)))Z_4 \times \\ &\quad \left(\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds\right) \\
&= \tilde{\varsigma}^T(t)[\tau_1^2 \tilde{e}_1 Z_3 \tilde{e}_1^T - \tau_1^2 \tilde{e}_8 Z_3 \tilde{e}_8^T + (\tau_2-\tau_1)^2 \tilde{e}_3 Z_4 \tilde{e}_3^T \\ &\quad - (\tau_2-\tau_1)(\tau(t)-\tau_1)\tilde{e}_9 Z_4 \tilde{e}_9^T - (\tau_2-\tau_1)(\tau_2-\tau(t))\tilde{e}_{10} Z_4 \tilde{e}_{10}^T]\tilde{\varsigma}(t). \end{aligned} \tag{29}$$

Taking the time-derivative of $\dot{V}_6(t)$, we have

$$\begin{aligned}\dot{V}_6(t) &= (\tau_1^4/4)\dot{x}^T(t)R_1\dot{x}(t) - (\tau_1^2/2)\int_{t-\tau_1}^{t}\int_{s}^{t} \dot{x}^T(u)R_1\dot{x}(u)duds \\ &\quad + ((\tau_2-\tau_1)^4/4)\dot{x}^T(t-\tau_1)R_2\dot{x}(t-\tau_2) \\ &\quad - ((\tau_2-\tau_1)^2/2)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds. \end{aligned} \tag{30}$$

Note that

$$\begin{aligned} &-(\tau_1^2/2)\int_{t-\tau_1}^{t}\int_{s}^{t} \dot{x}^T(u)R_1\dot{x}(u)duds \\ &\le -\left(\tau_1 x(t) - \int_{t-\tau_1}^{t} x(s)ds\right)^T R_1\left(\tau_1 x(t) - \int_{t-\tau_1}^{t} x(s)ds\right) \end{aligned}$$

$$= -\left(\tau_1 x(t) - \frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right)^T (\tau_1^2 R_1)\left(\tau_1 x(t) - \frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right), \tag{31}$$

and

$$\begin{aligned}
&-((\tau_2-\tau_1)^2/2)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds \\
&= -((\tau_2-\tau_1)^2/2)\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds \\
&-((\tau_2-\tau_1)^2/2)\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds \\
&\le -((\tau_2-\tau_1)^2/2)\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds \\
&-((\tau_2-\tau_1)^2/2)\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau(t)} \dot{x}^T(u)R_2\dot{x}(u)duds \\
&\le -((\tau_2-\tau_1)^2/2)(2/(\tau(t)-\tau_1)^2)\times \\
&\left(\int_{t-\tau(t)}^{t-\tau_1}\dot{x}(u)duds\right)^T R_2\left(\int_{t-\tau(t)}^{t-\tau_1}\dot{x}(u)duds\right) \\
&-((\tau_2-\tau_1)^2/2)(2/(\tau_2-\tau(t))^2)\times \\
&\left(\int_{t-\tau_2}^{t-\tau(t)}\dot{x}(u)duds\right)^T R_2\left(\int_{t-\tau_2}^{t-\tau(t)}\dot{x}(u)duds\right) \\
&= -(\tau_2-\tau_1)^2\left(x(t-\tau_1) - \frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}x(s)ds\right)^T R_2\times \\
&\left(x(t-\tau_1) - \frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}x(s)ds\right) \\
&-(\tau_2-\tau_1)^2\left(x(t-\tau(t)) - \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}x(s)ds\right)^T R_2\times \\
&\left(x(t-\tau(t)) - \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}x(s)ds\right).
\end{aligned} \tag{32}$$

From (31) and (32), $\dot{V}_6(t)$ can be estimated as

$$\begin{aligned}
\dot{V}_6(t) \le \tilde{\zeta}^T(t)[&(\tau_1^4/4)\tilde{e}_5R_1\tilde{e}_5^T - \tau_1^2(\tilde{e}_1-\tilde{e}_8)R_1(\tilde{e}_1^T-\tilde{e}_8^T) \\
&+((\tau_2-\tau_2)^4/4)\tilde{e}_6R_2\tilde{e}_6^T - (\tau_2-\tau_1)^2(\tilde{e}_3-\tilde{e}_9)R_2(\tilde{e}_3^T-\tilde{e}_9^T) \\
&-(\tau_2-\tau_1)^2(\tilde{e}_2-\tilde{e}_{10})R_2(\tilde{e}_2^T-\tilde{e}_{10}^T)]\tilde{\zeta}(t).
\end{aligned} \tag{33}$$

From (15), (19), and (28)–(33), an upper bound of $\dot{V}(t)$ can be

$$\dot{V}(t) \le \tilde{\zeta}^T(t)\tilde{\Xi}_{[\tau(t)]}\tilde{\zeta}(t) \tag{34}$$

where $\tilde{\Xi}_{[\tau(t)]}$ is defined in (27).
Note that $0 = \Pi\tilde{\zeta}(t)$ and the elements of $\tilde{\Xi}_{[\tau(t)]}$ are affinely dependent on $\tau(t)$. Therefore, by the use of Lemma 3 and convex-hull properties, $\tilde{\zeta}^T(t)\tilde{\Xi}_{[\tau(t)]}\tilde{\zeta}(t) < 0$ with $0 = \Pi\tilde{\zeta}(t)$, which means that system (1) is asymptotically stable and is satisfied if LMIs (25) and (26) hold. This completes our proof. ■

Remark 4. Unlike in Theorem 1, by taking the integral form of states as

$$\frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds,$$

$$\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds,$$

and

$$\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds,$$

different upper bounds of $\dot{V}_i(t)\,(i=1,5,6)$ are estimated in Theorem 2. In the upper bounds of $\dot{V}_1(t)$ and $\dot{V}_5(t)$, time-varying delays $\tau(t)$ exists. In obtaining an upper bound of the double integral form of

$$-\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds$$

in $\dot{V}_6(t)$ of Theorem 1, Lemma 3 is firstly applied to this integral term and one integral term,

$$\int_{t-\tau_2}^{t-\tau_1} x(s)ds$$

which are separated as two terms:

$$\int_{t-\tau(t)}^{t-\tau_1} x(s)ds$$

and

$$\int_{t-\tau_2}^{t-\tau(t)} x(s)ds.$$

.

However, in Theorem 2, the term

$$-\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds$$

is firstly separated as

$$-\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds$$

and

$$-\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)duds.$$

Lemma 3 is then applied on each term. These are the main differences between Theorems 1 and 2. Through numerical examples, we can show that Theorem 2 can also improve the feasible region of stability criterion by comparing maximum delay bounds.

Remark 5. If the information of time-derivative of $\tau(t)$ is unknown, one can obtain delay-dependent stability criteria by setting $Q_3 = 0$ in Theorems 1 and 2.

4. Numerical Examples

Example 1. Consider the well-known benchmark system with interval time-varying delays:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-\tau(t)). \tag{35}$$

For the different conditions of μ and τ_1, the results of maximum delay bounds for guaranteeing the system (35), which are asymptotically stable, are shown in Table 1. From Table 1, Theorem 1 gives larger delay bounds compared with the results of [13] and [14]. Theorem 2 also provides a larger feasible region compared with Theorem 1.

Table 1. Upper bounds of time-varying delays with different values of μ and τ_1 (Example 1)

τ_1	Methods	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.9$	μ is unknown
2	Shao [13]	2.6972	2.5048	2.5048	2.5048
	Sun *et al.* [14]	3.0129	2.5663	2.5663	2.5663
	Theorem 1	3.0129	2.6099	2.6099	2.6099
	Theorem 2	3.0129	2.6190	2.6188	2.6187
3	Shao [13]	3.2591	3.2591	3.2591	3.2591
	Sun *et al.* [14]	3.3408	3.3408	3.3408	3.3408
	Theorem 1	3.3891	3.3891	3.3891	3.3891
	Theorem 2	3.3904	3.3904	3.3904	3.3904
4	Shao [13]	4.0744	4.0744	4.0744	4.0744
	Sun *et al.* [14]	4.1690	4.1690	4.1690	4.1690
	Theorem 1	4.1978	4.1978	4.1978	4.1978
	Theorem 2	4.1980	4.1980	4.1980	4.1980
5	Shao [13]	-	-	-	-
	Sun *et al.* [14]	5.0275	5.0275	5.0275	5.0275
	Theorem 1	5.0332	5.0332	5.0332	5.0332
	Theorem 2	5.0332	5.0332	5.0332	5.0332

Example 2. Consider the following time-varying delay systems

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t-\tau(t)). \tag{36}$$

When μ is unknown and τ_1 is 1, 2, 3, 4, and 5, one can obtain maximum delay bounds by applying Theorems 1 and 2, as listed in Table 2. From the table, one can see that the results of Theorems 1 and 2 provide larger delay bounds than those of [13] and [14], which were recently published.

Table 2. Upper bounds of time-varying delays with unknown μ and various τ_1 (Example 2)

Methods	τ_1	1	2	3	4	5
Shao [13]	τ_2	1.6169	2.4798	3.3894	4.3250	5.2773
Sun *et al.* [14]	τ_2	1.6198	2.4884	3.4030	4.3424	5.2970
Theorem 1	τ_2	1.6867	2.5736	3.4862	4.4181	5.3646
Theorem 2	τ_2	1.6904	2.5685	3.4797	4.4126	5.3605

Remark 6. In Eq. (32), to derive the stability criterion into the form of affinely dependent on $\tau(t)$, integral terms

$$-((\tau_2 - \tau_1)^2 / 2)\int_{t-\tau_2}^{t-\tau(t)} \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(u) R_2 \dot{x}(u) du ds$$

are neglected. Thus, unlike the results of Table 1, one can see that Theorem 2 is more conservative than Theorem 1 when τ_1 is larger than 2.

Remark 7. In delay-dependent stability analysis, the delay decomposition method that divides the delay interval into N number is very effective to enlarge the feasible region of stability criteria. Zhu *et al.* [15] recently proposed improved delay-dependent stability criteria for linear systems with interval time-varying delays by utilizing the delay decomposition method. Comparing our methods with the ones in [15], our proposed methods provide larger delay when τ_1 is large. For example, in (36), the upper bound of maximum delay bounds when delay decomposition number is 4 and μ is unknown; thus, $\tau_2 = 2$ is 2.5608 in [15]. However, the results obtained by Theorems 1 and 2 are 2.5736 and 2.5685, as shown in Table 2. Therefore, the proposed new Lyapunov-Krasovskii functionals in Theorems 1 and 2 are very effective in enhancing the feasible region of stability criteria for systems with interval time-varying systems.

5. Conclusion

In the present paper, two delay-range-dependent stability criteria for linear systems with interval time-varying delays have been proposed using the Lyapunov method and LMI framework. In Theorem 1, based on the method of [14], the improved feasible region can be obtained by modifying three Lyapunov-Krasovskii functionals without introducing any free-weight matrices. With the same Lyapunov-Krasovskii functional considered in Theorem 1, the new stability condition that utilized more information of $\tau(t)$ than Theorem 1 are proposed by considering new augmented vectors and taking the upper bound of integral terms differently. Through two numerical examples, the improvements of the proposed stability criteria are successfully verified. Therefore, the proposed two theorems support that choosing Lyapunov-Krasovskii functionals and estimating their upper bounds of time-derivative play important roles in reducing the conservatism of stability criteria as mentioned in the Introduction.

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