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# State-Space Analysis on The Stability of Limit Cycle Predicted by Harmonic Balance

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**Abstract** – In this paper, a closed-loop system constructed with a linear plant and nonlinearity in the feedback connection is considered to argue against its planar orbital stability. Through a state space approach, a main result that presents a sufficient stability criterion of the limit cycle predicted by solving the harmonic balance equation is given. Preliminarily, the harmonic balance of the nonlinear feedback loop is assumed to have a solution that determines the characteristics of the limit cycle. Using a state-space approach, the nonlinear loop equation is reformulated into a linear perturbed model through the introduction of a residual operator. By considering a series of transformations, such as a modified eigenstructure decomposition, periodic averaging, change of variables, and coordinate transformation, the stability of the limit cycle can be simply tested via a scalar function and matrix. Finally, the stability criterion is addressed by constructing a composite Lyapunov function of the transformed system.

**Keywords**: Stability criterion, Limit cycle, Harmonic balance, Lyapunov function

## 1. Introduction

In many practical closed-loop systems, nonlinearity in the feedback connection often exists because of internal physical constraints. In these cases, a periodic signal may exist in the loop as an equilibrium state. Classically, in view of simplified planar dynamics, this periodic signal is often called a limit cycle. In addition, if the solution trajectory generates a closed positive semi-orbit as its limit set, this limit cycle can be effectively predicted via the describing function method. The describing function method provides an optimal quasi-linearization for the given nonlinearity in the feedback loop [1, 2], thereby enabling the prediction of the best approximated sinusoid in the loop. Thus, a number of engineering problems in such fields as biomedical engineering [3], switching system [4], resonant converters [5], fuzzy system prediction [6], and magnetic bearing system problem [7] used the describing function method to approximate the solution of loop equation.

In other aspects, to compliment its inherent approximation property, several results based on operator theory have attempted to verify the theoretical soundness of describing function method [8]–[12]. From these results, a compact existence domain and norm bound of the higher-order solution terms, as well as the information of the best predicted sinusoid are provided. In addition, a very insightful design process for analyzing and designing the feedback control loop has been established via a graphical method.

However, the stability problem of the limit cycle determined by harmonic balance has been relatively less studied. A classical approach for the stability analysis on the limit cycle takes advantage of quasi-static scheme. For instance, an extended Nyquist stability criterion [13] for stability test can be used. By checking the Nyquist condition, whether or not a steady-state limit cycle is stable under perturbation case can be determined. The methods of incremental describing function and Loeb condition [2] also provide the stability condition of the predicted limit cycle. However, quasi-static approaches simply focus on the harmonic balance; they do not consider the effect because of the neglected uncertain part of the original loop equation.

Meanwhile, a rigorous approach to treat the stability of periodic signal has been sought via an exact solution-based analysis, which often necessitates very complicated or nearly impractical mathematics. Examples include mathematical theories such as linearization combined with Floquet theory [14], linearization combined with a high-order Galerkin approximation and Hopf bifurcation [15], and the theory of integral manifold and stability invariance surface at perturbation by Pyatnitskiy and Rapoport [16] and Sastry [17]. However, these implicit methods are extremely difficult to apply, even in a simple problem case.

In the present paper, the stability problem of the limit cycle is revisited through a state-space analysis approach. With the extension of a preliminary formulation in [8] and systematic developments in [21], the claimed proposition provides a modified result investigating the stability of the

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limit cycle. Specifically, it relates to the stability criterion of the quasi-static solution considering the uncertainty term that appears during approximation of the exact loop equation around the limit cycle. In the analysis, the required assumptions are conventional, such as the Lipschitz continuity of the nonlinearity and the solution's existence at the harmonic balance. Under these conditions, it is claimed that the problem can be modified into conventional problem that regards the loop stability around its equilibrium. Using various transformations of modified eigenstructure decomposition, periodic averaging, change of variables, and coordinate transformation, stability of the limit cycle, which is predicted by the harmonic balance solution, is analyzed by checking a scalar function and matrix. In completing the stability proof, a composite Lyapunov function consisting of respective candidate Lyapunov functions for the isolated subsystems is introduced. The main results arguing against sufficiency for orbit stability are therefore drawn with a more compact and straightforward treatment on perturbation bounds, compared with the previous work [21]. In Section 2, system and required preliminaries are given. System transformations to yield a proper problem formulation are presented in Section 3. The main result and remark are shown in Section 4, followed by a conclusion in Section 5.

## 2. Preliminaries for System Analysis

### 2.1 Notation and Definition

The notations and definitions are given as follows:

$x = [x_1, x_2, ... x_n]' \in R^n$ : an $n$-tuple real vector

$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ : class of the $p$-norms

$\|x\|_\infty = \max_i |x_i|$ : class of the infinite norms

$L_p^n$ : a set of all piecewise continuous functions such that $p$-norm is bounded, that is,

$$\|x\|_{L_p^n} = (\int_0^\infty \|x(t)\|^p dt)^{1/p} < \infty, \quad 1 \le p < \infty$$

$L_\infty^n$ : a set of all piecewise continuous uniformly bounded functions such that infinite norm is bounded, that is,

$$\|x\|_{L_\infty^n} = \sup_{t \in [0,\infty)} \|x(t)\| < \infty$$

$d(x, \eta) = \min_{y \in \eta} \|x - y\|$ : distance function

Consider a smooth dynamical system represented as

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

on $R^n$. If $\eta$ is assumed as a $T$-periodic orbit of (1), an open set can be defined as $\varepsilon$ neighborhood of $\eta$:

$$\mathrm{B}_\varepsilon(\eta) = \{x \in R^n : d(x, \eta) < \varepsilon\} \tag{2}$$

The orbit $\eta$ is stable if trajectories starting near $\eta$ stay in the neighborhood of $\eta$. In other words, for every $\varepsilon > 0$, there exists $\delta > 0$, such that $x(t_o) \in B_\delta(\eta) \to x(t) \in B_\varepsilon(\eta)$ for all $t \ge t_o$. The orbit $\eta$ is asymptotically stable if it is stable and if the trajectories beginning near $\eta$ tend to $\eta$; in other words, if there exists $\delta > 0$ such that $d(x(t), \eta) \to 0$ as $t \to \infty$ for all $x(t_o) \in B_\delta(\eta)$ [20].

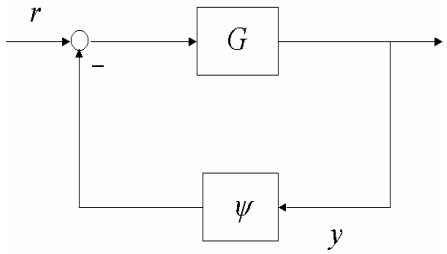


**Fig. 1.** Block diagram of the nonlinear feedback SISO system.

Typically, this orbit (asymptotic) stability considers stability problem around periodic orbit, whereas the general Lyapunov (asymptotic) stability considers behavior around equilibrium (e.g., origin).

### 2.2 System and Problem Formulation

Consider a single-input, single-output, and nonlinear feedback system, as shown in Fig. 1, consisting of linear plant and nonlinearity in the feedback connection. Assume that the plant belongs to $L_2^n[0,\infty)$ with a relative degree $n$ and, for simplicity, no zero dynamics. Assume also that the nonlinearity $\psi$ is an odd symmetric, sector-bounded, and piecewise continuous function, such that

$$S_{[c_1, c_2]} := \{\psi : [\psi(y) - c_1 y]^T [\psi(y) - c_2 y] \le 0\}$$

where $c_1$ and $c_2$ are the lower and higher bounds of the sector, respectively. In addition, $c_c = (c_1 + c_2)/2$ and $c_r = (c_2 - c_1)/2$ are defined. A classical describing function approximates the nonlinearity through an optimal quasi-linearization, thereby predicting the best approximated sinusoid in the feedback system. Using this concept, assume that there exists a sinusoidal limit cycle point, the amplitude-frequency pair of which is given as $(r_0, \omega_0)$. The problem studied in this paper is therefore to prove the orbit asymptotic stability of the quasi-static solution of the nonlinear system in Fig. 1.

To solve the stability problem, several preliminary steps are required [21]. With $r = 0$, the loop equation for the system in Fig. 1 is represented as follows:

$$y = -G\psi(y) \tag{3}$$

where the plant is assumed as $G = d_n/[s^n + n_1 s^{n-1} + n_2 s^{n-2} + \ldots + n_n]$ with $n$ distinct eigenvalues. Because $G$ is a linear operator, (3) can be represented as follows:

$$Ly + \psi(y) = 0\,, \tag{4}$$

where the differential operator $L$ is easily found. After introducing a residual operator, (4) can be decomposed into linear and nonlinear residual parts. If a residual operator is defined using nominal gain of the nonlinearity, the following nonlinear residual term is defined:

$$\tilde{\psi}(y) := \Psi(r_o)y - \psi(y)\,.$$

By introducing the describing function for the nonlinearity, defining a modified linear operator $\tilde{L}$ is also possible. Thus, the linear part of the differential operator equation can be obtained as follows:

$$\tilde{L}y := Ly + \Psi(r_o)y\,.$$

Next, substituting the residual term into (4), the resulting operator equation that uses both the residual term and the modified linear operator term is obtained. The loop equation in (4) can then be rearranged into

$$\tilde{L}y = \tilde{\psi}(y)\,. \tag{5}$$

For a state-space approach, state variables are defined as $\bar{y} = [y_1\ y_2\ y_3\ \ldots y_n]^T$. After normalizing coefficients of plant dynamics, (5) can be rearranged into the Brunovsky controllable by form

$$\frac{d\bar{y}}{dt} = A\bar{y} + B\tilde{\psi}(y)\,, \tag{6}$$

where $A$ and $B$ are given as

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -[n_n + d_n\Psi(r_o)] & -n_{n-1} & -n_{n-2} & \ldots & -n_1 \end{bmatrix},$$

$$B = [0\ \ 0\ \ \cdots\ \ 0\ \ d_n]^T\,.$$

Note that the harmonic balance method determines $\omega_o$ when $\tilde{L}y = 0$, which suggests that the matrix $A$ has $\pm i\omega_o$ as its spectrum. The latter part of this section demonstrates that the linear perturbed system in (6) can be transformed into the isolated equations consisting of principal term and higher-order residual terms.

Because $\lambda_{1,2} = \pm i\omega_o$ are eigenvalues of $A$ in (6), the following equations hold:

$$A\bar{v}_1 = \lambda_1\bar{v}_1 = i\omega_o\bar{v}_1\,,$$
$$A\bar{v}_2 = \lambda_2\bar{v}_2 = -i\omega_o\bar{v}_2\,.$$

By defining two real vectors $\bar{e}_1 \in R^n$ and $\bar{e}_2 \in R^n$ such that

$$\bar{e}_1 = \begin{bmatrix} 1 & 0 & -\omega_0^2 & 0 & \omega_0^4 & \cdots \end{bmatrix}^T,$$
$$\bar{e}_2 = \begin{bmatrix} 0 & \omega_0 & 0 & -\omega_0^3 & 0 & \cdots \end{bmatrix}^T,$$

the following equations hold:

$$A\bar{e}_1 = -\omega_0\bar{e}_2\,, \tag{7a}$$
$$A\bar{e}_2 = \omega_0\bar{e}_1\,, \tag{7b}$$

because $\bar{e}_1 \pm i\bar{e}_2$ turns out to be the eigenvector of $\pm i\omega_0$ for $A \in R^{n\times n}$. Because $A$ is a nonsingular matrix, (6) can be transformed into a suitable form by introducing a new variable: $\bar{\zeta} := V_1^{-1}Y$, where $V_1$ is given by

$$V_1 = \begin{bmatrix} | & | & : & | & : \\ \bar{e}_1 & \bar{e}_2 & \ldots & \bar{v}_i & \ldots \\ | & | & : & | & : \end{bmatrix},$$

and where $\bar{v}_i\,s$ are the remaining eigenvectors, except for $\bar{v}_1$ and $\bar{v}_2$. Applying the transformation matrix $V_1$ to the system in (6) yields

$$\frac{d\bar{\zeta}}{dt} = \Lambda_1\bar{\zeta} + K\tilde{\psi}(\zeta_1 + \bar{v}_{1i}\bar{\zeta}_r)\,, \tag{8}$$

where

$$\Lambda_1 = \begin{bmatrix} 0 & \omega_0 & & 0_{2\times n-2} & \\ -\omega_0 & 0 & & & \\ & & \lambda_3 & & 0 \\ 0_{n-2\times 2} & & & \ddots & \\ & & 0 & & \lambda_n \end{bmatrix},$$

and $K := V_1^{-1}B$. In (8), the variables are given as $\bar{\zeta} = [\zeta_1\ \zeta_2\ \bar{\zeta}_r^T]^T$ and $y = y_1 = \zeta_1 + \bar{v}_{1i}\bar{\zeta}_r$. Here, $\bar{v}_{1i}$ is an $n$-2-dimensional low vector composed of the first element of column $\bar{v}_i$ for $i = 3, 4, \ldots, n$. Finally, using the van der Pol transformation, a decoupled equation having principal and higher-order harmonic terms is achieved. First, $\varphi(t)$

is defined as

$$\varphi(t)=\begin{bmatrix}\cos\omega_0 t & \sin\omega_0 t\\ -\sin\omega_0 t & \cos\omega_0 t\end{bmatrix}.$$

By differentiating the following equation relating the variable $[\zeta_1\ \zeta_2]^T$ with a newly defined variable $[z_1\ z_2]^T$,

$$\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}=\varphi(t)\begin{bmatrix}z_1\\ z_2\end{bmatrix}, \tag{9}$$

and by using (8), a transformed equation containing principal terms is obtained as follows:

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}&=D_t\varphi(t)\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\varphi(t)\begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix}\\ &=\begin{bmatrix}0 & \omega_0\\ -\omega_0 & 0\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}+\begin{bmatrix}k_1\\ k_2\end{bmatrix}\tilde\psi(\zeta_1+\bar v_{1i}\bar\zeta_r).\end{aligned} \tag{10}$$

Thus, the system is formulated into the isolated equations of the principal terms and higher-order terms

$$\frac{d}{dt}\begin{bmatrix}z_1\\ z_2\end{bmatrix}=f(t,z_1,z_2,\bar\zeta_r), \tag{11a}$$

$$\frac{d\bar\zeta_r}{dt}=\tilde\Lambda\bar\zeta_r+g(t,z_1,z_2,\bar\zeta_r), \tag{11b}$$

where the nonlinear parts $f$ and $g$ are respectively defined as

$$f:=\begin{bmatrix}k_1\cos\omega_o t-k_2\sin\omega_o t\\ k_1\sin\omega_o t+k_2\cos\omega_o t\end{bmatrix}\tilde\psi\begin{pmatrix}z_1\cos\omega_o t+\\ z_2\sin\omega_o t+\bar v_{1i}\bar\zeta_r\end{pmatrix},$$

$$g:=\begin{bmatrix}k_3\\ \vdots\\ k_n\end{bmatrix}\tilde\psi\begin{pmatrix}z_1\cos\omega_0 t+\\ z_2\sin\omega_0 t+\bar v_{1i}\bar\zeta_r\end{pmatrix}.$$

## 3. Stability Criterion

Section 2 has shown that the system formulation is replaced via several basic transformations. In this section, three lemmas and one theorem that argue against the stability criterion are presented. In the first lemma, the system equation is modified into an equivalent form using the periodic averaging method. Particularly, the describing function representation is derived via the averaging technique. The second lemma states that the stability problem in Section 2 can be converted into a local stability problem of equilibrium at the interconnected ordinary differential equation. The final theorem provides a sufficient condition for the asymptotic stability of the sinusoidal limit cycle by taking advantage of the former two lemmas and an additional lemma.

[Lemma 1]

Consider the system equation in (11) and assume that $f$ and $g$ are $T$-periodic, Lipschitz continuous, and bounded. Using the periodic averaging method, the system in (11) can be rearranged into the interconnected system equations composed of linear averaged and nonlinear perturbed parts. The periodic averaging induces the describing function formula in the system equation.

*Proof)* For nonlinear periodic systems, the averaging method is reported as a proper approximation of the original system preserving the stability property [11]. Because $f$ is $T$-periodic, smooth, and bounded, an averaged term can be isolated from the nonlinear equation using the periodic averaging method. The method begins by isolating the higher-order terms in the periodic system equation:

$$f_o(z_1,z_2)=\frac{1}{T}\int_0^T f(t,z_1,z_2,0)dt. \tag{12}$$

Then, the error residual from the averaged function is given as

$$E(t,z_1,z_2)=\int_0^t[f(s,z_1,z_2,0)-f_o(z_1,z_2)]ds. \tag{13}$$

Clearly, the error function $E(t,z_1,z_2)$ is $T$-periodic from (12) and (13). A new variable is then defined

$$z:=x+E(t,x),$$

where $z:=[z_1\ z_2]^T$, $x:=[x_1\ x_2]^T$, and $E(t,x):=[E_1(t,x)\ E_2(t,x)]^T$. Differentiating each side yields

$$\frac{d}{dt}\begin{bmatrix}z_1\\ z_2\end{bmatrix}=D_t\begin{bmatrix}x_1\\ x_2\end{bmatrix}+D_tE(t,x)+D_xE(t,x)\frac{dx}{dt}. \tag{14}$$

where $D_{(\cdot)}$ denotes a partial derivative operator. Using (13) and (14), (11-a) is rearranged into

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}z_1\\ z_2\end{bmatrix}&=f(t,z_1,z_2,\bar\zeta_r)\\ &=\frac{dx}{dt}+D_xE\frac{dx}{dt}+f(t,x,0)-f_o(x)\\ &\quad+D_xEf_o(x)-D_xEf_o(x).\end{aligned}$$

Finally, the averaged system is given as

$$\frac{dx}{dt}=f_o(x)+f_1(t,x,\bar\zeta_r), \tag{15}$$

where

$$f_1(t,x,\bar{\zeta}_r) := [I + D_x E]^{-1}[f(t,x+E,\bar{\zeta}_r) - f(t,x,0) - D_x E f_o(x)] .$$

In deriving $f_1(t,x,\bar{\zeta}_r)$, note that the existence condition for the inverse of $I+D_xE$ is $(1+D_{x_1}E_1)(1+D_{x_2}E_2) - D_{x_2}E_1 \cdot D_{x_1}E_2 \neq 0$. The first term in the right-hand side of (15) represents the averaged equation and behaves as a dominant factor in determining the stability property. To obtain the averaged term in (15), consider (12) again in order to compute $f_o(x)$:

$$f_o(x) = \frac{1}{T}\int_0^T f(t,x,0)dt ,$$

where

$$f(t,x,0) = \begin{bmatrix} \begin{pmatrix} k_1 \cos\omega_0 t - \\ k_2 \sin\omega_0 t \end{pmatrix} \\ \begin{pmatrix} k_1 \sin\omega_0 t + \\ k_2 \cos\omega_0 t \end{pmatrix} \end{bmatrix} \tilde{\psi} \begin{pmatrix} x_1 \cos\omega_0 t + \\ x_2 \sin\omega_0 t \end{pmatrix} \tag{16}$$

Note that the definition of sinusoidal input describing function is restated as the following equation for a sinusoid signal input $r\cos(\tilde{\theta}+\alpha)$:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tilde{\theta}}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha))d\tilde{\theta} = \frac{r}{2}\tilde{\Psi}(r)e^{i\alpha}$$

From the above equation, the following equations can be obtained:

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\tilde{\theta}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha))d\tilde{\theta} = \frac{r}{2}\tilde{\Psi}(r)\cos\alpha \tag{17a}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\tilde{\theta}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha))d\tilde{\theta} = -\frac{r}{2}\tilde{\Psi}(r)\sin\alpha \tag{17b}$$

Using (16) and (17), the first term of the averaged equation can be found as follows

$$\begin{aligned} f_{o1}(x) &= \frac{1}{T}\int_0^T [k_1\cos\omega_0 t - k_2 \sin\omega_0 t]\times \\ &\quad \tilde{\psi}(x_1\cos\omega_0 t + x_2\sin\omega_0 t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} k_1\cos\tilde{\theta}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha)) - \\ k_2\sin\tilde{\theta}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha)) \end{bmatrix} d\tilde{\theta} \\ &= \frac{1}{2}k_1\tilde{\Psi}(r)x_1 - \frac{1}{2}k_2\tilde{\Psi}(r)x_2 . \end{aligned} \tag{18a}$$

For simplification, the relationships of $x_1 = r\cos(-\alpha)$ and $x_2 = r\sin(-\alpha)$ are applied. Similarly, the second term is given as

$$\begin{aligned} f_{o2}(x) &= \frac{1}{T}\int_0^T [k_2\cos\omega_0 t + k_1 \sin\omega_0 t] \\ &\quad \tilde{\psi}(x_1\cos\omega_0 t + x_2\sin\omega_0 t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} k_1\sin\tilde{\theta}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha)) + \\ k_2\cos\tilde{\theta}\tilde{\psi}(r\cos(\tilde{\theta}+\alpha)) \end{bmatrix} d\tilde{\theta} \\ &= \frac{1}{2}k_2\tilde{\Psi}(r)x_1 + \frac{1}{2}k_1\tilde{\Psi}(r)x_2 . \end{aligned} \tag{18b}$$

From the above results, the averaged term can be re-arranged as

$$f_o(x) = \begin{bmatrix} f_{o1}(x) \\ f_{o2}(x) \end{bmatrix} = \frac{1}{2}\tilde{\Psi}(r)\begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \tag{19}$$

Thus, the system in (11) is converted into the following isolated equations containing the describing function representation of the nonlinearity:

$$\frac{dx}{dt} = \frac{1}{2}\tilde{\Psi}(r)\begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + f_1(t,x,\bar{\zeta}_r) , \tag{20a}$$

$$\frac{d\bar{\zeta}_r}{dt} = \tilde{\Lambda}\bar{\zeta}_r + g_1(t,x,\bar{\zeta}_r) \tag{20b}$$ *Q.E.D.*

According to Lemma 1, the system equation is modified into a linear perturbed form that includes the describing function via the periodic averaging process. Lemma 2 claims that the stability of sinusoidal limit cycle in (3) turns out to be equivalent with the local stability problem around origin in (20).

[Lemma 2]

Assume the conditions in Lemma 1. The stability of the sinusoidal limit cycle with respect to the system equation $y = -G\psi(y)$ given in Fig. 1 is equivalent to the local stability of $(\delta r, \bar{\zeta}_r) = (0,0)$ in $R^1 \times R^{n-2}$ with respect to the following differential equations:

$$\frac{d\delta r}{dt} = c(r_o)\delta r + \tilde{f}(t,\delta r,\bar{\zeta}_r) \tag{21a}$$

$$\frac{d\bar{\zeta}_r}{dt} = \tilde{\Lambda}\bar{\zeta}_r + \tilde{g}(t,\delta r,\bar{\zeta}_r) \tag{21b}$$

where $c(r_o) := \frac{k_1 r_o}{2}\cdot\left.\frac{\partial\tilde{\Psi}(r)}{\partial r}\right|_{r=r_o}$.

*Proof)* The system equation in Fig. 1 has been developed into (20). Noting that the state variables representing the principal term can be transformed into radius and phase of sinusoidal limit cycle, $x_1$ and $x_2$ are represented as

$$x_1 = r\sin\theta, x_2 = r\cos\theta .$$

The time derivatives are given as

$$\frac{dx_1}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt},$$
$$\frac{dx_2}{dt} = \frac{dr}{dt}\cos\theta + r\sin\theta(-\frac{d\theta}{dt}).$$

Because the following relationships hold,

$$\frac{dx_1}{dt}\sin\theta + \frac{dx_2}{dt}\cos\theta = \frac{dr}{dt},$$
$$\frac{dx_1}{dt}\cos\theta - \frac{dx_2}{dt}\sin\theta = r\frac{d\theta}{dt},$$

the coordinate can be transformed into a polar form, which yields the following equations:

$$\begin{bmatrix} \frac{dr}{dt} \\ r\frac{d\theta}{dt} \end{bmatrix} = \frac{1}{r}\begin{bmatrix} \frac{dx_1}{dt}x_1 + \frac{dx_2}{dt}x_2 \\ \frac{dx_1}{dt}x_2 - \frac{dx_2}{dt}x_1 \end{bmatrix}. \tag{22}$$

By applying (15) and (19) into (22), systems (15) and (11-b) are depicted by the interconnected ordinary differential equations:

$$\begin{aligned} \frac{dr}{dt} &= \sin\theta\frac{dx_1}{dt} + \cos\theta\frac{dx_2}{dt} \\ &= \sin\theta\frac{\tilde{\Psi}(r)}{2}[k_1x_1 - k_2x_2] + \sin\theta f_{11}(\cdot) \\ &\quad + \cos\theta\frac{\tilde{\Psi}(r)}{2}[k_1x_2 + k_2x_1] + \cos\theta f_{12}(\cdot) \\ &= \frac{\tilde{\Psi}(r)}{2}k_1r + \sin\theta f_{11}(\cdot) + \cos\theta f_{12}(\cdot), \end{aligned} \tag{23a}$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{r}[-\sin\theta\frac{dx_2}{dt} + \cos\theta\frac{dx_1}{dt}] \\ &= \frac{\cos\theta}{r}\frac{\tilde{\Psi}(r)}{2}[k_1x_1 - k_2x_2] + \frac{\cos\theta}{r}f_{11}(\cdot) \\ &\quad - \frac{\sin\theta}{r}\frac{\tilde{\Psi}(r)}{2}[k_1x_2 + k_2x_1] - \frac{\sin\theta}{r}f_{12}(\cdot) \\ &= -\frac{\tilde{\Psi}(r)}{2}k_2 + \frac{1}{r}[\cos\theta f_{11}(\cdot) - \sin\theta f_{12}(\cdot)], \end{aligned} \tag{23b}$$

$$\frac{d\bar{\zeta}_r}{dt} = \tilde{\Lambda}\bar{\zeta}_r + g_1(t, x, \bar{\zeta}_r), \tag{23c}$$

where $f_{11}$ and $f_{12}$ are the first and second terms of $f_1$ in (15), respectively. Because the stability of sinusoidal limit cycle of amplitude $r_0$ (with additional information of $\omega_0$) is of interest, the case of concern is to inspect the stability of the equilibrium at $r = r_o$ and $\bar{\zeta}_r = 0$ with an arbitrary phase angle $\theta \in [0, 2\pi)$ with the given system in (23).

Noting that $\tilde{\Psi}(r)$ is zero when $r = r_o$, reconsider the system in (23). Setting $r = r_o + \delta r$ and extending equation via the Taylor series around $r_0$ converts (23-a) to

$$\frac{d\delta r}{dt} = \frac{k_1 r_o}{2}\frac{\partial\tilde{\Psi}(r)}{\partial r}\bigg|_{r=r_o} \cdot \delta r + \tilde{f}(t, \delta r, \bar{\zeta}_r) \tag{24a}$$

where $\tilde{f}(t, \delta r, \bar{\zeta}_r)$ is the sum of $\sin\theta \cdot f_{11}(\cdot) + \cos\theta \cdot f_{12}(\cdot)$ in (23-a) and higher-order perturbation is generated from the Taylor series representation of $\tilde{\Psi}(r)k_1r/2$ around $r_0$. The equation for higher-order residuals is given as follows:

$$\frac{d\bar{\zeta}_r}{dt} = \tilde{\Lambda}\bar{\zeta}_r + \tilde{g}(t, \delta r, \bar{\zeta}_r). \tag{24b}$$ *Q.E.D*

Note that, to determine the stability of the limit cycle, one only needs to show that (24) is stable. This is because the factored-out phase term can be neglected. Using Lemmas 1 and 2, the following theorem argues against a sufficient condition investigating the stability of sinusoidal limit cycle in (21).

[Theorem 1]

Consider the nonlinear feedback system in Fig. 1. Assume that the plant and nonlinearity are Lipschitz continuous and bounded. Assume also that there exists a sinusoidal limit cycle whose amplitude and frequency pair is $(r_0, \omega_0)$. If $c(r_o) < 0$, $\tilde{\Lambda}$ is Hurwitz and $S$ is an $M$ matrix,

$$\det(S) = \det\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} > 0,$$

whose entries are given as

$$\begin{aligned} s_{11} &= -c(r_o) - [\Delta_E\varepsilon_2\varepsilon_E + \Delta_E L_{D_xE} + \varepsilon_O] \\ s_{12} &= -\Delta_E\varepsilon_1 \\ s_{21} &= [\sum_{i=3}^{n}|k_i|^2]^{1/2}(c_r + |\Psi(r_o) - c_c|)\cdot\varepsilon_z \\ s_{22} &= -\lambda_{\max}(\tilde{\Lambda}) - [\sum_{i=3}^{n}|k_i|^2]^{1/2}(c_r + |\Psi(r_o) - c_c|)\cdot\|\bar{v}_{1j}\| \end{aligned},$$

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_E$, $\Delta_E$, $L_{D_xE}$, and $\varepsilon_O$ are arbitrary small constants that can be obtained from the continuity and bound condition of $\tilde{f}$ and $\tilde{g}$, then the equilibrium $(\delta r, \bar{\zeta}_r) = (0, 0)$ in $R^1 \times R^{n-2}$ is locally asymptotically stable. In other words, the approximated sinusoidal limit cycle is orbit asymptotically stable.

In the proof of Theorem 1, a lemma introducing an $M$ matrix property is required.

[Lemma 3]

There exists a positive diagonal $D$, such that

$DS + S^T D > 0$ if and only if $S$ is an $M$ matrix. In short, the leading principal minors of $S$ are positive.

*Proof)* [18]

*Proof of Theorem 1)* Following the results of Lemmas 1 and 2, one only needs to verify that (24) is asymptotically stable around $(\delta r, \bar{\zeta}_r) = (0,0)$ for the proof. The local stability problem can be argued by applying a composite Lyapunov function to the interconnected nonlinear system of (24).

For simplicity, a new state vector is defined as $u = [\delta r \ \bar{\zeta}_r]^T$. Because $k_1 r_o D_r \tilde{\psi}|_{r_o} < 0$ and $\tilde{\Lambda}$ is Hurwitz, the Lyapunov functions $V_1 = \frac{1}{2}\delta r^2$ and $V_2 = \frac{1}{2}\bar{\zeta}_r^T \bar{\zeta}_r$ can be chosen for each unperturbed system. In addition, certain positive constant pair ( $\alpha_i$ , $\beta_i$ ) and positive continuous function ( $\phi_i$ ) can be found such that

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial u_i} f_i(t,u_i) \le -\alpha_i \phi_i^2(u_i)$$
$$\left\| \frac{\partial V_i}{\partial u_i} \right\| \le \beta_i \phi_i(u_i) \tag{25a}$$

for each unperturbed subsystem $\dot{u}_i = f_i(t,u_i)$. Using only the unperturbed part of (21) and each Lyapunov function $V_1$ and $V_2$, $\alpha_1 = -c(r_0)$, $\phi_1(u_1) = |\delta_r|$, $\beta_1 = 1$, $\alpha_2 = -\lambda_{\max}(\tilde{\Lambda})$, $\phi_2(u_2) = \|\bar{\zeta}_r\|$, and $\beta_2 = 1$ can be obtained. In addition, from the Lipschitz continuity and boundedness of the plant and nonlinearity, some non-negative $\gamma_{ij}$'s are found, such that

$$\|g_i(t,u)\| \le \sum_{j=1}^{2} \gamma_{ij} \phi_j(u_j) \tag{25b}$$

for the given $\phi_j$'s . In fact, noting that the smooth functions $\tilde{f}$ and $g$ are bounded in a local neighborhood around the equilibrium, and using the relationships of (23), (24), and (11), the following inequalities hold around the equilibrium:

$$\begin{aligned} \|\tilde{f}\| &\le \|f_1\| \\ &\le [\Delta_E \varepsilon_2 \varepsilon_E + \Delta_E L_{D_x E} + \varepsilon_O]\|\delta_r\| + \Delta_E \varepsilon_1 \|\bar{\zeta}_r\| \end{aligned} \tag{26a}$$

$$\|g_1\| \le \sqrt{\sum_{j=3}^{n} |k_j|^2}\,[c_r + |\Psi(r_o) - c_c|](\varepsilon_z \|\delta_r\| + \|\bar{v}_{1i}\|\|\bar{\zeta}_r\|) . \tag{26b}$$

where each upper bound is derived from perturbation terms and consequently constitutes $\gamma_{ij}$'s [21]. A composite Lyapunov candidate for the entire system in (24) can now be defined as

$$V = w_1 V_1 + w_2 V_2 . \tag{27}$$

Using (25) and (27), the time derivative of $V$ along the trajectories of the interconnected system (24) satisfies

$$\frac{dV(t,x)}{dt} \le -\frac{1}{2}\phi^T (WS + S^T W)\phi \tag{28}$$

where

$$\phi = [\phi_1, \phi_2]^T, \quad W = diag(w_1, w_2)$$

and where $s_{ij}$ (i.e., the $ij^{\text{th}}$ element of $S$) is given as

$$s_{ij} = \begin{cases} \alpha_i - \beta_i \gamma_{ii}, & i = j \\ -\beta_i \gamma_{ij}, & i \ne j \end{cases} .$$

From Lemma 3, note that $WS + S^T W$ is positive definite because $S$ is an $M$ matrix and $W$ is a positive diagonal matrix. Thus, $V$ in (27) becomes the Lyapunov function of (24), which guarantees local stability of origin. Finally, following the results of Lemmas 1 and 2, a sinusoidal limit cycle with respect to (3) is asymptotically stable. *Q.E.D*

Note that the linear growth bound condition is essential in developing the Lyapunov-based stability analysis. Moreover, the smoothness condition of functions, $\tilde{f}$ and $\tilde{g}$, is natural in that a typical differential equation in many practices is assumed to satisfy the local Lipschitz continuity for the existence and uniqueness of the solution. This assumption can be further removed by developing the boundedness and piecewise continuity condition of the plant and feedback nonlinearity.

[Remark] Theorem 1 provides an extended result for stability test criterion from the quasi-static method. In practical application, the conditions that $c(r_o) < 0$ and $\tilde{\Lambda}$ is Hurwitz can be intuitively used to determine the steady-state limit cycle's stability, which is equivalent to the quasi-static analysis result via the Loeb criterion.

## 4. Conclusion and Discussion

The current paper presents a state-space result for the assumed stability of a limit cycle by solving the harmonic balance. Through a series of proper transformations and simplification, a relatively simple criterion that argues against the limit cycle's orbital stability is presented. Because the stability criterion is presented using linear system theory, an explicit procedure can be expected for the nonlinear loop analysis or synthesis compared with complex methodologies from mathematical theory.

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## References

[1] M. Vidyasagar, *Nonlinear systems analysis; 2nd Edition*, Englewood Cliffs, New Jersey: Prentice-Hall Inc., 1993.

[2] D. P. Atherton, *Nonlinear control Engineering*, Van Nostrand Reinhold Company Ltd., London, 1975.

[3] J. M. Hidler and W.Z. Rymer, "Limit cycle behavior in spasticity: Analysis and evaluation," IEEE Transactions on Biomedical engineering, Vol. 47, No. 12, December 2000, pp.1565-1575.

[4] S. R. Sanders, "On limit cycles and the describing function method in periodically switched circuits," IEEE Transactions on Circuits and Systems, Vol. 40, No. 9, September 1993, pp. 564-572.

[5] H. Pinheiro, P. Jain, and G. Joos, "Self-sustained oscillating resonant converters operating above the resonant frequency," *IEEE Transactions on Power Electronics*, Vol. 14, No. 5, September 1999, pp. 803-809.

[6] E. Kim, H. Lee and M. Park, " Limit cycle prediction of a fuzzy control system based on describing function method," IEEE Transactions on Fuzzy Systems, Vol. 8, No. 1, February 2000, pp. 11-22.

[7] A. M. Mohamed and F. P. Emad, "Nonlinear oscillations in magnetic bearing systems," IEEE Transactions on Automatic Control, Vol. 38, No. 8, August 1993, pp. 1242-1245.

[8] A. I. Mees and A. R. Bergen, "Describing functions revisited," IEEE Transactions on Automatic Control, Vol. 20, No. 4, August 1975, pp. 473-478.

[9] A. R. Bergen and R. L. Franks, "Justification of describing function method," SIAM Journal of Control, Vol. 9, No. 4, November 1971, pp. 568-589.

[10] F. L. Swern, "Analysis of oscillation in systems with polynomial-type nonlinearities using describing functions," IEEE Transactions on Automatic Control, Vol. 28, No. 1, January 1983, pp. 31-41.

[11] H. K. Khalil, *Nonlinear systems; 2nd edition*, Upper Saddle River, New Jersey: Prentice-Hall Inc., 1996.

[12] D. Williamson, "Periodic Motion in nonlinear systems," IEEE Transactions on Automatic Control, Vol. 20, No. 4, August 1975, pp. 479-486.

[13] J. E. Slotine and W. Li, *Applied nonlinear control*, Englewood Cliffs, New Jersey: Prentice-Hall Inc., 1991.

[14] Herbert Amann, *Ordinary differential equation: An introduction to nonlinear analysis*, Walter de Gruyter & Co., 1990.

[15] A. I. Mees and L. O. Chua, "The Hopf bifurcation theorem and its application to nonlinear oscillations in circuits and systems," IEEE Transactions on Circuits and Systems, Vol. 26, No. 4, April 1979, pp. 235-254.

[16] E. S. Pyatnitskiy and L. B. Rapoport, "Criteria of asymptotic stability of differential inclusions and periodic motions of time-varying nonlinear control systems," IEEE Transactions on Circuits and Systems, Vol. 43, No. 3, March 1996, pp. 219-229.

[17] Shankar Sastry, *Nonlinear systems : Analysis, Stability and Control*, Springer-Verlag, 1999.

[18] M. Fiedler and V. Ptak. "On matrices with nonnegative off-diagonal elements and positive principal minors," Czech. Mathematics Journal, Vol. 12, 1962, pp. 382-400.

[19] C. C. Chung and J. Hauser, "Nonlinear H-inf control around periodic orbits," Systems and Control Letters 30, 1997, pp.127-137.

[20] S. Lee and S. M. Meerkov, "Vibrational feedback control in the problem of absolute stability," IEEE Transactions on Automatic Control, Vol. 36, No. 4, April 1991, pp. 482-484.

[21] S. Sung, A Feedback Loop Design for MEMS Resonant Accelerometer Using a Describing Function Technique, *Ph.D Thesis*, Seoul National University, 2003.

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