On the Distribution Functions of Ratios Involving Gaussian Random Variables

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In this letter, we derive the distribution functions of five ratios involving two correlated Gaussian random variables by using the rotation of Cartesian coordinates. The results can be used in evaluating the various probability performances of wireless communications systems.

Keywords: Error probability, Gaussian distribution, Q-function.

I. Introduction

The distribution of the ratio involving two correlated Gaussian random variables (RVs) has been used in computing error and outage probabilities. As an example, the analytical expression for outage probability over a dual lognormal fading channel is obtained in the form of two-dimensional (2D) Gaussian *Q*-function by using the ratio of two Gaussian RVs [1]. The rotation of Cartesian coordinates was presented to compute the error probability of *M*-ary phase shift keying (MPSK) over an additive white Gaussian noise (AWGN) channel [2]-[5].

There are various error or outage probabilities characterized by the ratio involving Gaussian RVs. It is important to obtain the ratio distribution function (DF) to compute the probabilities. In order to evaluate the probability performances of wireless communications systems over a Gaussian channel, we focus on analytical expressions for the DFs of five ratio RVs: i) the DF of signal-to-noise (or interference) ratio, Y/X, ii) the DF of signal-to-received power ratio, $Y/X+Y$, iii) the ratio DF used to compute the symbol error probability (SEP) of binary differential phase shift keying (BDPSK), $|Y|/|X|$, iv) the phase DF employed when computing the SEP of MPSK, $\tan^{-1}(Y/X)$, and v) the ratio DF applied to evaluate the performance of dual switch antenna, $\min(X, Y)/\max(X, Y)$.

Thus, in this letter, we develop the DF of five ratio RVs to compute error or outage probabilities in the presence of two correlated Gaussian noises. The objective of this letter is to present new and simple derivations of the five DFs.

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II. Derivations of DFs

In this section, we consider that the distribution of X and Y has a correlated Gaussian distribution with two means, μ_X and μ_Y, two standard deviations, σ_X and σ_Y, and a correlation coefficient, ρ_{XY}.

1. DF of *Y/X*

The DF of signal-to-noise (or interference) ratio plays a key role in computing the outage probability of a wireless communications system over a correlated Gaussian channel. Hinkley [6], [7] first gave the derivation of $F_{Y/X}(z)$ through an algebraic-direct technique. Alternatively, we present a derivation of $F_{Y/X}(z)$ by computing wedge-shaped regions in two correlated Gaussian RVs.

The DF of Y/X is determined by the event $\{(x, y)|Y/X\le z\}$, which can be divided into two events: $\{(x, y)|Y\ge zX, X\le 0\}$ and $\{(x, y)|Y\le zX, X\ge 0\}$. Figure 1 shows that the event $\{(x, y)|Y/X\le z\}$ is expressed by two wedge-shaped regions, $\sphericalangle Y^{+}OU_1^{-}$ and $\sphericalangle Y^{-}OU_1^{+}$. The symbol '$\sphericalangle$' denotes the wedge-shaped region. In Fig. 1, the phase angle $\psi=\tan^{-1}(z)$ characterizes two wedge-shaped regions: $\sphericalangle Y^{+}OU_1^{-}$ and $\sphericalangle Y^{-}OU_1^{+}$. In order to obtain an analytical expression for $F_{Y/X}(z)$, we rotate the X-Y Cartesian coordinates counterclockwise about the origin through an angle $\psi=\tan^{-1}(z)$ in a way that

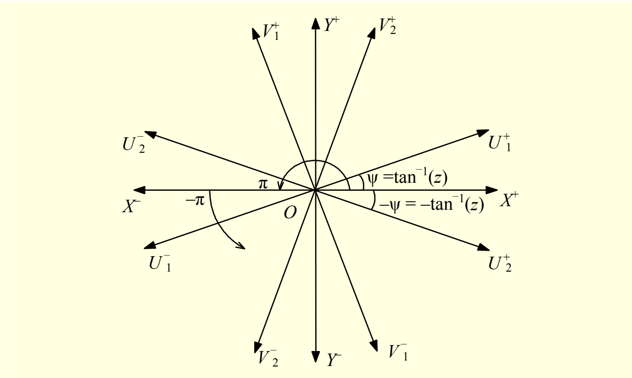

Fig. 1. Geometry of wedge-shaped regions generated by two rotated X-Y Cartesian coordinates.

$$\begin{cases} U_1 = X\cos\psi + Y\sin\psi, \\ V_1 = -X\sin\psi + Y\cos\psi. \end{cases} \tag{1}$$

According to the theory of linear combinations of Gaussian RVs, the distribution of two Gaussian RVs, U_1 and V_1, becomes Gaussian with two means and two standard deviations such that

$$\begin{cases} \mu_{U_1} = \mu_X \cos\psi + \mu_Y \sin\psi, \\ \mu_{V_1} = -\mu_X \sin\psi + \mu_Y \cos\psi, \\ \sigma_{U_1} = \sqrt{\sigma_X^2 \cos^2\psi + \sigma_Y^2 \sin^2\psi + \rho_{XY}\sigma_X\sigma_Y \sin 2\psi}, \\ \sigma_{V_1} = \sqrt{\sigma_X^2 \sin^2\psi + \sigma_Y^2 \cos^2\psi - \rho_{XY}\sigma_X\sigma_Y \sin 2\psi}. \end{cases} \tag{2}$$

Using the geometry of Fig. 1, we get

$$F_{Y/X}(z) = \Pr\{V_1 > 0, X < 0\} + \Pr\{V_1 < 0, X < 0\}. \tag{3}$$

The correlation coefficient between two Gaussian RVs, V_1 and X, ρ_{V_1X}, is computed as

$$\rho_{V_1X} = -\frac{\sigma_X^2 \sin\psi - \rho_{XY}\sigma_Y \cos\psi}{\sigma_{V_1}\sigma_X}. \tag{4}$$

Finally, applying normalization and applying (26.3.5) and (26. 3.6) of [8] to (3) gives the expression for $F_{Y/X}(z)$ as

$$F_{Y/X}(z) = Q\left(-\frac{\mu_{V_1}}{\sigma_{V_1}}, \frac{\mu_X}{\sigma_X}; -\rho_{V_1X}\right) + Q\left(\frac{\mu_{V_1}}{\sigma_{V_1}}, -\frac{\mu_X}{\sigma_X}; -\rho_{V_1X}\right), \tag{5}$$

where the 2D Gaussian Q-function $Q(x, y; \rho)$ is defined as

$$Q(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_x^\infty \int_y^\infty \exp\left[-\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)}\right] du dv. \tag{6}$$

Note that Simon [9] and Park and others [10] presented a single finite integral representation of the 2D Gaussian Q-function. Approximations for the 2D Gaussian Q-function are provided in [11], [12]. As in (3) of [1], the derived DF can be used to compute the outage probability in a log-normal fading channel because Y (in decibels)/X (in decibels) is represented by the ratio of two Gaussian RVs.

2. DF of $X/(X+Y)$

We consider the outage probability of a pilot channel in the mobile cellular system using code division multiple access. We assume that the RV X is the received pilot channel power and the RV Y is the received total power excluding the received pilot channel power X at mobile station over a Gaussian channel. Thus, the DF of $X/(X+Y)$, $F_{X/(X+Y)}(z)$, can be applied to compute the outage probability of pilot channel, as presented in (3.61) of [13]. We easily obtain the DF of $X/(X+Y)$ as

$$F_{X/(X+Y)}(z) = \Pr\left\{\frac{X}{X+Y} \le z\right\} = F_{Y/X}\left(\frac{1-z}{z}\right), \tag{7}$$

where $F_{Y/X}(\cdot)$ is in (5).

3. DF of $|Y|/|X|$

As in (7.6.19) of [14], the DF of $|Y|/|X|$, $F_{|Y|/|X|}(z)$, can be used in evaluating the SEP performance of a BDPSK system over an AWGN channel. The $F_{|Y|/|X|}(z)$ is computed by using the geometry of Fig. 1 as

$$\begin{aligned} F_{|Y|/|X|}(z) &= \Pr\{|Y| \le z|X|\} \\ &= \Pr\{\sphericalangle U_1^+ O U_2^+\} + \Pr\{\sphericalangle U_1^- O U_2^-\}. \end{aligned} \tag{8}$$

In Fig. 1, two U_1-V_1 and U_2-V_2 Cartesian coordinates are obtained by rotating the X-Y Cartesian coordinates clockwise and counterclockwise through the phase angle $\psi = \tan^{-1}(z)$, respectively. Here, we focus on two Gaussian RVs, V_1 and V_2:

$$\begin{cases} V_1 = -X\sin\psi + Y\cos\psi, \\ V_2 = X\sin\psi + Y\cos\psi. \end{cases} \tag{9}$$

Employing the theory of Gaussian RVs to (9) gives a bivariate Gaussian distribution with two means, μ_{V_1} and μ_{V_2}, two standard deviations, σ_{V_1} and σ_{V_2}, and a correlation coefficient, $\rho_{V_1V_2}$, such that

$$\begin{cases} \mu_{V_1} = -\mu_X \sin\psi + \mu_Y \cos\psi, \\ \mu_{V_2} = \mu_X \sin\psi + \mu_Y \cos\psi, \\ \sigma_{V_1} = \sqrt{\sigma_X^2 \sin^2\psi + \sigma_Y^2 \cos^2\psi - \rho_{XY}\sigma_X\sigma_Y \sin 2\psi}, \\ \sigma_{V_2} = \sqrt{\sigma_X^2 \sin^2\psi + \sigma_Y^2 \cos^2\psi + \rho_{XY}\sigma_X\sigma_Y \sin 2\psi}, \\ \rho_{V_1V_2} = \dfrac{\sigma_Y^2 \cos^2\psi - \sigma_X^2 \sin^2\psi}{\sigma_{V_1}\sigma_{V_2}}. \end{cases} \tag{10}$$

Equation (8) can be rewritten by using the geometry of two wedge-shaped regions, $\sphericalangle U_1^+OU_2^+$ and $\sphericalangle U_1^-OU_2^-$, in Fig. 1 as

$$F_{|Y|/|X|}(z)=\Pr\{V_1<0,V_2>0\}+\Pr\{V_1>0,V_2<0\}. \quad (11)$$

Thus, by using the same step done in the previous subsection, we obtain an exact expression for $F_{|Y|/|X|}(z)$ as

$$F_{|Y|/|X|}(z)=Q\left(\frac{\mu_{V_1}}{\sigma_{V_1}},-\frac{\mu_{V_2}}{\sigma_{V_2}};-\rho_{V_1V_2}\right)+Q\left(-\frac{\mu_{V_1}}{\sigma_{V_1}},\frac{\mu_{V_2}}{\sigma_{V_2}};-\rho_{V_1V_2}\right), \quad (12)$$

where five parameters, $\mu_{V_1},\mu_{V_2},\sigma_{V_1},\sigma_{V_2}$, and $\rho_{V_1V_2}$, in the 2D Gaussian Q-function are in (10).

4. DF of $\psi=\tan^{-1}(Y/X)$

There have been several expressions to compute the probability of a wedge-shaped region in the presence of two Gaussian RVs. Case I of Pawula F-function [15], [16] has been used in the case of two uncorrelated Gaussian RVs. Park and others [17] extended Case I of Pawula F-function to correlated Gaussian quadratures. Aalo and others [18] derived a new expression for the probability density function (pdf) of phase angle for correlated Gaussian quadratures. Shmaliy [19] also presented the pdf of random RF pulse. However, motivated by the possibility to compute the probability of the wedge-shaped region through the well-known function, we derive an alternative expression for the DF of phase angle in the presence of two correlated Gaussian RVs.

We define that

$$\Psi=\begin{cases}\tan^{-1}(Y/X) & \text{if } X\ge 0\\ \pi+\tan^{-1}(Y/X) & \text{if } X<0\end{cases};\quad -\pi\le\Psi\le\pi. \quad (13)$$

Using the geometry in Fig.1, $F_\Psi(\psi)$ is expressed as

$$F_\Psi(\psi)=\begin{cases}\Pr\left\{\sphericalangle X^-OU_1^-\right\}, & -\pi\le\Psi\le 0,\\ \Pr\{Y<0\}+\Pr\left\{\sphericalangle X^+OU_1^+\right\}, & 0<\Psi\le\pi.\end{cases} \quad (14)$$

Next, we consider the similar transformation given in the previous subsections:

$$\begin{bmatrix}U_1\\V_1\end{bmatrix}=\begin{bmatrix}\operatorname{sgn}(\psi)\cos\psi & \operatorname{sgn}(\psi)\sin\psi\\ -\operatorname{sgn}(\psi)\sin\psi & \operatorname{sgn}(\psi)\cos\psi\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}, \quad (15)$$

where sgn(u) is +1 if $u\ge 0$, and -1 if $u<0$. From (15), we get that the RV, V_1, has Gaussian distribution with mean μ_{V_1} and standard deviation σ_{V_1} such that

$$\begin{cases}\mu_{V_1}=-\operatorname{sgn}(\psi)\mu_X\sin\psi+\operatorname{sgn}(\psi)\mu_Y\cos\psi,\\ \sigma_{V_1}=\sqrt{\sigma_X^2\sin^2\psi+\sigma_Y^2\cos^2\psi-\rho_{XY}\sigma_X\sigma_Y\sin(2\psi)}.\end{cases} \quad (16)$$

The DF of Ψ, $F_\Psi(\psi)$, can be expressed by using (14) and (15) as

$$F_\Psi(\psi)=\begin{cases}\Pr\{V_1>0,Y<0\}, & -\pi\le\Psi\le 0,\\ \Pr\{Y\le 0\}+\Pr\{V_1<0,Y>0\}, & 0<\Psi\le\pi.\end{cases} \quad (17)$$

The correlation coefficient of V_1 and Y is

$$\rho_{V_1Y}=\frac{\operatorname{sgn}(\psi)\sigma_Y\cos\psi-\operatorname{sgn}(\psi)\rho_{XY}\sigma_X\sin\psi}{\sigma_{V_1}}. \quad (18)$$

Then, applying normalization and (26.2.6), (26.3.5), and (26.3.6) of [8] to (17) gives the expression for $F_\Psi(\psi)$ as

$$F_\Psi(\psi)=\frac{1+\operatorname{sgn}(\psi)}{2}Q\left(\frac{\mu_Y}{\sigma_Y}\right)+Q\left(\operatorname{sgn}(\psi)\frac{\mu_{V_1}}{\sigma_{V_1}},-\operatorname{sgn}(\psi)\frac{\mu_Y}{\sigma_Y};-\rho_{V_1Y}\right), \quad (19)$$

where the mean μ_Y and standard deviation σ_Y are given, the mean μ_{V_1} and standard deviation σ_{V_1} are in (16), and the correlation coefficient ρ_{V_1Y} is in (18).

5. DF of min(X, Y)/max(X, Y)

The DF of min(X, Y)/max(X, Y), $F^*(z)$, is obtained as

$$F^*(z)=\Pr\left\{\frac{\min(X,Y)}{\max(X,Y)}\le z\right\}=\Pr\left\{Y\ge\frac{X}{z},Y\ge X\right\}+\Pr\{Y\le zX,Y\le X\}. \quad (20)$$

Figure 2 represents two wedge-shaped regions, $\sphericalangle U_0^-OU_1^+$ and $\sphericalangle U_0^-OU_2^+$, which determine the DF $F^*(z)$. Next, we consider the rotated X-Y Cartesian coordinates determined by three phase angles $\psi_i;\ i=0,1,2$:

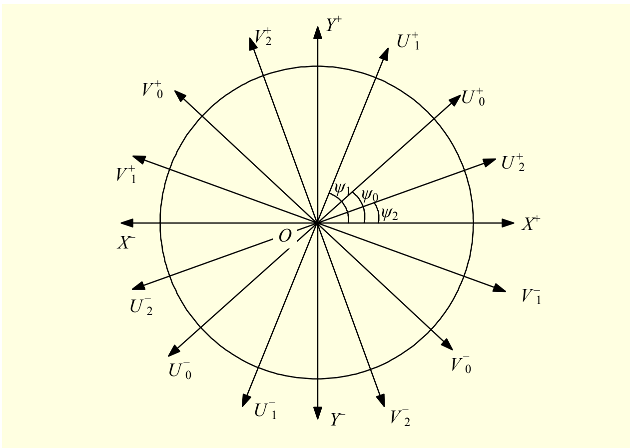

Fig. 2. Geometry of wedge-shaped regions generated by three rotated X-Y Cartesian coordinates.

$$\begin{cases} U_i = X\cos\psi_i + Y\sin\psi_i \\ V_i = -X\sin\psi_i + Y\cos\psi_i \end{cases}; \quad i = 0,1,2, \tag{21}$$

where $\psi_0 = \pi/4$, $\psi_1 = \cot^{-1}(z)$, and $\psi_2 = \tan^{-1}(z)$. Using the geometry of Fig. 2, we get

$$\begin{aligned} F^*(z) &= \Pr\{\sphericalangle U_0^- O U_1^+\} + \Pr\{\sphericalangle U_0^- O U_2^+\} \\ &= \Pr\{V_0 > 0, V_1 > 0\} + \Pr\{V_0 < 0, V_2 < 0\}. \end{aligned} \tag{22}$$

Finally, applying normalization and (26.3.3) and (26.3. 11) of [8] to (22) yields an expression for $F^*(z)$ as

$$F^*(z) = Q\left(-\frac{\mu_{V_0}}{\sigma_{V_0}}, -\frac{\mu_{V_1}}{\sigma_{V_1}}; \rho_1\right) + Q\left(\frac{\mu_{V_0}}{\sigma_{V_0}}, \frac{\mu_{V_2}}{\sigma_{V_2}}; \rho_2\right), \tag{23}$$

where

$$\mu_{V_0} = -\frac{\mu_X}{\sqrt{2}} + \frac{\mu_Y}{\sqrt{2}}, \tag{24}$$

$$\sigma_{V_0} = \sqrt{\frac{\sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y}{2}}, \tag{25}$$

$$\mu_{V_i} = -\mu_X \sin\psi_i + \mu_Y \cos\psi_i;\ i = 1,2, \tag{26}$$

$$\sigma_{V_i} = \sqrt{\sigma_X^2 \sin^2\psi_i + \sigma_Y^2 \cos^2\psi_i - \rho_{XY}\sigma_X\sigma_Y \sin(2\psi_i)};\ i = 1,2 \tag{27}$$

and

$$\rho_i = \frac{\sigma_X^2 \sin\psi_i - (\sin\psi_i + \cos\psi_i)\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2 \cos\psi_i}{\sigma_{V_i}\sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho_{XY}\sigma_X\sigma_Y}}; \quad i = 1,2. \tag{28}$$

III. Conclusion

In this letter, we have derived five DFs of the ratios involving two correlated Gaussian RVs by using the rotation of Cartesian coordinates. The derived functions are represented as only the known Gaussian *Q*-function. The results can be used in evaluating the probability performances of various analyses determined by the ratio of two Gaussian RVs.

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