A Polynomial Complexity Optimal Multiuser Detection Algorithm Based on Monotonicity Properties

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An optimal multiuser detection algorithm with a computational complexity of $O(K \log K)$ is proposed for the class of linear multiple-access systems which have constant cross-correlation values. Here the optimal multiuser detection is implemented by searching for a monotone sequence with maximum likelihood, under the ranking of sufficient statistics. The proposed algorithm is intuitive and concise. It is carried out in just two steps, and at each step only one kind of operation is performed. Also, the proposed algorithm can be extended to more complex systems having more than a single cross-correlation value.

Keywords: optimal multiuser detection, polynomial complexity, monotone sequence, multiple-access system.

I. Introduction

The optimal multiuser detection for linear a multiple-access system has, in general, a computational complexity that is exponential in the number of users. For this reason, most studies have focused on designing sub-optimal multiuser detectors that have low computational complexity and as close as possible performance to the optimal one. Meanwhile, a few researchers reviewed the optimal multiuser detection problem and found that optimal multiuser detection, in very special cases, may be solved with polynomial complexity. Two classes of optimal multiuser detection problems have been solved with polynomial complexity [1]-[4]. One has constant cross-correlation values [1], [2], and the other has nonpositive cross-correlation values [3], [4]. Those optimal multiuser detection problems that can be transformed into one of the two classes are also solved with polynomial complexity [5].

A class of symbol-synchronous code division multiple-access (CDMA) systems has been developed [1], [2] where all cross-correlation values are equal. Two polynomial complexity optimal multiuser detection algorithms were developed applicable for that system.

This letter proposes a more concise polynomial complexity optimal multiuser detection algorithm for that system formulated in [1]. Compared with the two existing algorithms addressed in [1], [2], the proposed algorithm is more intuitive and requires fewer steps.

II. System Model and Problem Formulation

The symbol-synchronous CDMA system model of [1] is considered, assuming K users and binary modulation. The matched filter outputs, which form sufficient statistics for the decision, are given by $\mathbf{y} = \mathbf{Rd} + \mathbf{z}$, where $\mathbf{R}$ is the $K \times K$ correlation matrix between the signature signals of the different users, the elements of the correlation matrix $\mathbf{R}$ are given by

$$R_{i,j} = \begin{cases} 1, & i = j, \\ \rho, & i \neq j, \end{cases}$$

where $\mathbf{d}$ is a binary column vector of length K, and $\mathbf{z}$ is a noise vector with convariance matrix $\mathbf{E}[\mathbf{zz}^*] = \sigma^2\mathbf{R}$.

The optimal multiuser detector will output hypothesis $\mathbf{d}$, which maximizes the likelihood function $p(\mathbf{y}|\mathbf{d})$. In an additive Gaussian noise channel, this is equivalent to minimizing the quadratic as in

$$\hat{\mathbf{d}} = \arg \min_{\mathbf{d} \in \{-1,+1\}^K} (\mathbf{d}^T\mathbf{Rd} - 2\mathbf{d}^T\mathbf{y}). \quad (1)$$

In general, solving (1) requires 2^K function evaluations (for

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all possible binary sequences of length K), and so the computational complexity grows exponentially, that is, $O(2^K)$, with the number of users K.

III. A Concise Polynomial Complexity Algorithm

Proposition. For a symbol-synchronous CDMA system with binary modulation, K users, and the constant cross-correlation values between them, the optimal multiuser detector output $\hat{\mathbf{d}}$ satisfies $\hat{d}_{\pi(1)} \le \hat{d}_{\pi(2)} \le \cdots \le \hat{d}_{\pi(K)}$, where π is a permutation of $(1, 2, \cdots, K)$ so that $y_{\pi(1)} \le y_{\pi(2)} \le \cdots \le y_{\pi(K)}$.

Proof. Assume, to the contrary, that there is i and j $(1 \le i < j \le K)$ so that $\hat{d}_{\pi(i)} > \hat{d}_{\pi(j)}$. Let vector $\check{\mathbf{d}}$ be $\hat{\mathbf{d}}$ with rows $\pi(i)$ and $\pi(j)$ interchanged, thus

$$2\hat{\mathbf{d}}^T\mathbf{y} \le 2\check{\mathbf{d}}^T\mathbf{y}, \tag{2}$$

where $\mathbf{y} = [y_1, y_2, \cdots, y_K]^T$. The constant cross-correlation values in $\mathbf{R}$ stand so that

$$\hat{\mathbf{d}}^T\mathbf{R}\hat{\mathbf{d}} = \check{\mathbf{d}}^T\mathbf{R}\check{\mathbf{d}}. \tag{3}$$

It follows from (2) and (3) that

$$\hat{\mathbf{d}}^T\mathbf{R}\hat{\mathbf{d}} - 2\hat{\mathbf{d}}^T\mathbf{y} \ge \check{\mathbf{d}}^T\mathbf{R}\check{\mathbf{d}} - 2\check{\mathbf{d}}^T\mathbf{y}. \tag{4}$$

Expression (4) means that $\hat{\mathbf{d}}$ is not the optimal multiuser detector output. Note that $\check{\mathbf{d}}$ and $\hat{\mathbf{d}}$ are both optimal multiuser detector outputs when equality is achieved in (4). This case is the result of $y_{\pi(i)} = y_{\pi(j)}$. □

The proposition shows that the detection result corresponding to a larger statistic is not less in value than that corresponding to a smaller one. After the permutation π, the optimal multiuser detector output $\{\hat{d}_{\pi(1)}, \hat{d}_{\pi(2)}, \cdots, \hat{d}_{\pi(K)}\}$ becomes an ascending sequence of binary numbers. Therefore, the hypothesis tests in (1) only need to be performed over binary ascending sequences, not all possible binary sequences of length K.

The number of the binary ascending sequences of length K is K+1, which grows linearly with the sequence length (that is, the number of users). They can be enumerated as follows. The first sequence is $\{-1, -1, \cdots, -1\}$, which consists of the element "-1" only. The i-th ($2 \le i \le K+1$) sequence is derived from the first sequence by replacing the last $i-1$ elements with "$+1$." As an example, binary ascending sequences of length 2 are listed as follows: $\{-1, -1\}$, $\{-1, +1\}$, and $\{+1, +1\}$.

Now we are ready to present the concise polynomial complexity algorithm.

Algorithm 1.

1. Determine a permutation π of $(1, 2, \cdots, K)$ so that $y_{\pi(1)} \le y_{\pi(2)} \le \cdots \le y_{\pi(K)}$.
2. Perform $\hat{\mathbf{d}}_\pi = \arg\min_{\mathbf{d}_\pi \in S} (\mathbf{d}_\pi{}^T\mathbf{R}\mathbf{d}_\pi - 2\mathbf{d}_\pi{}^T\mathbf{y}_\pi)$, where $\mathbf{y}_\pi = [y_{\pi(1)}, y_{\pi(2)}, \cdots, y_{\pi(K)}]^T$, $\mathbf{d}_\pi = [d_{\pi(1)}, d_{\pi(2)}, \cdots, d_{\pi(K)}]^T$, and S is the set of binary ascending sequences of length K.

The proposed algorithm consists of two steps. The first step provides initial detection of the transmitted bit vector $\mathbf{d}$ through ranking the sufficient statistics $\mathbf{y}$. After ranking the sufficient statistics, the K+1 ascending sequences are determined as the initial detection results for $\mathbf{d}_\pi$. The computational complexity of the ranking operation performed here grows, in the worst case, $O(K \log K)$, with the number of users K. The second step produces the final decision for the $\mathbf{d}$ by hypothesis tests within the K+1 ascending sequences. Its computational complexity is $O(K)$, which grows linearly with the number of users, K. Therefore, the computational complexity of the proposed algorithm is dominated by the ranking operation performed at the first step.

IV. Generalization

Algorithm 1 can be extended to a more complex multiple-access system as defined in section IV in [1]. In that system, all signature signals are partitioned into L groups according to the cross-correlation values between them. The transmitted binary vector and sufficient statistics are grouped accordingly and represented by $\mathbf{d} = [\mathrm{d}_1{}^T, \mathrm{d}_2{}^T, \cdots, \mathrm{d}_L{}^T]^T$ and $\mathbf{y} = [\mathrm{y}_1{}^T, \mathrm{y}_2{}^T, \cdots, \mathrm{y}_L{}^T]^T$, respectively, and $\mathbf{d}_i$ is a binary column vector of length K_i (i=1, 2,$\cdots$, L). The correlation matrix of that system has the following properties: the cross-correlation values between different signature signals of group i are equal to ρ_{ii} (i=1, 2,$\cdots$, L), and the cross-correlation values between any signature signal of group i and any signature signal of group j are equal to ρ_{ij} (j=1, 2,$\cdots$, L; $j \ne i$), with $\rho_{ij} = \rho_{ji}$. These properties of the correlation matrix stand so that in each group of L, the detection result corresponding to a larger statistic is not less in value than that corresponding to a smaller one. It can be proved in the same way as the proposition in section III.

Algorithm 2.

1. Determine L permutations π_i of $(1, 2, \cdots, K_i)$ so that $\mathbf{y}_{\pi_i}$ permutated from $\mathbf{y}_i$ becomes an ascending sequence, where i=1, 2,$\cdots$, L.

2. Perform $\hat{\mathbf{d}}_\pi = \arg \min_{\mathbf{d}_\pi \in \{S_1 \times S_2 \times \cdots \times S_L\}} (\mathbf{d}_\pi{}^T \mathbf{R} \mathbf{d}_\pi - 2\mathbf{d}_\pi{}^T \mathbf{y}_\pi)$, where $\mathbf{y}_\pi = [\mathrm{y}_{\pi_1}{}^T, \mathrm{y}_{\pi_2}{}^T, \cdots, \mathrm{y}_{\pi_L}{}^T]^T$, $\mathbf{d}_\pi = [\mathrm{d}_{\pi_1}{}^T, \mathrm{d}_{\pi_2}{}^T, \cdots, \mathrm{d}_{\pi_L}{}^T]^T$, and S_i is the set of binary ascending sequences of length $K_i (i = 1, 2, \cdots, L)$.

The size of the set $\{S_1 \times S_2 \times \cdots \times S_L\}$ is $\prod_{i=1}^{L}(K_i + 1)$. Therefore, taking into account the ranking operation performed in the first step, the worst case complexity is in order

$$o\ (\sum_{i=1}^{L} K_i \log K_i + \prod_{i=1}^{L}(K_i + 1)\ ).$$

Now we illustrate algorithm 2 with an example. Consider a symbol-synchronous CDMA system with binary modulation, four users, and the correlation matrix

$$\mathrm{R} = \begin{bmatrix} 1 & 0.1 & 0.2 & 0.2 \\ 0.1 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.1 \\ 0.2 & 0.2 & 0.1 & 1 \end{bmatrix}.$$

According to the properties of the **R**, the transmitted binary vector **d** is partitioned into 2 groups as $\mathbf{d} = [\mathrm{d}_1{}^T, \mathrm{d}_2{}^T]^T$, where $\mathbf{d}_1 = [d_1, d_2]^T$ and $\mathbf{d}_2 = [d_3, d_4]^T$. Assume the sufficient statistics $\mathbf{y} = [-1, 0.2, 1, 2]^T$, then the optimal multiuser detector output $\hat{\mathbf{d}}$ should satisfy $\hat{d}_1 \le \hat{d}_2$ and $\hat{d}_3 \le \hat{d}_4$. Therefore, the candidates for the optimal multiuser detector output are the following nine sequences: $\{-1,-1,-1,-1\}$, $\{-1,+1,-1,-1\}$, $\{+1,+1,-1,-1\}$, $\{-1,-1,-1,+1\}$, $\{-1,+1,-1,+1\}$, $\{+1,+1,-1,+1\}$, $\{-1,-1,+1,+1\}$, $\{-1,+1,+1,+1\}$, and $\{+1,+1,+1,+1\}$. Finally, the sequence $\{-1,-1,+1,+1\}$ is detected by hypothesis tests over the nine sequences.

V. Conclusion

The proposed algorithm is based on the monotonicity implicit in optimal multiuser detector output. With the monotonicity, the hypothesis tests involved in optimal multiuser detection are performed over monotone sequences instead of all possible binary sequences. It is proved that the monotonicity in the optimal detector output can be obtained just by ranking the sufficient statistics for the multiuser systems with constant cross-correlation values.

So far only two basic classes of optimal multiuser detection problems have been solved in polynomial complexity. One is with constant cross-correlation values, and the other with nonpositive cross-correlation values. Their computational complexities are in order $O(K \log K)$ and $O(K^3)$, respectively.

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