Approximation for the Two-Dimensional Gaussian Q-Function and Its Applications

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In this letter, we present a new approximation for the two-dimensional (2-D) Gaussian Q-function. The result is represented by only the one-dimensional (1-D) Gaussian Q-function. Unlike the previous 1-D Gaussian-type approximation, the presented approximation can be applied to compute the 2-D Gaussian Q-function with large correlations.

Keywords: Gaussian distribution, probability, Q-function.

I. Introduction

The two-dimensional (2-D) Gaussian Q-function has been used in evaluating the error and outage probability performance of wireless communication systems. The 2-D Gaussian Q-function representation has been proposed to compute the probability of an arbitrary wedge-shaped region in the presence of additive white Gaussian noise (AWGN) [1]-[5]. Alouini and others developed the 2-D Gaussian Q-function expression to compute various outage probabilities of dual diversity systems over correlated lognormal fading channels [6]. Simon gave the Craig-form representation for the 2-D Gaussian Q-function [7], and Park and others presented the alternative form of the representation [8]. Here, the Craig-form representation requires numerical integral operations. Chiani and others derived the exponential-type approximation with a square of the argument in the exponent for the one-dimensional (1-D) Gaussian Q-function [9]. The series expansion of the 2-D Gaussian Q-function uses the 1-D Gaussian Q-function and the derivatives of the Gaussian probability density function (pdf) as in equation (26.3.29) in [10]. Several series expansions of the 1-D Gaussian Q-function have recently been published [11]-[15]. Lin gave the 1-D Gaussian Q-function-type approximation for the 2-D Gaussian Q-function in equation (6) in [16]. We notice that the relative errors of the approximation obtained by Lin increase as the correlation coefficient approaches -1. Therefore, given that it would be valuable to obtain a new 1-D Gaussian Q-function-type approximation for the 2-D Gaussian Q-function in a way that increases accuracy, we focus on the new approximation represented by the 1-D Gaussian Q-function.

In this letter, we derive a new approximation for the 2-D Gaussian Q-function by using the 1-D Gaussian Q-function approximation developed by Chiani and others.

II. Derivation of Approximation

It is known that the 2-D Gaussian Q-function is defined by

$$Q(x,y;\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_x^\infty\int_y^\infty \exp\left[-\frac{u^2+v^2-2\rho uv}{2\left(1-\rho^2\right)}\right]dudv. \tag{1}$$

The 2-D Gaussian Q-function can be also rewritten as in equation (26.3.20) in [10] as

$$Q(x,y;\rho)=Q\left(x,0,\rho_{xy}\right)+Q\left(y,0,\rho_{yx}\right)-\delta_{xy}, \tag{2}$$

where

$$\rho_{xy}=\mathrm{sgn}(x)(\rho x-y)/\sqrt{x^2-2\rho xy+y^2}, \tag{3}$$

$$\rho_{yx}=\mathrm{sgn}(y)(\rho y-x)/\sqrt{x^2-2\rho xy+y^2}, \tag{4}$$

and

$$\delta_{xy}=\begin{cases}0 & \text{if } xy>0 \text{ or } (xy=0 \text{ and } x+y\geq 0),\\ 1/2 & \text{otherwise,}\end{cases} \tag{5}$$

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in which sgn(u)=1 if $u \geq 0$, and sgn(u)= −1 if $u < 0$. Furthermore, letting y=0 in $Q(x, y; \rho)$ and using equations (26.3.7) and (26.3.8) in [10], the following two relations are given:

$$Q(-x,0;\rho) = 1/2 - Q(x,0;-\rho)\,, \tag{6}$$

$$Q(x,0;\rho) = Q(x) - Q(x,0;-\rho)\,. \tag{7}$$

As given in [16], we consider the case of $Q(x, 0; \rho)$, when $x \geq 0$ and $\rho < 0$, to compute the general 2-D Gaussian Q-function. The relation $Q(x, 0; \rho)$ can be expressed using equation (26.3.2) in [10] and equation (4) in [16] as

$$Q(x,0;\rho) = \int_x^{\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}} Q\left(-\frac{\rho t}{\sqrt{1-\rho^2}}\right) dt, \tag{8}$$

where $x \geq 0$ and $\rho < 0$. The 1-D Gaussian Q-function $Q(x)$ is defined by

$$Q(u) = \int_u^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt. \tag{9}$$

There are various series expansions of the 1-D Gaussian Q-function $Q(\cdot)$ [11]-[15].

Finally, substituting equation (14) from [9] into (8) gives the new approximation for the $Q(x, 0; \rho)$ as

$$Q(x,0;\rho) \approx \frac{\sqrt{1-\rho^2}}{12} Q\left(\frac{x}{\sqrt{1-\rho^2}}\right) + \frac{1}{4}\sqrt{\frac{3(1-\rho^2)}{3+\rho^2}} Q\left(\sqrt{\frac{3+\rho^2}{3(1-\rho^2)}}x\right), \tag{10}$$

where $x \geq 0$ and $\rho < 0$. Note that the numerical validation of (10) depends on that of equation (14) in [9].

III. Numerical Results

In this section, we demonstrate the validity of the presented approximation. The relative errors were computed to compare the new approximation, (10), with the previous approximation, equation (6) in [16], for four special cases: ρ= −0.5, ρ= −0.6, ρ= −0.7, and ρ= −0.8.

Figure 1 shows that the relative errors of previous approximation increase as the correlation coefficient approaches −1, whereas the new approximation, (10), has small relative errors for $\rho \geq -0.6$. That is, the relative errors of the new approximation are smaller than those of the previous one when the correlation coefficient approaches −1. For $\rho \geq -0.6$, the new approximation also becomes available by using (7). Therefore, the presented expression can be used for an approximation in computing the 2-D Gaussian Q-function with large correlations.

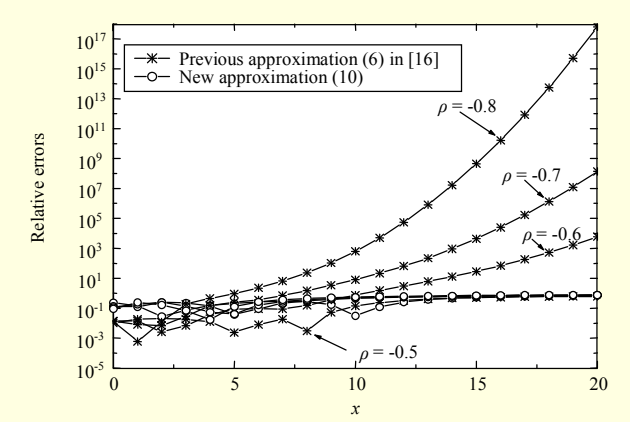

Fig. 1. Comparison of the relative errors in (10) and the results in equation (6) in [16].

IV. Applications

In this section, we present three applications of the derived approximation. First, we consider the analytical expression for the symbol-error probability (SEP) of an M-ary phase shift keying (MPSK) system over AWGN channels. The approximation for the SEP of MPSK can be obtained by using equation (20) in [1] and (10) as

$$P_{\mathrm{E}} \approx 2Q\left[\sqrt{\frac{2E_S}{N_0}}\sin(\pi/M)\right] - \frac{\sin(\pi/M)}{6} Q\left(\sqrt{\frac{2E_S}{N_0}}\right) - \frac{\sin(\pi/M)}{2}\sqrt{\frac{3}{3+\cos^2(\pi/M)}} Q\left[\sqrt{\frac{2E_S}{N_0}\left(1+\frac{\cos^2(\pi/M)}{3}\right)}\right], \tag{11}$$

where E_S is the signal energy, N_0 is the noise power, and M is the size of constellation with M>2. Figure 2 shows good agreement between equation (20) in [1] and (11) for various values of M.

Also, as a second application, two 2-D Gaussian Q-function representations for both the probability of a wedge-shaped region over AWGN channels and the outage probability of a dual-branch selection combining system over correlated lognormal fading channels [5]-[6] have been recently presented. Here, applying (1), (6), and (7) to equation (12) in [5] and equation (42) in [6] and using (10) results in two corresponding approximations which are computed by only the 1-D Gaussian Q-function.

Next, we will consider the other application. The integral of the standardized circular normal distribution over the bounded triangle, $V(h, ah)$, plays a key role in computing the probability of a polygon over AWGN channels as seen in example 9 presented in [10]. The quantity of $V(h, ah)$ is computed as in equation (26.3.23) in [10] as

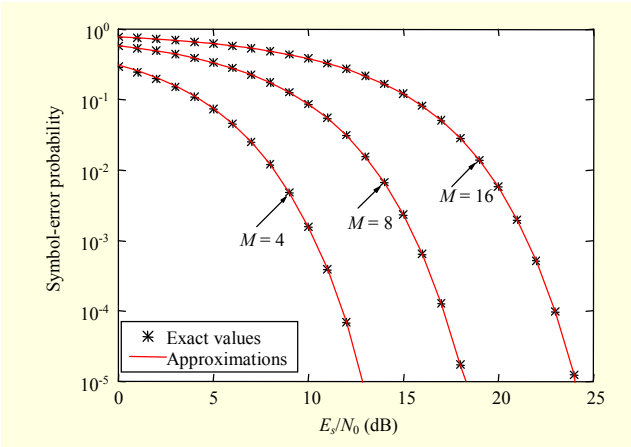

Fig. 2. Comparison of the SEP between exact values in equation (20) in [1] and results in (11) for various M.

$$V(h,ah)=\frac{1}{2\pi}\int_0^h\int_0^{au}\exp\left[-\frac{1}{2}(u^2+v^2)\right]dudv$$
$$=\frac{1}{4}+Q(h,0;r)-Q(0,0;r)-\frac{Q(h)}{2}, \qquad (12)$$

where $r=-a/\sqrt{1+a^2}$.

Finally, applying equation (26.3.19) in [10] and (10) to (12) leads to the following approximation for $V(h, ah)$:

$$V(h,ah)\approx\frac{\sqrt{1-r^2}}{12}Q\left(\frac{h}{\sqrt{1-r^2}}\right)+\frac{1}{4}\sqrt{\frac{3(1-r^2)}{3+r^2}}Q\left(\sqrt{\frac{3+r^2}{3(1-r^2)}}h\right)$$
$$+\frac{\sin^{-1}r}{2\pi}-\frac{Q(h)}{2}. \qquad (13)$$

Note that the relation between $V(h, ah)$ and the 2-D Gaussian Q-function is provided in equation (46.51) in [17].

V. Conclusion

A new 1-D Gaussian-type approximation for the 2-D Gaussian Q-function has been presented. The numerical results have shown that the presented approximation can be applied to compute the 2-D Gaussian Q-function with large correlations. The new approximation is numerically more useful in computing the 2-D Gaussian Q-function by employing the built-in series expansion of 1-D Gaussian Q-function without various numerical integration techniques.

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