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# A Batch Sequential Sampling Scheme for Estimating the Reliability of a Series/Parallel System

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**Abstract.** It is desired to estimate the reliability of a system that has two subsystems connected in series where each subsystem has two components connected in parallel. A batch sequential sampling scheme is introduced. It is shown that the batch sequential sampling scheme is asymptotically optimal as the total number of units goes to infinity. Numerical comparisons indicate that the batch sequential sampling scheme performs better than the balanced sampling scheme and is nearly optimal.

**Key Words :** *Reliability, Batch sequential scheme, Balanced sampling scheme, Optimal allocation scheme, Allocation procedure, Stopping rule.*

## 1. INTRODUCTION

Suppose that a system is composed of two independent subsystems I and II connected in series, where each subsystem has two components connected in parallel; see Figure 1.1 below.

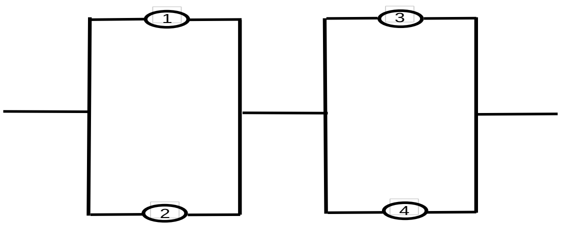


Figure 1.1. Series/parallel system diagram.

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Denoting by $1 - q_i$ the reliability of component $i$ where $q_i$ is the probability that a unit of component $i$ fails, then the reliability of subsystem I is $(1 - q_1 q_2)$ and the reliability of subsystem II is $(1 - q_3 q_4)$. Therefore, the reliability of the system is the product of the reliability of each subsystem

$$R = (1 - q_1 q_2)(1 - q_3 q_4).$$

Sampling schemes for estimating system reliabilities have been proposed by Berry (1977), Rekab (1993), Enaya et al. (2006). These papers dealt with series systems with many components. The system we consider in this paper is more general, since it consists of subsystems in series where each subsystem has parallel components. In this article, a batch sequential sampling scheme is introduced and compared with the balanced and the optimal allocation schemes. Theoretical results show that the proposed batch sequential sampling scheme is asymptotically optimal. Monte Carlo studies indicate that it performs better than the balanced sampling scheme, and as well as the optimal sampling scheme.

To estimate the product of means (which has an application in reliability) using a Bayesian approach, Page (1995) determined an allocation procedure (what component to test) and a stopping rule (when to stop the experiment). The allocation procedure was shown to be asymptotically optimal as the cost of testing tends to zero. A similar formulation without a stopping rule was investigated earlier by Berry (1977). Exact optimal sampling schemes were determined by Hardwick and Stout (1994). However, exact optimality could only be achieved using dynamic programming.

The rest of the paper is organized as follows. In the next section, we compute the variance of the overall reliability. The batch sequential sampling scheme is described in Section 3. Section 4 presents the results of the Monte Carlo simulation. Finally, the paper is concluded in Section 5 where some directions for future research are also given.

## 2. SYSTEM RELIABILITY VARIANCE

The optimal sampling scheme is to select $M_1, M_2, M_3$, and $M_4$, which represent the number of units to test from each component, so that the variance of the system reliability estimate $\hat{R}$ is minimized. Since the reliability of the system is

$$R = P(\text{I works})\, P(\text{II works}) = (1 - q_1 q_2)(1 - q_3 q_4),$$

then,

$$\hat{R} = (1 - \hat{q}_1 \hat{q}_2)(1 - \hat{q}_3 \hat{q}_4),$$

where $\hat{q}_i$ is the maximum likelihood estimator (MLE) of the probability of failure of component $i$, that is

$$\hat{q}_i = \sum_{j=1}^{M_i} \frac{X_{ij}}{M_i}$$

where

$$X_{ij} = \begin{cases} 0, & \text{if the } j\text{th unit of component } i \text{ functions,} \\ 1, & \text{if the } j\text{th unit of component } i \text{ fails.} \end{cases}$$

The selection of $M_1, M_2, M_3$, and $M_4$ will be determined so that the variance of $\hat{R}$ is minimized. Since $\text{Var}(\hat{R}) = E[\hat{R}^2] - E^2[\hat{R}]$ and $\hat{R} = (1 - \hat{q}_1\hat{q}_2)(1 - \hat{q}_3\hat{q}_4)$, so

$$\text{Var}(\hat{R}) = E[(1 - \hat{q}_1\hat{q}_2)^2(1 - \hat{q}_3\hat{q}_4)^2] - E^2[(1 - \hat{q}_1\hat{q}_2)(1 - \hat{q}_1\hat{q}_2)],$$

and after dropping higher order terms, then the variance can be approximated by

$$\begin{aligned}\text{Var}(\hat{R}) \quad \approx \quad & \frac{p_1q_1q_2^2(1 - q_3q_4)^2}{M_1} + \frac{p_2q_2q_1^2(1 - q_3q_4)^2}{M_2} + \frac{p_3q_3q_4^2(1 - q_1q_2)^2}{M_3} \\ & + \frac{p_4q_4q_3^2(1 - q_1q_2)^2}{M_4},\end{aligned}$$

where $p_i = 1 - q_i$. The following is the main result of this section.

**Theorem 2.1.** Let $T$ denote the total number of units to test. The variance of the optimal scheme is given by

$$\begin{aligned}\text{Var}(O) \quad = \quad & \frac{1}{T}\Big[q_2(1 - q_3q_4)\sqrt{p_1q_1} + q_1(1 - q_3q_4)\sqrt{p_2q_2} + q_4(1 - q_1q_2)\sqrt{p_3q_3} \\ & + q_3(1 - q_1q_2)\sqrt{p_4q_4}\Big]^2.\end{aligned}$$

**Proof**: Treating the $M_i$'s as continuous variables, then by differentiation with respect to $M_i$'s leads to a system of linear equations under the constraint $\sum_{i=1}^4 M_i = T$.

## 3. BATCH SEQUENTIAL SAMPLING SCHEME

The system of equations yields the following expressions which will be referred to as the optimal sampling scheme

$$M_i = T\frac{\sqrt{p_i/q_i}}{\sum_{j=1}^4 \sqrt{p_j/q_j}}, \quad i = 1, \cdots, 4.$$

However, since $p_i$'s ($i = 1, \cdots, 4$) are unknown, then the optimal sampling scheme is not practical. To overcome this problem, we propose a batch sequential sampling scheme that will mimic the optimal sampling scheme and will satisfy

$$\frac{M_i}{T} \to \frac{\sqrt{p_i/q_i}}{\sum_{j=1}^4 \sqrt{p_j/q_j}} \qquad \text{as } t \to \infty.$$

Let $T = \sum_{i=1}^4 M_i$. The tester starts with an initial sample of 4 units, and one unit from each component is tested. The remaining $T - 4$ units will be tested in $B$ batches, where each batch $B$ is composed of 3 units. The scheme is defined as follows:

1. Test 1 unit from each component.
2. If at any stage $k$ the tester has tested $M_{ik}$ units of component $i$, then he will test one unit from each component except the $i$th component if for all $j$:

$$M_{ik}\hat{C}_{jk} > M_{jk}\hat{C}_{ik},$$

   where

$$\hat{C}_{jk} = \sqrt{\hat{p}_{ik}/\hat{q}_{ik}},$$

   and $\hat{p}_{ik}$ is the MLE of $p_i$ based on $M_{ik}$ units of component $i$.

3. The test proceeds sequentially until all $T$ units are tested.

The following is the main result of this section.

**Theorem 3.1.** The excess variance incurred by the batch sequential sampling scheme over the variance incurred by the optimal sampling scheme is in the order of $\frac{1}{T}$.
**Proof**: The proof of the theorem will follow if we establish

$$\frac{M_i}{M_j} \to \frac{\sqrt{p_i/q_i}}{\sqrt{p_j/q_j}} \quad \text{as } t \to \infty.$$

Let $B = \frac{[T-4]}{3}$ where $[x]$ represents the integer part of $x$. Then $B \to \infty$ as $T \to \infty$.

Let $b^{(i)} = \sup\{k \leq b : \frac{M_{ik}}{\hat{C}_{ik}} > \sup_{j \neq i} \frac{M_{jk}}{\hat{C}_{jk}}\}$.

It is clear that $b^{(i)} \to \infty$ as $b \to \infty$.

The adaptive rule can be rewritten as: Test one unit from each component excluding the $i$th component if

$$M_{ib}\frac{\hat{C}_{jb}}{\hat{C}_{ib}+\hat{C}_{jb}} > M_{jb}\frac{\hat{C}_{ib}}{\hat{C}_{ib}+\hat{C}_{jb}}$$

for all $j \neq i$. Suppose that $M_{ib}$ is bounded, then $M_{jb}$ is also bounded and, therefore, $b$ is bounded, which is a contradiction since $b \to \infty$ as $T \to \infty$. Consequently, $M_{ib}$ goes to infinity as $T$ goes to infinity.

By definition of $b^{(i)}$ and an argument similar to the one used in Rekab (1993), the proof follows.

## 4. MONTE CARLO SIMULATION

In this section, we use Monte Carlo simulation with 5,000 replications to compare the variance incurred by the batch sequential sampling scheme with the variance of the balanced sampling scheme and the variance of the optimal sampling scheme. For example, consider a system that has two subsystems connected in series with equal reliabilities $p_i = 0.95, (i = 1, \cdots, 4)$, and suppose that a total of 100 units will be tested. Balanced allocation would test 25 units of each component and gives a ratio of the variance of the optimal scheme over the variance of the balanced scheme equal to 1. However, the ratio of the variance of the optimal scheme over the variance of the batch scheme is 0.998245; see Table 4.1 below.

Table 4.1. Comparison of the batch sequential with optimal and balanced for equal reliabilities ($T = 100$ and 5,000 replications).

| $p_1, p_2, p_3, p_4$ | Var(O)/Var(BS) | Var(O)/Var(BA) |
|---|---|---|
| 0.10, 0.10, 0.10, 0.10 | 0.992642 | 1 |
| 0.30, 0.30, 0.30, 0.30 | 0.931834 | 1 |
| 0.50, 0.50, 0.50, 0.50 | 0.905024 | 1 |
| 0.70, 0.70, 0.70, 0.70 | 0.954846 | 1 |
| 0.90, 0.90, 0.90, 0.90 | 0.996248 | 1 |
| 0.95, 0.95, 0.95, 0.95 | 0.998245 | 1 |

Consider now a system that has two subsystems connected in series with unequal reliabilities $p_1 = 0.70$, $p_2 = 0.80$, $p_3 = 0.80$, $p_4 = 0.90$, and suppose that a total of 100 units will be tested. Balanced allocation would test 25 of each component and gives a ratio of the variance of the optimal scheme over the variance of the balanced scheme equal to 0.857368. However, the ratio of the variance of the optimal scheme over the variance of the batch scheme is 0.905682; see Table 4.2 below.

Table 4.2. Comparison of the batch sequential with optimal and balanced for different reliabilities ($T = 100$ and 5,000 replications)

| $p_1, p_2, p_3, p_4$ | Var(O)/Var(BS) | Var(O)/Var(BA) |
|---|---|---|
| 0.10, 0.20, 0.30, 0.30 | 0.962931 | 0.909532 |
| 0.10, 0.20, 0.30, 0.40 | 0.938457 | 0.868681 |
| 0.30, 0.40, 0.40, 0.50 | 0.967158 | 0.965328 |
| 0.60, 0.60, 0.70, 0.80 | 0.923695 | 0.888417 |
| 0.60, 0.60, 0.70, 0.80 | 0.944807 | 0.928482 |
| 0.60, 0.70, 0.80, 0.90 | 0.851184 | 0.742754 |
| 0.70, 0.80, 0.80, 0.90 | 0.905682 | 0.857368 |
| 0.70, 0.80, 0.90, 0.95 | 0.676451 | 0.648021 |
| 0.90, 0.90, 0.95, 0.95 | 0.845747 | 0.819044 |

Tables 4.1 and 4.2 also indicate that the variance incurred by our sampling scheme is always less than the variance of the balanced sampling scheme. They also indicate that the variance of the batch sequential scheme is very close to the variance of the optimal sampling scheme.

## 5. CONCLUSION

We have described in this paper a batch sequential sampling scheme to estimate the reliability of a series/parallel system. Monte Carlo simulation is conducted to verify the theoretical results obtained that the introduced scheme performs better than the balanced sampling scheme and is asymptotically optimal. A possible extension of this paper would consider $N$ subsystems connected in series where each subsystem $n$ consists of $T_n$ components connected in parallel. It is to be noted that the overall reliability of this system is given by

$$R = \prod_{n=1}^{N} \left( 1 - \prod_{j=1}^{T_n} q_j \right).$$

Also, the variance of the estimate $\hat{R}$ of $R$ is given by

$$\mathrm{Var}(\hat{R}) = \sum_{n=1}^{N} S_n,$$

where

$$S_n = \prod_{\substack{\ell=1 \\ \ell \neq n}}^{N} \left( 1 - \prod_{j=1}^{T_\ell} q_{j\ell} \right)^2 \sum_{j=1}^{T_n} \left( \frac{p_{jn} q_{jn}}{M_{jn}} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{T_n} q_{\ell n}^2 \right).$$

The problem here is to determine $M_{jn}$, $j = 1, \cdots, T_n$ and $n = 1, \cdots, N$ under the constraint $\sum_{n=1}^{N} \sum_{j=1}^{T_n} M_{jn} = T$, where $T$ is the total fixed number of units.

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## REFERENCES

Enaya, T., Rekab, K., and Whittaker, A. (2006). A two-stage sampling scheme for estimating the product of means with application in reliability, *Engineering Simularion*, **28**.

Berry, D.A. (1977). Optimal sampling schemes for simulating system reliability by testing components I: Fixed sample size, *J. Amer. Stat. Assoc.* **69**, 485-491.

Hardwick, J.P. and Stout, F.Q. (1994). Optimal allocation for estimating the product of two means. *Computing Science and Statistics.* **24**, 592-596.

Page, A. (1995). Adaptive allocation for estimations. *IMS Lecture Notes.* **25**, 213-222.

Rekab, K. (1993). A sampling scheme for estimating the reliability of a series system. *IEEE Trans. on Reliability.* **42**, 287-290.