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GENERALIZED sg-OPEN SETS IN STRONG GENERALIZED NEIGHBORHOOD SPACES

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Abstract. We introduce the notions of sg-semiopen set and other kinds of generalized sg-open sets and we investigate some properties for such generalized sg-open sets.

1. Introduction

Császár introduced the notions of generalized topological spaces and generalized open sets [1]. In [3], the author studied some properties for generalized topologies. Also the author introduced the strong generalized neighborhood system which induces a strong generalized neighborhood space (briefly SGNS) and investigated topological properties for SGNS's [4, 5].

In this paper, we introduce the notions of generalized sg-open sets on SGNS; sg-semiopen set, sg-preopen set, sg-regular open set, sg-α open set and sg-β open set. And we investigate characterizations and some properties for such generalized sg-open sets.

2. Preliminaries

Let X be a nonempty set, $exp(X)$ the power set of X, and $\psi : X \to exp(exp(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* of $x \in X$ and ψ is called a *generalized neighborhood system* [1] on X.

Definition 2.1 ([4]). Let $\psi : X \to exp(exp(X))$. Then ψ is called a *strong generalized neighborhood system* on X if it satisfies the following:

(1) $x \in V$ for $V \in \psi(x)$;

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(2) for $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.

Then the pair (X, ψ) is called a *strong generalized neighborhood space* (briefly SGNS). Then $V \in \psi(x)$ is called a *strong generalized neighborhood* of $x \in X$.

Definition 2.2 ([4])**.** Let (X, ψ) be an SGNS and $G \subseteq X$. Then G is called an *sg_ψ-open* set if for each $x \in G$, there is $V \in \psi(x)$ such that $V \subseteq G$.

Let us denote $sg_\psi(X)$ the collection of all *sg_ψ-open* sets on an SGNS (X, ψ). We call the collection $sg_\psi(X)$ a strong generalized topology on X. The complements of *sg_ψ-open* sets are called *sg_ψ-closed* sets. The *sg_ψ-interior* of A (denoted by $i_\psi(A)$) is the union of all $G \subseteq A$, $G \in sg_\psi(X)$, and the *sg_ψ-closure* of A (denoted by $c_\psi(A)$) is the intersection of all sg_ψ-closed sets containing A.

Theorem 2.3 ([4])**.** *Let (X, ψ) be an SGNS.*

(1) *The empty set is an sg_ψ-open set;*

(2) *The intersection of two sg_ψ-open sets is an sg_ψ-open set;*

(3) *The arbitrary union of sg_ψ-open sets is an sg_ψ-open set.*

Theorem 2.4 ([4])**.** *Let (X, ψ) be an SGNS. We get the following things.*

(1) $i_\psi(A) \subseteq A \subseteq c_\psi(A)$ *for* $A \subseteq X$.

(2) $c_\psi(A \cup B) = c_\psi(A) \cup c_\psi(B)$ *and* $i_\psi(A \cap B) = i_\psi(A) \cap i_\psi(B)$ *for* $A, B \subseteq X$.

(3) $i_\psi(i_\psi(A)) = i_\psi(A)$ *and* $c_\psi(c_\psi(A)) = c_\psi(A)$ *for* $A \subseteq X$.

(4) $c_\psi(A) = X - i_\psi(X - A)$ *and* $i_\psi(A) = X - c_\psi(X - A)$ *for* $A \subseteq X$.

Let X be a nonempty set and $\tau \subseteq exp(X)$:

(1) τ is called a *generalized topology* [1] on X iff (i) $\emptyset \in \tau$ and (ii) $G_i \in \tau$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \tau$.

(2) τ is called a *strong generalized topology* [4] (or *quasi-topology* [2]) on X iff (i) $\emptyset \in \tau$, (ii) $G_1, G_2 \in \tau$ implies $G_1 \cap G_2 \in \tau$ and (iii) $G_i \in \tau$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \tau$.

(3) τ is called a *supratopology* [6] on X iff (i) $\emptyset, X \in \tau$ and (ii) $G_i \in \tau$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \tau$.

3. Main Results

Definition 3.1. Let (X, ψ) be an SGNS and $A \subseteq X$. Then A is said to be

(1) *sg-α open* if $A \subseteq i_\psi(c_\psi(i_\psi(A)))$,

(2) *sg-semiopen* if $A \subseteq c_\psi(i_\psi(A))$,

(3) *sg-preopen* if $A \subseteq i_\psi(c_\psi(A))$,

(4) *sg-regular open* if $A = i_\psi(c_\psi(A))$,

(5) *sg-β open* if $A \subseteq c_\psi(i_\psi(c_\psi(A)))$.

Denote the set of all sg-α open (resp., sg-semiopen, sg-preopen, sg-regular open, sg-β open) by $sg\alpha(X)$ (resp., $sgSO(X)$, $sgPO(X)$, $sgRO(X)$, $sg\beta(X)$).

The complement of sg-α open (resp., sg-semiopen, sg-preopen, sg-regular open, sg-β open) is called *sg-α closed* (resp., *sg-semiclosed*, *sg-preclosed, sg-regular closed, sg-β closed*)

sg-semiopen

↗ ↘

sg-regular open → sg-open → sg-α open $\qquad\qquad$ sg-β open

↘ ↗

sg-preopen

The converse implications are not always true:

Example 3.2. Let $X = \{a, b, c, d\}$. Consider a strong generalized topology $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ on X.

(1) Let $A = \{a, b\}$. Then A is an sg-open set but since $i_\psi(c_\psi(A)) = \{a, b, c\}$, A is not sg-regular open.

(2) Let $A = \{a, c\}$. Since $i_\psi(c_\psi(i_\psi(A))) = i_\psi(c_\psi(\{a\})) = i_\psi(X) = \{a, b, c\}$, A is an sg-α open set but not sg-open.

Consider a strong generalized topology $\tau_2 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ on X.

(1) Let $A = \{b\}$. Then $c_\psi(i_\psi(c_\psi(A))) = \{b, c, d\}$ and $c_\psi(i_\psi(A)) = \{d\}$, so A is an sg-β open set but not sg-semiopen.

(2) Let $A = \{a, d\}$. Since $c_\psi(i_\psi(A)) = A$ and $i_\psi(c_\psi(i_\psi(A)) = \{a\}$, A is an sg-semiopen set but not sg-α open.

(3) Let $A = \{a, b\}$. Since $i_\psi(c_\psi(A)) = \{a, b, c\}$, A is an sg-preopen set but not sg-α open.

(4) Let $A = \{a, d\}$. From (2), A is an sg-β open set. But since $i_\psi(c_\psi(A)) = \{a\}$, A is not an sg-preopen set.

Theorem 3.3. *Let (X, ψ) be an SGNS and $A \subseteq X$.*

(1) *A is sg-α closed if and only if $c_\psi(i_\psi(c_\psi(A))) \subseteq A$,*

(2) *A is sg-semiclosed if and only if $i_\psi(c_\psi(A)) \subseteq A$,*

(3) *A is sg-preclosed if and only if $c_\psi(i_\psi(A)) \subseteq A$,*

(4) *A is sg-regular closed if and only if $A = c_\psi(i_\psi(A))$,*

(5) *A is g-β closed if and only if* $i_\psi(c_\psi(i_\psi(A))) \subseteq A$.

Proof. Straightforward. □

Lemma 3.4. *Let* (X, ψ) *be an SGNS and* $A \subseteq X$.

(1) $c_\psi(i_\psi(c_\psi(i_\psi(A)))) = c_\psi(i_\psi(A))$.

(2) $i_\psi(c_\psi(i_\psi(c_\psi(A)))) = i_\psi(c_\psi(A))$.

Proof. (1) $i_\psi(A) \subseteq c_\psi(i_\psi(A))$ implies $c_\psi(i_\psi(i_\psi(A))) \subseteq c_\psi(i_\psi(c_\psi(i_\psi(A))))$. By Theorem 2.4, we have $c_\psi(i_\psi(A)) \subseteq c_\psi(i_\psi(c_\psi(i_\psi(A))))$. Similarly, the other inclusion is obtained.

(2) It is similar to (1). □

Theorem 3.5. *Let* (X, ψ) *be an SGNS. Then* $sgSO(X)$ *and* $sg\beta(X)$ *are supratopologies on* X.

Proof. (i) Clearly $\emptyset \in sgSO(X)$.

(ii) Suppose that $i_\psi(X) \subseteq c_\psi(i_\psi(X)) = F \neq X$. Then $\emptyset \neq X - F$ and $X - F$ is sg-open. Since $i_\psi(X)$ is the largest sg-open set in X, it has to contain $X - F$. So it is impossible that F contains $i_\psi(X)$: a contradiction. Hence $X = c_\psi(i_\psi(X))$, that is, $X \in sgSO(X)$.

(iii) Let $G_i \in sgSO(X)$ for $i \in I \neq \emptyset$. From Theorem 2.4, $i_\psi(G_i) \subseteq i_\psi(\cup_{i\in I} G_i)$. This implies $c_\psi(i_\psi(G_i)) \subseteq c_\psi(i_\psi(\cup_{i\in I} G_i))$. Since G_i is an sg-open set for $i \in I$, $\cup_{i\in I} G_i \subseteq c_\psi(i_\psi(\cup_{i\in I} G_i))$. Hence $\cup_{i\in I} G_i \in sgSO(X)$.

Similarly, $sg\beta(X)$ is also a supratopology on X. □

Example 3.6. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ a strong generalized topology on X. Then $c_\psi(i_\psi(X)) = X$ and $i_\psi(c_\psi(i_\psi(X))) = \{a, b, c\}$, so X is neither $sg\alpha$ open nor sg-preopen.

Theorem 3.7. *Let* (X, ψ) *be an SGNS. Then* $sg\alpha(X)$ *is a strong generalized topology on* X.

Proof. Clearly $\emptyset \in sg\alpha(X)$. Since every $sg\alpha$-open set is sg-semiopen, the arbitrary union of $sg\alpha$ open sets is $sg\alpha$ open.

Now let $U, V \in sg\alpha(X)$. Then from Theorem 2.4, it follows $U \cap V \subseteq i_\psi(c_\psi(i_\psi(U))) \cap i_\psi(c_\psi(i_\psi(V))) \subseteq c_\psi(i_\psi(U)) \cap i_\psi(c_\psi(i_\psi(V))) \subseteq c_\psi(i_\psi(U) \cap i_\psi(c_\psi(i_\psi(V)))) \subseteq c_\psi(i_\psi(U) \cap c_\psi(i_\psi(V))) \subseteq c_\psi(c_\psi(i_\psi(U) \cap i_\psi(V))) = c_\psi(i_\psi(U) \cap i_\psi(V)) = c_\psi(i_\psi(U \cap V))$. From Theorem 2.3, it follows $A \cap B \subseteq i_\psi(c_\psi(i_\psi(U \cap V)))$. Hence $U \cap V \in sg\alpha(X)$.

□

Theorem 3.8. *Let (X, ψ) be an SGNS. Then $sgPO(X)$ is a generalized topology on X.*

Proof. Clearly $\emptyset \in sgPO(X)$. Let $G_i \in sgPO(X)$ for $i \in I \neq \emptyset$. From Theorem 2.4, $G_i \subseteq c_\psi(i_\psi(G_i)) \subseteq i_\psi(c_\psi(\cup_{i\in I} G_i))$. Hence $\cup_{i\in I} G_i \in sgPO(X)$. □

Theorem 3.9. *Let (X, ψ) be an SGNS. Then $sg\alpha(X) = sgPO(X) \cap sgSO(X)$.*

Proof. Let A be a nonempty set and $A \in sg\alpha(X)$. Then $A \subseteq i_\psi(c_\psi(i_\psi(A)))$, so $A \subseteq i_\psi(c_\psi(A))$ and $A \subseteq c_\psi(i_\psi(A))$. Thus $A \in sgPO(X) \cap sgSO(X)$.

For the converse, let $A \in sgPO(X) \cap sgSO(X)$. Then $A \subseteq i_\psi(c_\psi(A))$ and $A \subseteq c_\psi(i_\psi(A))$. So

$$A \subseteq i_\psi(c_\psi(A)) \subseteq i_\psi(c_\psi(c_\psi(i_\psi(A)))) \subseteq i_\psi(c_\psi(i_\psi(A))).$$

Thus $A \in sg\alpha(X)$. □

Theorem 3.10. *In an SGNS (X, ψ), set $\tau = \{G \subseteq X : G \cap U \in \mathcal{T}$ for all $U \in \mathcal{T}\}$ for $\mathcal{T} = sgPO(X), sgSO(X), sg\beta(X), sg\alpha(X)$. Then τ is a topology on X.*

Proof. In case $\mathcal{T} = sgPO(X)$, we show that τ is a topology on X. Clearly $\emptyset$, $X \in \tau$. Let $G_i \in \tau$ for $i \in J$. Then for all $U \in \mathcal{T}$, since $(G_i \cap U) \in \mathcal{T}$ and $\mathcal{T} = sgPO(X)$ is a general topology, $\cup G_i \cap U = \cup(G_i \cap U) \in \mathcal{T}$. Hence $\cup G_i \in \tau$. Finally, let $G_1, G_2 \in \tau$ and $U \in \mathcal{T}$; since $(G_2 \cap U) \in \mathcal{T}$, $(G_1 \cap G_2) \cap U = G_1 \cap (G_2 \cap U) \in \mathcal{T}$. So $G_1 \cap G_2 \in \tau$.

The other cases are similar to $\mathcal{T} = sgPO(X)$. □

References

1. Á. Császár: Generalized Topology, Generalized Continuity. *Acta Math. Hungar.* **96** (2002), 351-357.
2. ______: Remarks on quasi-topologies. *Acta Math. Hungar.* **119** (2008), no. 1-2, 197-200.
3. W.K. Min: Some Results on Generalized Topological Spaces and Generalized Systems. *Acta Math. Hungar.* **108** (2005), no. 1-2, 171-181.
4. ______: On strong generalized neighborhood systems and sg-open sets. *Commun. Korean Math. Soc.* **23** (2008), no. 1, 125-131
5. ______: Results on strong generalized neighborhood spaces. *J. Korea Soc. Math. Edu. Ser. B: Pure Appl. Math.* **15** (2008), no. 3, 221-227.
6. A.S. Mashhourr, A.A. Allam, F.S. Mahmoud & F.H. Khadr: On supratopological spaces. *Indian J. Pure Appl. Math.* **14** (1983), no. 3, 502-510.

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