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A Discussion to Dimensions of Spline Spaces Over Unconstricted Triangulations

Na Yi[1] and Huan-Wen Liu[2]

[1]Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China

[2]School of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R. China

Abstract - Let $S_n^r(\Omega)$ be the spline space of degree n and smoothness r with respect to Ω where Ω is a triangulation of a planner polygonal domain. Dimensions of $S_n^r(\Omega)$ over the so-called unconstricted triangulation were given by Farin in [J. Comput. Appl. Math. 192(2006), 320-327]. In this paper, a counter example is given to show that the condition used in the main result in Farins paper is not correct, and then an improved necessary and sufficient condition is presented.

***Keywords*: Bezier-net technique, Unconstricted triangulation, Bivariate spline space, Minimaldetermining set, Dimension.**

1. A counter example

By introducing two kinds of construction operations, called a flap and a pair of triangles respectively, the so-called unconstricted triangulation was first defined in [1], which can be obtained by recursively adding a flap or a pair of triangles to a subtriangulation started from a single triangle. In Section 3 in [1], the dimension of $S_3^1(\Omega)$ over the unconstricted triangulation was given. Then in Section 4, the construction of a minimal determining set for the spline space $S_n^r(\mathbf{v}^*)$ over a star $\mathbf{v}^*$ was further considered. And finally in Section 5, the dimension of the spline space $S_n^r(\Omega)$ over the unconstricted triangulation Ω was determined by recursively using the results over stars presented in Section 4.

Given a star $\mathbf{v}^*$ which is obtained from $\mathbf{v}^{*-2}$ by adding a pair of triangles, $\delta_n^r(b)$ was defined in [1] as

$$\delta_n^r(b)=\dim S_n^r(\mathbf{v}^*)-\dim S_n^r(\mathbf{v}^{*-2}) \tag{1}$$

where b is the valence of an interior vertex $\mathbf{v}$.

The key step in the proof of the theorem in Section 5 in [1] is based on the statement "if $\delta_n^r(b)\geq 0$ then a minimal determining set for $S_n^r(\mathbf{v}^*)$ can be obtained by adding some other Bézier ordinates to the minimal determining set for $S_n^r(\mathbf{v}^{*-2})$". However, it is found in this section that this statement is not always true. The following is a counter example.

Let us take $b=5$, $n=5$ and $r=2$, and let $\mathbf{v}^{*-2}=\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_2\cup\Delta\mathbf{v}_2\mathbf{v}\mathbf{v}_3\cup\Delta\mathbf{v}_3\mathbf{v}\mathbf{v}_4$ with $\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_3\in(\frac{\pi}{2},\pi)$. The star $\mathbf{v}^*$ is obtained by adding a pair of triangles $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_5\cup\mathbf{v}_5\mathbf{v}\mathbf{v}_4$ to $\mathbf{v}^{*-2}$, where $\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4\in(0,\frac{\pi}{2})$ and $\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5\in(0,\frac{\pi}{2})$, as shown in Fig. 1.

*Corresponding author:
Tel: +86-771-3264839
Fax: +86-771-3260186
E-mail: mengtian29@163.com

Firstly we consider the spline space $S_5^2(\mathbf{v}^{*-2})$. It follows from [4, 5] that dim $S_5^2(\mathbf{v}^{*-2})=33$, and a minimal determining set for $S_5^2(\mathbf{v}^{*-2})$ can be easily chosen as the Bézier ordinates with respect to the domain points marked by "•" as shown in Fig. 1(a), which is denoted by $P_5^2(\mathbf{v}^{*-2})$.

We now consider the spline space $S_5^2(\mathbf{v}^*)$. It follows from [3] or [4] that dim $S_5^2(\mathbf{v}^*)=36$. Thus

$$\delta_5^2(5)=\dim S_5^2(\mathrm{V}^*)-\dim S_5^2(\mathrm{V}^{*-2})=36-33=3>0\,. \tag{2}$$

However, for this case, we can show that any minimal determining set for $S_5^2(\mathbf{v}^*)$, denoted by $P_5^2(\mathbf{v}^*)$, cannot be obtained by adding some Bézier ordinates to the minimal determining set $P_5^2(\mathbf{v}^{*-2})$.

In fact, for the space $S_5^2(\mathbf{v}^*)$, let us consider C^2 smoothness conditions in $D_3(\mathbf{v})$, where $D_3(\mathbf{v})$ is the third disk around the vertex $\mathbf{v}$, as shown in Fig. 1(b). Let

$$\mathbf{v}_4=\alpha_1\mathbf{v}+\beta_1\mathbf{v}_1+\gamma_1\mathbf{v}_5, \tag{3}$$

$$\mathbf{v}_5=\alpha_2\mathbf{v}+\beta_2\mathbf{v}_3+\gamma_2\mathbf{v}_4, \tag{4}$$

and $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$ denote the corresponding Bézier ordinates of $s\in S_5^2(\mathbf{v}^*)$ with respect to four domain points A, B, C and D, respectively. If we assume that all the Bézier ordinates with respect to all domain points marked by "•" in $D_3(\mathbf{v})$ in Fig. 1(b) vanish, then it follows from C^2 smoothness conditions [2] that

$$\begin{pmatrix} \gamma_1 & -1 & 0 & 0 \\ \gamma_1^2 & 0 & -1 & 0 \\ 0 & 0 & -1 & \gamma_2 \\ 0 & -1 & 0 & \gamma_2^2 \end{pmatrix}\begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \\ \bar{d} \end{pmatrix}=0 \tag{5}$$

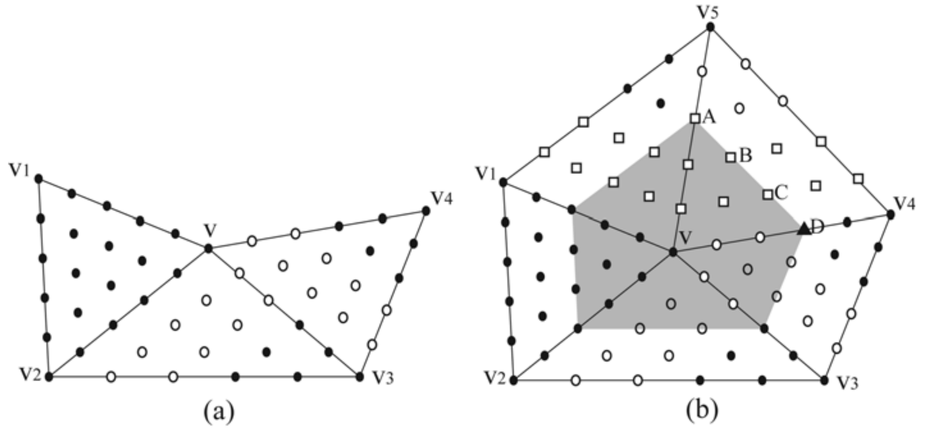

Fig. 1. (a) A minimal determining set P_5^2 ($\mathbf{v}^{*-2}$) for the space S_5^2 ($\mathbf{v}^{*-2}$), (b) A minimal determining set P_5^2 ($\mathbf{v}^*$) for the space S_5^2 ($\mathbf{v}^*$).

Because of the assumption in [1] that the triangulation does not contain any degenerated (or called singular) edge, we have $\gamma_1 = \frac{S_{\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_4}}{S_{\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_5}} \neq 0, \gamma_2 = \frac{S_{\Delta\mathbf{v}_3\mathbf{v}\mathbf{v}_5}}{S_{\Delta\mathbf{v}_3\mathbf{v}\mathbf{v}_4}} \neq 0$. In addition,

$$\gamma_1\gamma_2 = \frac{S_{\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_4}}{S_{\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_5}} \cdot \frac{S_{\Delta\mathbf{v}_3\mathbf{v}\mathbf{v}_5}}{S_{\Delta\mathbf{v}_3\mathbf{v}\mathbf{v}_4}} = \frac{\sin\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_4}{\sin\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5} \cdot \frac{\sin\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_5}{\sin\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4}$$

$$= \frac{\sin\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_4}{\sin\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4} \cdot \frac{\sin\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_5}{\sin\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5} \tag{6}$$

It is noted that $\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_3 \in (\frac{\pi}{2}, \pi)$, so $\pi < \angle\mathbf{v}_1\mathbf{v}\mathbf{v}_4 + \angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4 = \angle\mathbf{v}_3\mathbf{v}\mathbf{v}_5 + \angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5 < \frac{3\pi}{2}$, which together with the assumptions $\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5 \in (0, \frac{\pi}{2})$, and $\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4 \in (0, \frac{\pi}{2})$ yield that

$$\frac{\pi}{2} < \pi - \angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4 < \angle\mathbf{v}_1\mathbf{v}\mathbf{v}_4 < \pi, \tag{7}$$

$$\frac{\pi}{2} < \pi - \angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5 < \angle\mathbf{v}_3\mathbf{v}\mathbf{v}_5 < \pi. \tag{8}$$

Since the function $\sin x$ decreases monotonously in the interval $(\frac{\pi}{2}, \pi)$, we have

$$\sin\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_4 < \sin(\pi - \angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4) = \sin\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_4, \tag{9}$$

$$\sin\angle\mathbf{v}_3\mathbf{v}\mathbf{v}_5 < \sin(\pi - \angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5) = \sin\angle\mathbf{v}_1\mathbf{v}\mathbf{v}_5. \tag{10}$$

So we have $\gamma_1\gamma_2 < 1$, and thus

$$det\begin{pmatrix} \gamma_1 & -1 & 0 & 0 \\ \gamma_1^2 & 0 & -1 & 0 \\ 0 & 0 & -1 & \gamma_2 \\ 0 & -1 & 0 & \gamma_2^2 \end{pmatrix} = \gamma_1\gamma_2(1-\gamma_1\gamma_2) \neq 0 \tag{11}$$

This means that $\bar{a} = \bar{b} = \bar{c} = \bar{d} = 0$. Therefore the Bézier ordinate with respect to domain point D marked by "▲" must be excluded from the minimal determining set $P_5^2(\mathbf{v}^*)$ for $S_5^2(\mathbf{v}^*)$ though it is in $P_5^2(\mathbf{v}^{*-2})$. A correct minimal determining set for $S_5^2(\mathbf{v}^*)$ with respect to domain points marked by "●" is displayed in Fig. 1(b).

The counter example reveals that the condition $\delta_n^r(b)$ 0 is not a sufficient condition to guarantee that there exists a minimal determining set for $S_n^r(\mathbf{v}^*)$ which can be obtained by adding some Bézier ordinates to a minimal determining set for $S_n^r(\mathbf{v}^{*-2})$. In the next section, we shall give a necessary and sufficient condition.

2. An improved necessary and sufficient condition

Let $P_n^r(\mathbf{v}^*)$ be a minimal determining set for $S_n^r(\mathbf{v}^*)$ which consists of Bézier ordinates with respect to domain points taken ring by ring from $R_0(\mathbf{v})$ to $R_n(\mathbf{v})$, where $R_i(\mathbf{v})$ is the i-th ring around the vertex $\mathbf{v}$ in the star $\mathbf{v}^*$. We have the following

Lemma 1. *Let $N_n^r(b, i) = |P_n^r(\mathbf{v}^*) \cap R_i(\mathbf{v})|$, $i = 0, 1, ..., n$, where $|\cdot|$ denotes the cardinality of the set. Then*

$$N_n^r(b,i) = \begin{cases} i+1 & 0 \le i \le r, \\ (i-r)b + (i+1-(i-r)e)_+, & r < i \le n, \end{cases} \tag{12}$$

where e is the number of edges with different slopes attached to the vertex $\mathbf{v}$.

Proof. For $0 \le i \le r$, it is well-known that

$$\dim S_i^r(\mathbf{v}^*) = \binom{i+2}{2}. \tag{13}$$

For $r < i \le n$, it follows from [3] or [4] that

$$\dim S_i^r(\mathbf{v}^*) = \binom{r+2}{2} + \binom{i-r+1}{2} b + \sum_{j=1}^{i-r} (r+j+1-je)_+. \tag{14}$$

Thus,

$$N_n^r(b,i)=\dim S_i^r(\mathbf{v}^*)-\dim S_{i-1}^r(\mathbf{v}^*)$$

$$=\begin{cases}1, & i=0,\\ i+1, & 1\le i\le r\\ b+(r+2-e)_+, & i=r+1,\\ (i-r)b+(i+1-(i-r)e)_+, & r+1<i\le n\end{cases}$$

$$=\begin{cases}i+1, & 0\le i\le r,\\ (i-r)b+(i+1-(i-r)e)_+, & r<1\le n,\end{cases} \tag{15}$$

where $\dim S_{-1}^r(\mathbf{v}^*)=0$. The proof of the lemma is completed.

Similarly, let $P_n^r(\mathbf{v}^{*-2})$ be a minimal determining set for $S_n^r(\mathbf{v}^{*-2})$ which consists of Bézier ordinates with respect to domain points taken ring by ring from $R_0'(\mathbf{v})$ to $R_n'(\mathbf{v})$, where $R_i'(\mathbf{v})$ is the i-th ring around the vertex $\mathbf{v}$ in the triangulation $\mathbf{v}^{*-2}$. We also have

Lemma 2. *Let* $\overline{N_n^r}(b,i)=|P_n^r(\mathbf{v}^{*-2})\cap R_i'(\mathbf{v})|$, $i=0,1,\ldots,n$. *Then*

$$\overline{N_n^r}(b,i)=\begin{cases}i+1, & 0\le i\le r,\\ i+1+(i-r)(b-3), & r<1\le n.\end{cases} \tag{16}$$

Proof. For $0\le i\le r$, it is well-known that

$$\dim S_i^r(\mathbf{v}^{*-2})=\binom{i+2}{2}. \tag{17}$$

For $r<i\le n$, it follows from the dimensional formula for cross-cut partition given by [4,5] that

$$\dim S_i^r(\mathbf{v}^{*-2})=\binom{i+2}{2}+\binom{i-r+1}{2}(b-3), \tag{18}$$

where $b-3$ is the number of the cross-cut edges. Thus we have

$$\overline{N_n^r}(b,i)=\dim S_i^r(\mathbf{v}^{*-2})-\dim S_{i-1}^r(\mathbf{v}^{*-2})$$

$$=\begin{cases}1, & i=0\\ i+1, & 1\le i\le r\\ r+b-1, & i=r+1,\\ i+1+(i-r)(b-3), & r+1<i\le n,\end{cases}$$

$$=\begin{cases}i+1, & 0\le i\le r,\\ i+1+(i-r)(b-3) & r<i\le n,\end{cases} \tag{19}$$

where $\dim S_{-1}^r(\mathbf{v}^{*-2})=0$. The proof of the lemma is completed.

Based on Lemma 1 and 2, we have the following

Theorem. *Let Ω be an unconstricted triangulation with nonsingular vertices, that is, triangulations without vertices with collinear edges emanating from them. And let* $A_n^r(b,i)=N_n^r(b,i)-\overline{N_n^r}(b,i)-i=0,1,\ldots,n$. *A necessary and sufficient condition for existing a pair of minimal determining sets* $P_n^r(\mathbf{v}^{*-2})$ *for* $S_n^r(\mathbf{v}^{*-2})$ *and* $P_n^r(\mathbf{v}^*)$ *for* $S_n^r(\mathbf{v}^*)$ *to satisfy* $P_n^r(\mathbf{v}^{*-2})\subseteq P_n^r(\mathbf{v}^*)$ *is*

$$A_n^r(b,i)=\begin{cases}0, & 0\le i\le r,\\ 2i-3r-1+(i+1-(i-r)e)_+>0, & r<i\le n.\end{cases} \tag{20}$$

Proof. Suppose that a pair of minimal determining sets $P_n^r(\mathbf{v}^{*-2})$ for $S_n^r(\mathbf{v}^{*-2})$ and $P_n^r(\mathbf{v}^*)$ for $S_n^r(\mathbf{v}^*)$ exist to satisfy $P_n^r(\mathbf{v}^{*-2})\subseteq P_n^r(\mathbf{v}^*)$. Then

$$N_n^T(b,i)\ge\overline{N}_n^r(b,i),\quad i=0,1,\ldots,n,$$

i.e.,

$$A_n^r(b,i)\ge 0,\quad i=0,1,\ldots n,$$

Thus, the inequality (20) holds.

Suppose that the inequality (20) holds. If we take $i=r+1$, the inequality (20) becomes

$$A_n^r(b,r+1)=1-r+(r+2-e)_+\ge 0.$$

Specifically,

$$1-r\ge 0,\quad \text{when } r+2-e\le 0,$$

or

$$3-e\ge 0,\quad \text{when } r+2-e\ge 0.$$

That is to say, if we suppose that the inequality (20) holds, then there exist at most five possibilities as follows

Case 1) $r=0$, when $e=2$,
Case 2) $r=0$, when $e\ge 3$,
Case 3) $r=1$, when $e\ge 3$,
Case 4) $e=3$, when $r\ge 1$,
Case 5) $e=2$, when $r\ge 0$.

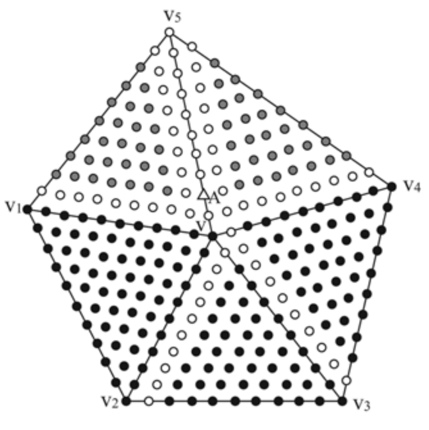

Fig. 2. The star $\mathbf{v}^*$ for Case 2).

It is noted that Ω does not contain any degenerated edge, both Case 1) and Case 5) can be discarded. In addition, the proof for Case 2) is trivial, we only consider Case 3) and Case 4).

For Case 3), if $n=r=1$, then both spline spaces $S_n^r(\mathbf{v}^{*-2})$ and $S_n^r(\mathbf{v}^*)$ are degenerated into the polynomial space P_1, thus the minimal determining set $P_n^r(\mathbf{v}^*)$ can take to be the same to $P_n^r(\mathbf{v}^{*-2})$, the conclusion holds.

If $n > r = 1$, as shown in Fig. 2, when we add a pair of triangles $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_5 \cup \Delta\mathbf{v}_5\mathbf{v}\mathbf{v}_4$ to $\mathbf{v}^{*-2}$, by using the C^1 smoothness conditions along the two edges $\mathbf{v}_1\mathbf{v}$ and $\mathbf{v}\mathbf{v}_4$, all the B$\acute{e}$zier ordinates associated with domain points in $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_5 \cup \Delta\mathbf{v}_5\mathbf{v}\mathbf{v}_4$ with distance to two edges $\mathbf{v}_1\mathbf{v}$ and $\mathbf{v}\mathbf{v}_4$ being 1 can be determined by $P_n^r(\mathbf{v}^{*-2})$. Further, since there is no degenerated edge in the triangulation Ω, the B$\acute{e}$zier ordinate associated with domain point A can be also determined by $P_n^r(\mathbf{v}^{*-2})$. Next, by using the C^1 smoothness conditions along the edge $\mathbf{v}\mathbf{v}_5$, we can obtain other $n-2$ equations. Therefore the total number of independent B$\acute{e}$zier ordinates in $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_5 \cup \Delta\mathbf{v}_5\mathbf{v}\mathbf{v}_4$ is at least

$$((n-1)^2-1)-(n-2)=(n-1)(n-2), \tag{21}$$

which is nonnegative as $n \geq r = 1$. So we can construct a minimal determining set $P_n^r(\mathbf{v}^*)$ for $S_n^r(\mathbf{v}^*)$ which contains $P_n^r(\mathbf{v}^{*-2})$ as its subset, i.e., $P_n^r(\mathbf{v}^{*-2}) \subseteq P_n^r(\mathbf{v}^*)$.

We now consider Case 4). Since the unconstricted triangulation Ω does not contain any vertex with collinear edges emanating from it, it follows from $e=3$ that $b=3$, i.e., $b=3$ and $r \geq 1$.

In this case, the star $\mathbf{v}^*$ is formed by adding a pair of triangles $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_3 \cup \Delta\mathbf{v}_3\mathbf{v}\mathbf{v}_2$ to $\mathbf{v}^{*-2} = \Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_2$, see Fig. 3. On one hand, a minimal determining set $P_n^r(\mathbf{v}^{*-2})$ for $S_n^r(\mathbf{v}^{*-2})$ can be taken all the B$\acute{e}$zier ordinates associated with all ($\binom{n+2}{2}$) domain points in $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_2$.

On the other hand, we have from [4, 5] that

$$\dim \boldsymbol{S}_n^r(\mathbf{v}^*) = \binom{n+2}{2} + d_n^r(3), \tag{22}$$

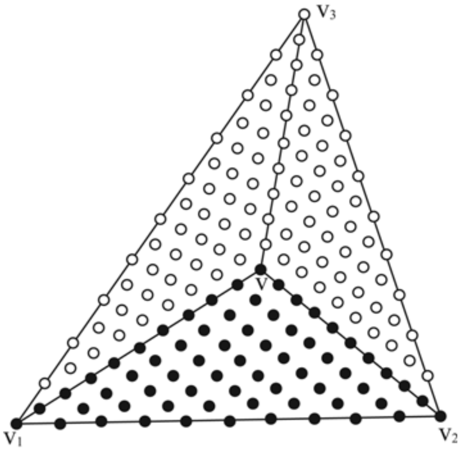

Fig. 3. The star $\mathbf{v}^*$ for $n=10$, $r=5$, $e=3$ and $b=3$.

where

$$d_n^r(3) = \left(n-r-\left[\frac{r+1}{2}\right]\right)_+ \left(n-2r-\left[\frac{r+1}{2}\right]\right)_+ \geq 0$$

is the dimension of the solution space of the system consisting from the conformality conditions around the vertex $\mathbf{v}$ with $[x]$ denoting the maximal integer which is not greater than x, and $\binom{n+2}{2}$ is the dimension contributed by the bivariate polynomial of degree n in the source triangle $\Delta\mathbf{v}_1\mathbf{v}\mathbf{v}_2$ which is exactly the cardinality of the minimal determining set $P_n^r(\mathbf{v}^{*-2})$. Hencea minimal determining set $P_n^r(\mathbf{v}^*)$ for $S_n^r(\mathbf{v}^*)$ can be constructed by adding $d_n^r(3)$ independent B$\acute{e}$zier ordinates to $P_n^r(\mathbf{v}^{*-2})$, i.e., $P_n^r(\mathbf{v}^{*-2}) \subseteq P_n^r(\mathbf{v}^*)$.

We thus finish the proof of the theorem.

Further, for $0 \leq m \leq n$, let us introduce $\delta_m^r(b)$ – dim $S_m^r(\mathbf{v}^*)$ – dim $S_m^r(\mathbf{v}^{*-2})$, then

$$\begin{aligned}\delta_m^r(b) &= \sum_{i=0}^{m}(\dim S_i^r(\mathbf{v}^*) - \dim S_{i-1}^r(\mathbf{v}^*)) \\ &\quad - \sum_{k=0}^{m}(\dim S_i^r(\mathbf{v}^{*-2}) - \dim S_{i-1}^r(\mathbf{v}^{*-2})) \\ &= \sum_{i=0}^{m} N_n^r(b,i) - \sum_{i=0}^{m} \overline{N}_n^r(b,i) \\ &= \begin{cases} 0, & m \leq r, \\ \sum_{i=r+1}^{m} A_n^r(b,i), & r < m \leq n. \end{cases}\end{aligned} \tag{23}$$

Therefore, another necessary and sufficient condition equivalent to Eq.(20) is

$$\delta_m^r(b) = \begin{cases} 0, & 0 \leq m \leq r, \\ \geq 0, & r < m \leq n. \end{cases} \tag{24}$$

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Na Yi received her BSc degree at Ocean University of China in 2002 and MSc degree at Guangxi University for Nationalities in China in 2008, respectively. She is now a PhD student at Shanghai University. Her research interests include bivariate spline functions and computational molecular biology.

Huan-Wen Liu received his MSc degree at Xiangtan University in China in 1988 and PhD degree at University of Wollongong in Australia in 2001, respectively. He is now a professor in Guangxi University for Nationalities. His research interests include bivariate spline functions and the propagation of ocean waves.

Na Yi

Huan-Wen Liu