한국해양환경공학회지
Journal of the Korean Society
for Marine Environmental Engineering
Vol. 13, No. 3. pp. 174-180, August 2010

비균질 Helmholtz 방정식을 이용한 변동 수심에서의 파랑변형

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Inhomogeneous Helmholtz equation for Water Waves on Variable Depth

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요 약

변동 수심에서의 파랑변형을 비균질 Helmholtz 방정식을 이용하여 계산하였다. 포텐셜 함수가 존재한다고 가정하였으며, 변수분리를 적용하였다. 본 논문에서는 조화파만을 고려하였다. 포텐셜 함수로 구성된 지배방정식을 정수면에 직접 적용하였고, 변동 수심에 대한 비균질 Helmholtz 방정식을 얻었다. 파랑의 진폭과 위상차로 얻어진 복합 포텐셜 함수의 지배방정식을 실수형 변수로 된 두 방정식으로 분리하였다. 분리된 방정식들은 각각 1차와 2차 상미분 방정식이며, 이 방정식들을 단순한 형태의 중앙차분 수치기법을 이용하여 차분식으로 변형하였다. 측면 경계조건에서의 파랑의 진폭, 진폭경사, 그리고 위상경사를 경계면에 적용하여 전방진행방법으로 전 영역에서 해를 구하였다. Booij의 경사면 있는 저면의 경우와 Bragg의 물결모양이 있는 저면의 경우에 적용하였다. 본 연구로 도출된 비균질 Helmholtz 방정식은 완전 선형방정식 계산 결과, Massel의 수정 완경사 방정식, 그리고 Berkhoff의 완경사 방정식의 적용 결과와 비교하였으며, 만족스러운 결과를 얻었다.

Abstract − The inhomogeneous Helmholtz equation is introduced for variable water depth and potential function and separation of variables are introduced for the derivation. Only harmonic wave motions are considered. The governing equation composed of the potential function for irrotational flow is directly applied to the still water level, and the inhomogeneous Helmholtz equation for variable water depth is obtained. By introducing the wave amplitude and wave phase gradient the governing equation with complex potential function is transformed into two equations of real variables. The transformed equations are the first and second-order ordinary differential equations, respectively, and can be solved in a forward marching manner when proper boundary values are supplied, i.e. the wave amplitude, the wave amplitude gradient, and the wave phase gradient at a side boundary. Simple spatially-centered finite difference numerical schemes are adopted to solve the present set of equations. The equation set is applied to two test cases, Booij's inclined plane slope profile, and Bragg's wavy bed profile. The present equations set is satisfactorily verified against other theories including the full linear equation, Massel's modified mild-slope equation, and Berkhoff's mild-slope equation etc.

Keywords: inhomogeneous Helmholtz equation(비균질 Helmholtz 방정식), variable water depth(변동 수심), separation of variables(변수 분리), complex potential function(복합 포텐셜 함수), spatially-centered finite difference numerical scheme(중앙차분 수치기법)

1. INTRODUCTION

Wave equations for harmonic waves with potential function are distinguished by applicability on variable water depth. The inhomogeneous Helmholtz equation has been used for solving geological wave propagation in inhomogeneous media, see Manolis and Shaw (1997). Hsiao *et al.* (1998) also simplified the mild-slope equation to the inhomogeneous Helmholtz equation, and treated the variable water depth as modification of

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constant water depth by introducing a perturbation method. The inhomogeneous Helmholtz equation has not often been used for wave transformation over variable water depth, while the homogeneous Helmholtz equation has been widely used for description of wave transformation over uniform depth since it was proposed by Helmholtz.

Wave transformation over sloped sea bed has been described by the mild-slope equation. The "mild slope" has been defined by a slope smaller than 1/3. The mild-slope equation was developed from either the continuity equation or the principle of stationary action with the variational principle.

The starting point of both the Helmholtz equation and the mild-slope equation is the same. Velocity potential function, Φ, is used to describe the irrotational wave motion. The continuity of mass flow in the x-z domain is the Laplace equation:

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial z^2}=0 \tag{1}$$

where x is the horizontal coordinate, and z is the upward vertical coordinate, the origin of which is the still sea level. The continuity of mass flow should be satisfied at every point in the computational domain at every instant. Then, the integrated continuity equation will also be satisfied at every section in the computational domain at every instant.

At free surface boundary nonlinear terms of the momentum equation are ignored, and the following condition in a linear form is applied:

$$\frac{\partial \Phi}{\partial z}=-g\frac{\partial^2 \Phi}{\partial t^2} \tag{2}$$

where t is time, g is the acceleration due to gravity. At the bed the following zero fluid flux condition is applied:

$$\frac{\partial \Phi}{\partial z}=-\frac{dh}{dx}\frac{\partial \Phi}{\partial x} \tag{3}$$

where h is the water depth relative to the still water level, and varies along x. Considering harmonic motions only, variables are assumed to be separated as:

$$\Phi = Re(Z\phi\Omega) \tag{4}$$

where complex ϕ is dependent on x only, and complex Ω is dependent on time only as:

$$\Omega = \exp(-iwt) \tag{5}$$

where $i=\sqrt{-1}$, and is the wave angular velocity, and the function Z is assumed to have the following form:

$$Z=\frac{\cosh k(z+h)}{\cosh kh} \tag{6}$$

Then, the bed boundary condition becomes:

$$\frac{\partial \phi}{\partial z}=\frac{dh}{dx}\frac{\partial (Z\phi)}{\partial x} \tag{7}$$

and

$$\int_{-h}^{0} Zdz=\frac{k_0}{k^2} \tag{8}$$

where k_0 is the wave number at deep sea ($=w^2/g$). Then, the free surface boundary condition, Eq. (9), produces the following dispersion relationship:

$$\frac{k_0}{k}=\tanh kh \tag{9}$$

where both k and h are dependent on x, and Z is dependent on both x and z.

If the water depth is uniform, dh/dx is zero, and we obtain the homogeneous Helmholtz equation:

$$\frac{d^2\phi}{dx^2}+k^2\phi=0 \text{ for one-dimensional problems,}$$

$$\text{or } \nabla^2\phi+k^2\phi=0 \text{ for two-dimensional problems.} \tag{10}$$

where ∇ is the gradient vector in the and directions.

Berkhoff (1973) has derived the mild-slope equation for variable water depth problems. The integration of the equation of continuity equation multiplied by an arbitrary weight function at any selected section should be satisfied at every instant. Berkhoff chose a hyperbolic cosine function as the weight function in the vertical direction to take into account vertical distribution of wave energy flux, and integrated the equation in the same direction. Berkhoff then took the vertical integration process of the equation which is the multiplication of the continuity equation and the weight function, Z of Eq. (6), in the vertical direction, which is:

$$\int_{-h}^{0} Z\left(\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial z^2}\right)dz=0 \tag{11}$$

Berkhoff made use of Green's theorem in order to reflect the bed boundary condition in the middle of his derivation of the mild-slope equation. The mild-slope equation has also been proposed in different types of partial differential equation by Radder (1979) and Copeland (1985).

More recently the modified mild-slope equation was proposed by Massel (1993), and Chamberlain and Porter (1995). Two time-dependent forms of the modified mild-slope equation were presented by Suh *et al.* (1997) by using Green's theorem and the variational principle. Suh *et al.*'s equations are transformed into the modified mild-slope equation of Massel when the time-dependent term is replaced by time-invariant term. The modified mild-slope equation is reduced to the mild-slope equation when some higher-order terms of the modified mild-slope equation are turned off. The modified mild-slope equation reproduces more accurate reflection coefficients for Bragg's test cases than the mild-slope equation with the aid of additional higher-order terms. Kim *et al.* (2009) adopted another uniform weight function in the vertical direction as follows:

$$\int_{-h}^{0}\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right)dz = 0 \qquad (12)$$

However, the above equations, the mild-slope equation, modified mild-slope equation, and Kim *et al.*'s equation, have a common defect that they don't satisfy the bed boundary condition because they explain horizontally propagating mode only, and this discrepancy is passed over to the other vertical mode for perfect satisfaction of the bed boundary condition.

The continuity should be strictly satisfied in every fluid position within the computational domain including the still water level, because the wave equation is valid from the still water level to the bed level. If we pickup a level instead of integration of the continuity through the water depth, it corresponds to a case that a delta function is chosen as the weight function at the still water level.

We derive the inhomogeneous Helmholtz equation for variable water depth in Section 2, and the equation composed of complex potential function is transformed into two other equations composed of real wave amplitude and wave phase gradient function in Section 3. The system of equations is applied to two topographies for comparison with other theories in Section 4.

2. DERIVATION OF INHOMOGENEOUS HELMHOLTZ EQUATION FOR WATER WAVES

We apply Eq. (3) to the still water level ($z = 0$). Then,

$$\frac{\partial Z}{\partial x} = \frac{(k(z+h))'\sinh(k(z+h))\cosh(kh) - (kh)'\cosh(k(z+h))\sinh(kh)}{\cosh^2(kh)} \qquad (13)$$

$$\frac{\partial Z}{\partial x}(z=0) = 0 \qquad (14)$$

Similarly, we obtain the second derivative of the function Z at the still water level as follows:

$$\frac{\partial^2 Z}{\partial x^2}(z=0) = 0 \qquad (15)$$

The above results are obvious from the fact that is constant along the still water level, i.e. is always unity from its definition, Eq. (6), and its partial derivative and second partial derivative in the axis on the still water level are also zero. Then, we obtain the inhomogeneous Helmholtz equation with variable:

$$\frac{\partial^2\phi}{\partial x^2} + k^2\phi = 0 \qquad (16)$$

This equation has the same form as Eq. (10), but is avariable in this equation. Eq. (16) can be considered as an extreme view in which 100% of weight is concentrated on the free surface. Interestingly, Eq. (16) also quite closely reflects the flow characteristics of the short-wave transformation with the help of the consideration of the bathymetric change through the variable k in spite of its relatively simple form. Eq. (15) can be extended to a three-dimensional form by including the other horizontal coordinate, y, as:

$$\nabla^2\phi + k^2\phi = 0 \qquad (17)$$

Now Eq. (16) satisfies the governing equation, and the free surface boundary condition is satisfied by the dispersion relationship. Here we examine whether the bed boundary condition could be reflected in the governing equation, the inhomogeneous Helmholtz equation. When the hyperbolic cosine function of Eq. (6) is applied, the left side of the bed boundary condition, Eq. (7), becomes zero, which leads complete zero horizontal and vertical velocities at the bed. The right side of the bed boundary condition, Eq. (7), reads:

$$\frac{d\phi}{dx} = -k\tanh kh\frac{dh}{dx}\phi = g_1(k,h)\phi \qquad (18)$$

Replacing the second and first differential terms of the governing equation, Eq. (16), by the non-differential term of Eq. (18) repeatedly, we obtain:

$$g_2(k,h)\phi = 0 \qquad (19)$$

$$g_2 = \left[(kh)''\tanh kh + (kh)'^2\frac{1}{\cosh^2 kh}\right] + (kh)'\tanh kh^2 + k^2 \qquad (20)$$

where g_2 includes $k(x)$ and $h(x)$. Since Eq. (18) should always be satisfied, either ϕ or g_2 should be zero. Zero ϕ constitutes a trivial solution. The other equation, $g_2=0$, composes another relationship between k and h. This new relationship between k and h comes into conflict with the dispersion relationship between k and h derived from the free surface boundary condition. Therefore, we convey this mismatch of the mass conservation at the bed to evanescent modes instead of applying of this bed boundary condition to the horizontally propagating mode. We can also see that the mild-slope equation and the modified mild-slope equation have the same problem as the present equation. Trials have been attempted to incorporate evanescent modes in dealing with wave propagation problems over sloped beds, see Massel (1993). However the evanescent modes are not of main interest of this paper.

As far as the assumptions of cosine hyperbolic distribution function is adopted for Z function, and Eq. (7) is to be perfectly satisfied, then, both the horizontal velocity and the vertical velocity at the bed should be zero, and the final solution becomes inaccurate. At this point we examine whether Kim and Bae's (2006) complementary mild-slope equation satisfies the bed boundary condition. They introduced a hyperbolic sine function for Z:

$$Z=\frac{\sinh k(z+h)}{\sinh kh} \tag{21}$$

As far as the dispersion relationship is valid, another relationship between k and h develops, and the bed boundary condition cannot be satisfied with propagating mode only, either. In summary the mild-slope equations group to date including the mild-slope equation, the modified mild-slope equation, the complementary mild-slope equation, and the present inhomogeneous Helmholtz equation can comply with the bed boundary condition only with the help of the non-propagating evanescent modes.

The one-dimensional versions of both the Helmholtz equation and the previous equations including the mild-slope equation, modified mild-slope equation, and Kim *et al.*'s equation can be arranged in the following form:

$$\frac{d^2\phi}{dx^2}+A\frac{dh}{dx}\frac{d\phi}{dx}+k^2+B\frac{d^2h}{dx^2}+C\left(\frac{dh}{dx}\right)^2\phi=0 \tag{22}$$

While A, B, or C are non-zero functions in the mild-slope equation, modified mild-slope equation, or Kim *et al.*'s (2009) equation, A, B, and C are zero in the inhomogeneous Helmholtz equation.

Here we introduce two real functions, the wave amplitude, a, the wave phase function, S. And, b is defined as the spatial gradient of the wave phase function as:

$$\phi=ae^{iS} \text{ and } b=\frac{dS}{dx} \tag{23}$$

The wave amplitude and wave phase function are dependent on x for one-dimensional problems. Then, Eq. (16) is split into the following two equations as:

$$\frac{\partial^2 a}{\partial x^2}+(k^2-b^2)=0 \tag{24}$$

and

$$\frac{\partial b}{\partial x}+\frac{2\partial a}{a\partial x}b=0 \tag{25}$$

3. NUMERICAL SOLUTION

A set of real Eqs. (24) and (25) for wave transformation problems over mild-sloped beds are solved. We adopt explicit finite difference schemes for Eqs. (24) and (25). First, Eq. (24) is discretized as:

$$\frac{a_{i-1}-2a_i+a_{i+1}}{\Delta x^2}+(k_i^2-b_i^2)a_i=0 \tag{26}$$

and Eq. (25) is discretized as:

$$\frac{b_i-b_{i-1}}{\Delta x}+\frac{4(a_i-a_{i-1})}{\Delta x(a_i+a_{i-1})}\frac{b_i+b_{i-1}}{2}=0 \tag{27}$$

Boundary conditions at a side point are needed. Here a, da/dx, and b at the right end of the computational domain in the x direction are provided. These are discrete values a_M, $a_{M\text{-}1}$, and b_M. The two finite difference equations are alternately solved: Eq. (26) for $a_{i\text{-}1}$, and Eq. (27) for $b_{i\text{-}1}$. Both difference equations are centered in space, see Fig. 1:

$$a_{i-1}=\frac{1}{2}[4-2(k_i^2-b_i^2)\Delta x^2 a_i-2a_{i+1}] \tag{28}$$

and

$$b_{i-1}=\frac{2+G_{i-0.5}}{2-G_{i-0.5}}b_i \tag{29}$$

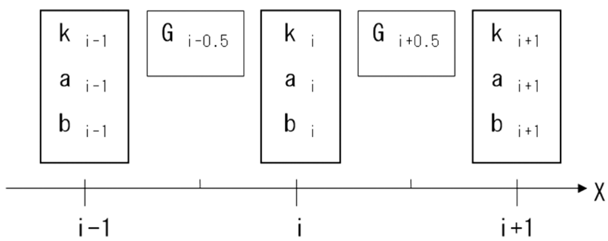

Fig. 1. Variable numbering system in finite difference equations set.

where

$$G_{i-0.5} = \frac{4(a_i - a_{i-1})}{\Delta x(a_i + a_{i-1})} \tag{30}$$

Eqs. (28) and (29) are solved in a forward progressive manner from the right side to the left side.

4. VERIFICATION OF THE INHOMOGENEOUS HELMHOLTZ EQUATION

The new set of equations is applied to a series of profiles used by Booij (1983). The bathymetry is composed of an inclined plane with a variable slope or variable width, B, which connects the two flat beds at off-shore and near-shore sides. A step with a bed of a constant slope separates two flat beds, see Fig. 2. The offshore water depth is 60 cm, the near-shore water depth is 20 cm, and the wave period of the incident waves is 2 s.

The boundary values at the right end of the computation domain are simply provided because only outgoing waves exist at the near-shore end. The wave amplitude, a, at the near-shore boundary is chosen as 0.1 m, and the value of the phase function derivative, b, is given the wave number at the shallow zone. The two dependent variables, a and b, are alternately computed by half grid size advancement at each computation of Eqs. (28) and (29) from the near-shore end to the off-shore end.

Computed spatial distribution of the relative wave amplitude to the outgoing wave amplitude for a specific plane inclination, $B = 1$ m, is shown in Fig. 3. The wave amplitude shows undulation off-shore side from the inclined plane because of the superposition of the incident and reflected waves.

Computed spatial distribution of the wave phase gradient function for a specific plane slope width of 1 m is shown in Fig. 4. The wave phase gradient function also has undulation off-shore side from the inclined plane because of the superposition of incident and reflected waves in the region.

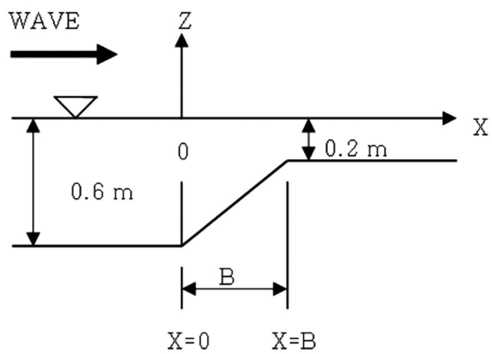

Fig. 2. Booij's test step with inclined plane.

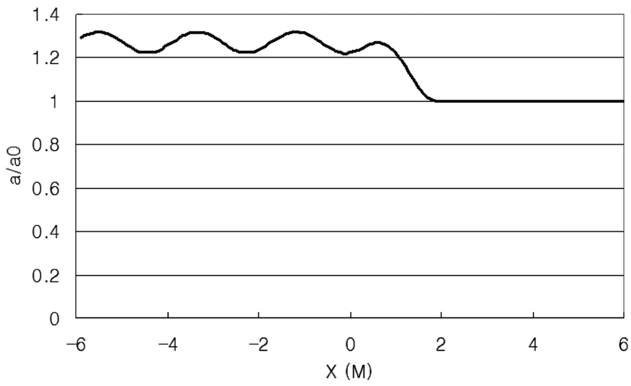

Fig. 3. Computed spatial distribution of relative amplitude for Booij's bed profile of B=1 m.

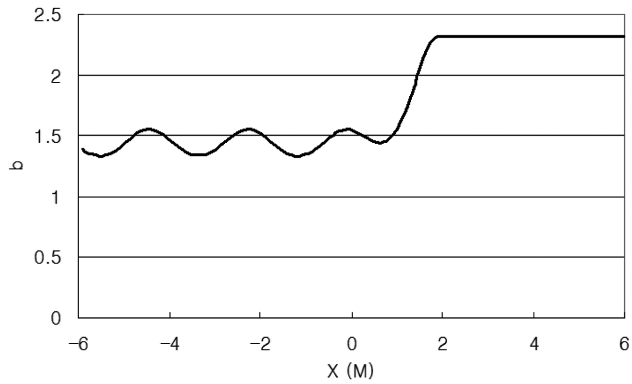

Fig. 4. Computed spatial distribution of phase gradient for Booij's bed profile of B=1 m.

Reflection coefficient, K_r, can be obtained from the computed wave amplitude field from the computed maximum and minimum wave amplitudes, $a_{\max}$ and $a_{\min}$, at the off-shore flat bed zone, that is:

$$K_r = \frac{a_{\max} - a_{\min}}{a_{\max} + a_{\min}} \tag{31}$$

Computed reflection coefficients from the inhomogeneous Helmholtz equation for Booij's test profiles are shown in Fig. 5. In general the reflection coefficients of the inhomogeneous Helmholtz equation are close to those of the full linear equation which does not involve separation of variables (Park *et al.*, 1991). The reflection coefficients of the present equation are smaller than those of the mild-slope equation or the modified mild-slope equation for inclined plane of slope between 0.4 and 4. The origin of these discrepancies should be the differences on the weight functions.

For the plane slopes of greater than or equal to 1, which corresponds to B≤0.4 m, the computed reflection coefficients from the inhomogeneous Helmholtz equation agree well with those

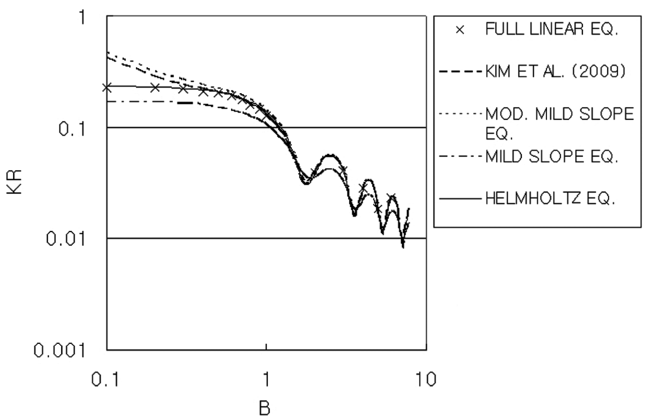

Fig. 5. Comparison of computed reflection coefficients for Booij's bed profile.

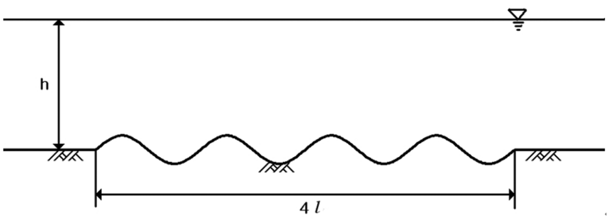

Fig. 6. Bragg's sinusoidal bathymetry with 4 ripples.

from the full linear equation, while the mild-slope equation and the modified mild-slope equation and Kim *et al.*'s (2009) equation show quite large gaps from the full linear equation. The reflection coefficients from the full linear equation, the inhomogeneous Helmholtz present equation, and the modified mild-slope equation are very close for B larger than or equal to 4 m, because the higher order terms become negligible when the bed slope is small.

Next the inhomogeneous Helmholtz equation is applied to Bragg's bathymetry to examine its applicability for a different bathymetry, see Fig. 6. It has been known that sinusoidal bathymetry can cause high reflection depending on the ratio between the bed form length and the wave length. The bathymetry is expressed by the following equation:

$$
\begin{aligned}
&h_1 = 0.156\\
&h = h_1 - 0.05\sin(2\pi/l) \qquad 0 \le x \le 4l\\
&h_2 = 0.156
\end{aligned}
\tag{32}
$$

where h is the water depth in meter, h_1 and h_2 are the offshore water depths from the bed forms, respectively, and l is the ripple length.

Incident waves propagate in the positive direction. Calling the wave length L, the computed reflection coefficient for $2l/L = 0.98$ from the Helmholtz equation is 0.748, which closely

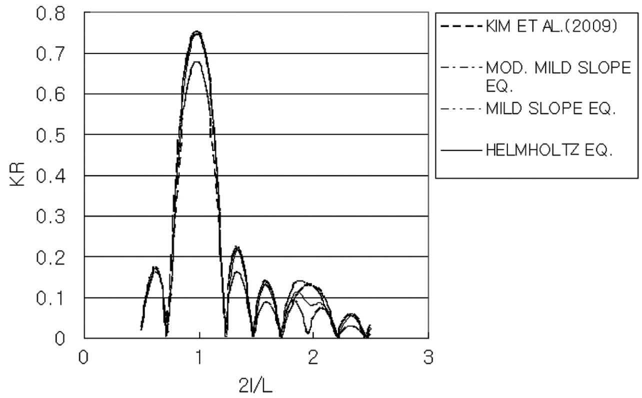

Fig. 7. Comparison of computed reflection coefficients for Bragg's bed profile.

agrees with 0.752 from the modified mild-slope equation, and 0.745 from Kim *et al.*(2009)'s equation, while the reflection coefficient of 0.678 from the mild-slope equation is smaller than the other results, see Fig. 7. The gaps between the reflection coefficients from the mild-slope equation and the other equations are non-negligible. It may be too early to conclude that any one equation is superior to the other equations just from these tests over Bragg's bathymetry because of difficulty in accurate measurements.

An interesting feature is the distribution of the reflection coefficient around the second resonance point, i.e. $2l/L = 2$. The compared equations produce different reflection coefficients around the point. The computed reflection coefficients from the inhomogeneous Helmholtz equation around the second resonance point are smaller than those from the mild-slope equation and the modified mild-slope equation, and greater than those from Kim et al.'s equation, see Fig. 8. The explanation for this difference could be possible in the future works.

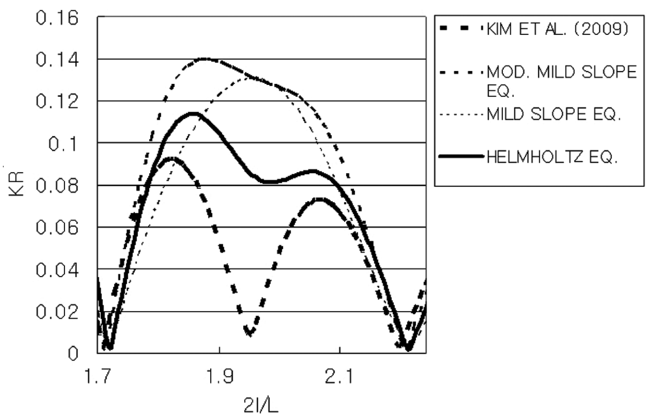

Fig. 8. Zoomed computed reflection coefficients for Bragg's bed profile around second resonance.

5. CONCLUSIONS

The inhomogeneous Helmholtz equation is adopted and examined for description of harmonic wave transformation over variable depth. The continuity equation has a weight of a delta function along the still water level, and is assumed to be separated by two functions, a vertical distribution function Z, and a spatial potential function Φ in the x axis. The inhomogeneous Helmholtz equation has two forms: a form with the complex velocity potential function, and the other form in a set of equations with the wave amplitude and the wave phase function.

The inhomogeneous Helmholtz was applied to two bed profiles, Booij's inclined bed profile, and Bragg's wavy ripple bed profile. The inhomogeneous Helmholtz equation was verified against other theories.

The test of the present equation on Booij's steps reveals that the inhomogeneous Helmholtz equation provides better accurate reflection coefficient with respect to the solutions from the full linear equation than the modified mild-slope equation or the mild-slope equation. Moreover the inhomogeneous Helmholtz equation shows good agreement over wide range of bed slopes.

The test of the present equation on Bragg's sinusoidal ripples confirms that the inhomogeneous Helmholtz equation produces correct reflection coefficient when the ripple length is about half of the wave length compared with other theories, e.g. the modified mild-slope equation, and the modified mild-slope equation, or Kim *et al.*'s (2009) equation.

The numerical experiments over two kinds of bathymetry confirm the accuracy and applicability of the inhomogeneous Helmholtz equation based on the delta weight function along the still water level.

ACKNOWLEDGEMENTS

The work was supported by Korean Ministry of Land, Transport and Maritime Affairs, and Korea Institute of Marine Science & Technology Promotion under the title of "Marine and Environmental Prediction System (MEPS)" in 2009-2010, and by Kookmin University as a university grant in 2010.

REFERENCES

[1] Berkhoff, J.C.W. 1973. Computation of combined refraction-diffraction. Proc. 13th Intl. Conf. on Coastal Engineering, July 1972, Vancouver, Canada, pp. 471-490. ASCE.

[2] Booij, N. 1983. A note on the accuracy of the mild-slope equation. Coastal Engineering. 7, 191-203.

[3] Chamberlain, P.G., Porter, D. 1995. The modified mild-slope equation. J. Fluid Mech. 291, 393-407.

[4] Copeland, G.J.M. 1985. A practical alternative to the mild-slope wave equation. Coastal Engineering. 9, 125-149.

[5] Hsiao, S.S., Weng, W.K. and Chiu, Y.F. 1998. Wave forces on a circular island using perturbation - DRBEM. Eighth International Offshore and Polar Engineering Conference, Montreal, Canada.

[6] Kim, H., Jung, B. and Lee, Y. 2009. A linear wave equation over mild-sloped bed from double integration. J. Korean Society for Marine Environmental Engineering. 12(3), 165-172.

[7] Kim, J.W., Bai, K.J. 2004. A new complementary mild-slope equation. J. Fluid Mech. 511, 25-40.

[8} Kirby, J.T. 1986. A general wave equation for waves over rippled beds. J. Fluid Mechanics. 162, 171-186.

[9] Luke, J.C. 1967. A variational principle for a fluid with a free surface. J. Fluid Mech. 27, 395-397.

[10] Manolis, G.D. and Shaw, R.P. 1997. Fundamental solutions to Helmholtz's equation for inhomogeneous media by a first-order differential equation system. Soil Dynamics and Earthquake Engineering. 16, 81-94.

[11] Massel, S.R. 1993. Extended reflection-diffraction equation for surface waves. Coastal Engineering. 19, 97-126.

[12] Porter, D., Staziker, D.J. 1995. Extentions of the mild-slope equation. J. Fluid Mech. 300, 367-382.

[13] Park, W.S., Lee, D.S., Oh, Y.M., Jeong, W.M. 1991. Infinite elements for analysis of diffraction and radiation problems in the vertical plane. J. Korean Soc. Coastal Ocean Engineers, 3 (4), 235-243 (in Korean).

[14] Rather, A.C. 1979. On the parabolic equation method for water-wave propagation. J. Fluid Mech. 95, 159-176.

[15] Suh, K.D., Lee, C., Park, W.S. 1997. Time-dependent equations for wave propagation on rapidly varying topography. Coastal Engineering. 32, 91-117.

2010년 5월 28일 원고접수
2010년 7월 1일 심사완료
2010년 7월 1일 수정본 채택