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The least squares estimation for failure step-stress accelerated life tests

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Abstract

The least squares estimation method for model parameters under failure step-stress accelerated life tests is studied and a numerical example will be given to illustrate the proposed inferential procedures under the compound linear plans proposed as an alternative to the optimal quadratic plan, assuming that the exponential distribution with a quadratic relationship between stress and log-mean lifetime. The proposed compound linear plan for constant stress accelerated life tests and 4:2:1 plan are compared for various situations. Even though the compound linear plan was proposed under constant stress accelerated life tests, we found that this plan did well relatively in failure step-stress accelerated life tests.

Keywords: Compound Linear plan, exponential distribution, failure step-stress accelerated life tests, transformed least squares estimates.

1. Introduction

Accelerated life tests (ALTs) use to get informations on the life distribution of test unit quickly. The various accelerated life testings are used to avoid this problem. Meeker and Nelson (1975) and Meeker (1984) suggested the design for Type I censored constant-stress ALTs. Nelson and Miller (1983), Bai *et al.* (1989) obtained the stress change time which minimizes the asymptotic variance of maximum likelihood estimate of the log scale parameter at the use stress level. Bai and Chung (1992) compared the performances of step-stress and constant-stress partially ALTs by the tampered random variable (TFR) model proposed DeGroot and Goel (1979). Khamis and Higgins (1996) evaluated compound linear plan under three step-stress ALTs. Kahn (1979) discussed least square estimation for constant-stress ALTs. Teng and Yeo (2002) proposed the a transformed least squares (TLS) approach for analyzing failure step-stress ALTs. Kim (2006) proposed the compound linear plan and compared the efficiencies with other compromise plans. Moon and Kim (2006) studied confidence interval estimation of the two-parameter exponential distribution under three step-stress ALTs. Moon (2008) obtained the optimal plans based on grouped and Type I censored data for

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three step-stress ALTs. Moon and Park (2009) also studied optimal plans based on periodic inspection with Type I censoring.

In this paper, we study a TLS approach for a log-quadratic life stress relationship under failure step-stress ALTs. Also we apply compound linear plan proposed by Kim (2006) under 3-level constant stress ALTs to three failure step-stress ALTs and show the proposed compound linear plan useful in this case. In Section 2, we describe some necessary assumptions and present the TLS estimates (TLSE) of regression parameters. The proposed inferential procedures and the efficiencies to compare the compound linear plan with the 4:2:1 plan are illustrated in Section 3.

2. Transformed least squares estimates

Let n_i be the number of failed units at the stress level s_i, $i = 1, 2, \cdots, m$ and $T_{i,j}$ be the j-th observation at stress level s_i, $j = 1, 2, \cdots, n_i$. The life distribution of the test unit for any stress is assumed to be exponential with mean life θ_i at stress level s_i, $i = 1, 2, \cdots, m$.

Under the failure-step stress ALTs, suppose that all n test units are initially placed at the lowest stress level s_1, and the number of failed units $n_i, i = 1, 2, \cdots, m$ at each stress level are preassigned. If n_1 failures are observed at stress level s_1, then the stress is changed to s_2 at $T_{1,n1}$ and the test runs until n_2 failures are observed. After n_2 units have failed at stress level s_2, the stress is changed to s_3 at $T_{2,n2}$, and test process is terminated after n_m failures are occurred.

Suppose that there are m stress levels with $s_1 < s_2 < \cdots < s_m$. In the presentation of our results and without loss of generality, we use the standardized stress levels given by

$$x_i = \frac{s_i - s_0}{s_m - s_0}, \quad i = 1, 2, \cdots, m,$$

where s_0 is the usual stress. The log (mean lifetime) ($\log\theta_i$) at stress x_i is assumed to be given by

$$\log\theta_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2, \quad i = 1, 2, \cdots, m, \tag{2.1}$$

where β_0, β_1 and β_2 are unknown parameters.

Lemma 1. Let $T_{(1)}, T_{(2)}, \cdots, T_{(n)}$ be the ordered observations of a size n random sample from the exponential distribution with mean θ. Let $T_{(0)} = 0$. Then for $i = 1, 2, \cdots, n$,

$$Z_i = (n - i + 1)(T_{(i)} - T_{(i-1)})$$

are independent and identically distributed as an exponential distribution with mean θ.

Lemma 2. Let Z have an exponential distribution with mean θ. Then $\log Z$ has an extreme-value distribution with location parameter $\log\theta$ and scale parameter 1. Then the mean and variance of $\log Z$ are $\log\theta - 0.5772$ and 1.283^2, respectively.

Proof: Proofs of Lemma 1 and Lemma 2 are in Lawless (1982) and many other books. □

Let $T_{1,0} = 0, T_{i,0} = T_{i-1,n_{i-1}}$ for $i = 1, 2, \cdots, m$. Let for $i = 2, 3, \cdots, m$,

$$Z_{1,1} = nT_{1,1}, Z_{1,2} = (n-1)(T_{1,2} - T_{1,1}), \cdots, Z_{1,n_1} = (n - n_1 - 1)(T_{1,n_1} - T_{1,n_1-1}), \cdots$$

$$Z_{i,1} = \left(n - \sum_{k=1}^{i-1} n_k\right)\left(T_{i,1} - T_{i-1,n_{i-1}}\right), Z_{i,2} = \left(n - \sum_{k=1}^{i-1} n_k - 1\right)\left(T_{i,2} - T_{i,1}\right), \cdots,$$

$$Z_{i,n_i} = \left(n - \sum_{k=1}^{i} n_k + 1\right)\left(T_{i,n_i} - T_{i,n_{i-1}}\right).$$

Then the time differences between two consecutive failures

$$Z_{i,j} = (n - [\sum_{k=1}^{i-1} n_k + j] + 1)(T_{i,j} - T_{i,j-1}) \tag{2.2}$$

for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n_i$ are independent and identically distributed as an exponential distribution with mean θ_i by Lemma 1. The $\log Z_{i,j}$ has therefore extreme-value distribution by Lemma 2.

Under the assumption that $\log \theta_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2$, the regression model is

$$W_{i,j} = \log \theta_i + e_{i,j},$$
$$W_{i,j} = \log Z_{i,j} + 0.5772,$$

where $e_{i,j}$ are independent extreme-value random variable with mean 0 and variance 1.283^2.

By LS estimation, the TLSE of β_i, $i = 0, 1, 2$, are given by

$$\widehat{\boldsymbol{\beta}} = \left(X^{'}X\right)^{-1} X^{'}W = \begin{pmatrix} n & \sum n_i x_i & \sum n_i x_i^2 \\ \sum n_i x_i & \sum n_i x_i^2 & \sum n_i x_i^3 \\ \sum n_i x_i^2 & \sum n_i x_i^3 & \sum n_i x_i^4 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i,j} W_{i,j} \\ \sum_{i,j} x_i W_{i,j} \\ \sum_{i,j} x_i^2 W_{i,j} \end{pmatrix}, \tag{2.3}$$

where

$$X = \begin{pmatrix} 1, & \cdots, & 1 & \cdots\cdots & 1 \\ x_1, & \cdots, & x_1 & \cdots\cdots & x_m \\ x_1^2, & \cdots, & x_1^2 & \cdots\cdots & x_m^2 \end{pmatrix}^{'}, \quad W = \left(W_{1,1}, \cdots, W_{1,n_1} \cdots\cdots W_{m,1}, \cdots, W_{m,n_m}\right)^{'}.$$

The TLS approach makes much easier to compute the estimates of $\beta_0, \beta_1, \beta_2$ compared to using maximum likelihood methods and we can also obtain the closed-form estimates of $\beta_0, \beta_1, \beta_2$ by this approach.

In particular, the asymptotic variance of the TLSE of logarithm of the mean lifetime, $\log \theta_0$, at use stress level x_0 is $aVar(\log \widehat{\theta}_0) = aVar(\widehat{\beta}_0)$, where $aVar(\widehat{\beta}_0)$ is 1.283^2 times the first element of the matrix $\left(X^{'}X\right)^{-1}$, that is,

$$aVar(\log \widehat{\theta}_0) = \frac{1.283^2}{n - \dfrac{a_1^2 a_4 + a_2^3 - 2a_1 a_2 a_3}{a_2 a_4 - a_3^2}} \tag{2.4}$$

with $a_l = \sum_{i=1}^{m} n_i x_i^l$, $l = 1, 2, 3, 4$.

To find the optimal stress change times T_{i,n_i}, $i = 1, 2, \cdots, m$ for failure step-stress ALTs minimizing $aVar(\log \widehat{\theta}_0)$ in (2.4) is to find the number of optimal failures n_i at each stress level x_i, subject to $\sum_{i=1}^{m} n_i = n$, which minimizes

$$\frac{a_1^2 a_4 + a_2^3 - 2a_1 a_2 a_3}{a_2 a_4 - a_3^2} \tag{2.5}$$

By differentiating (2.5) with respect to n_i, $i = 1, 2, \cdots, m$ and equating to zero, the number of optimal failures n_i^* at stress level x_i can be found.

3. Numerical example

The 40 simulated failure times from failure three step-stress ALT is given in Table 3.1 to illustrate TLSE of β_0, β_1 and β_2, based on three stress level $x_1 = 0.3$, $x_2 = 0.65$, $x_3 = 1$ and the log-quadratic relationship, $\log \theta_i = 1 - 2x_i - 5x_i^2$, $i = 1, 2, 3$ in (2.1).

The numbers of failures n_1 and n_2 at stress level x_1 and x_2 are solutions in (3.1).

$$\begin{aligned} f_1(n_1, n_2) = -\frac{1}{(a_2 a_4 - a_3^2)^2}((x_1^4 - x_3^4)(a_1 a_3 - a_2^2)^2 - 2(x_1^3 - x_3^3)(a_1 a_3 - a_2^2)(a_1 a_4 - a_2 a_3) \\ +(x_1^2 - x_3^2)(2(a_1 a_3 - a_2^2)(a_2 a_4 - a_3^2) + (a_1 a_4 - a_2 a_3)^2) \\ -2(x_1 - x_3)(a_2 a_4 - a_3^2)(a_1 a_4 - a_2 a_3)). \qquad (3.1) \\ f_2(n_1, n_2) = -\frac{1}{(a_2 a_4 - a_3^2)^2}((x_2^4 - x_3^4)(a_1 a_3 - a_2^2)^2 - 2(x_2^3 - x_3^3)(a_1 a_3 - a_2^2)(a_1 a_4 - a_2 a_3) \\ +(x_2^2 - x_3^2)(2(a_1 a_3 - a_2^2)(a_2 a_4 - a_3^2) + (a_1 a_4 - a_2 a_3)^2) \\ -2(x_2 - x_3)(a_2 a_4 - a_3^2)(a_1 a_4 - a_2 a_3)). \end{aligned}$$

But, finding the numbers of optimal failures n_1 and n_2 is very troublesome. So, the compound linear plan proposed by Kim (2006) is used and all failures are generated according to TFR model. We showed the compound linear plan was nearly as good as the optimum quadratic plan. The numbers of failures at each stress level by the compound linear plan are $n_1 = 17$, $n_2 = 15$ and $n_3 = 8$. On the other hand, $n_1 = 18$, $n_2 = 16$ and $n_3 = 6$ by optimal quadratic plan and $n_1 = 22$, $n_2 = 11$ and $n_3 = 7$ by the 4:2:1 plan.

The computed $Z_{i,j}$ in (2.2) from 40 failure times are listed in Table 3.2. By these $Z_{i,j}$ and (2.3), the TLSE of β_0, β_1 and β_2 are $\widehat{\beta}_0 = 1.087$, $\widehat{\beta}_1 = -2.489$ and $\widehat{\beta}_2 = -5.204$, respectively.

The 4:2:1 plan by Khamis and Higgins (1996) for three step-stress ALTs did well relatively over a range of test situations. But we showed the compound linear plan did better than 4:2:1 plan for constant stress ALTs by their efficiencies for various x_1 and x_2. We examine that the compound linear plan do well relatively compared to the 4:2:1 plan in failure step-stress accelerated life tests various x_1 and x_2 by $ratio = aVar_c/aVar_k$, ratio of $aVar(\log \widehat{\theta}_0)$ asymptotic variances of the TLSE of logarithm of the mean lifetime at use stress level x_0 and the results are given in Table 3.3, where $aVar_c$ and $aVar_k$ indicate variances $aVar(\log \widehat{\theta}_0)$ by the compound linear plan and the 4:2:1 plan, respectively.

Table 3.1 40 simulated failure times $T_{i,j}$

i	x_i	n_i	$T_{i,j}$					
1	0.3	17	0.04567	0.05213	0.06533	0.06816	0.09131	0.10500
			0.10586	0.16113	0.16818	0.23991	0.24551	0.26459
			0.30268	0.33121	0.34009	0.42073	0.44885	
2	0.65	15	0.45454	0.45594	0.4606	0.46086	0.46964	0.48216
			0.48450	0.48567	0.48628	0.48756	0.48943	0.48965
			0.49358	0.50258	0.51674			
3	1	8	0.51678	0.51819	0.51819	0.51823	0.51934	0.51964
			0.51988	0.52173				

Table 3.2 The computed for TLS estimation $Z_{i,j}$

i	x_i	n_i	$Z_{i,j}$					
1	0.3	17	1.82674	0.25191	0.50186	0.10468	0.83317	0.47917
			0.02946	1.82392	0.22557	2.22348	0.16813	0.55317
			1.06657	0.77041	0.23074	2.01593	0.67492	
2	0.65	15	0.13083	0.03088	0.09797	0.00507	0.16688	0.22531
			0.03977	0.01875	0.00916	0.01796	0.0243	0.0026
			0.04321	0.09002	0.12742			
3	1	8	0.00034	0.00985	0.00005	0.00019	0.00442	0.0009
			0.00049	0.00185				

From these results, we can see that when the stress x_2 is less than 0.6, the 4:2:1 plan do better slightly than the compound linear plan, regardless of x_1 and when the stress x_1 is less than 0.2, the 4:2:1 plan is better for the larger stress x_2, but the proposed compound linear plan do better relatively than the 4:2:1 plan for over most testing situations.

Table 3.3 The ratios of variances

x_1	x_2	ratio	x_1	x_2	ratio
	.55	1.01758		.55	1.02781
.20	.60	.97112	.35	.60	.95330
	.65	.96756		.65	.93262
	.70	.99964		.70	.89047
	.75	1.12962		.75	.88219
	.55	1.01286		.55	1.04943
.25	.60	.95643	.40	.60	.96661
	.65	.94644		.65	.94146
	.70	.93335		.70	.87748
	.75	1.01091		.75	.85149
	.55	1.01593		.55	1.04281
.30	.60	.97447	.45	.60	.99120
	.65	.93449		.65	.96213
	.70	.90554		.70	.88159
	.75	.93296		.75	.84172

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