

Convergence of a Power Variation Related to a Standard Brownian Sheet

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Abstract

By using a Malliavin calculus, we prove the central limit theorem on the power variation of the product of the unordered rectangles associated with a standard Brownian sheet.

Keywords: Malliavin calculus, standard Brownian sheet, Malliavin derivative, central limit theorem, power variation.

1. Introduction

For the study of the behavior on single path of stochastic processes, their power variations are often considered. There exists extensive literature on the subject. The p -power variation of a process $X = (X_t, t \in [0, 1])$ is defined to be the sum

$$\sum_{i=0}^{n-1} \left| X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right|^p.$$

Let us review some known results concerning the p -power variation. By using the classical central limit theorem, we immediately have

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[n^{\frac{p}{2}} (B_{\frac{i+1}{n}} - B_{\frac{i}{n}})^p - \mu_p \right] \xrightarrow{\mathcal{L}} \mathbb{N}(0, \mu_{2p} - \mu_p^2),$$

where $B = (B_t, t \in [0, 1])$ be a standard Brownian motion, μ_p denote the p -moment of a standard Gaussian distribution, and the notation $\xrightarrow{\mathcal{L}}$ denotes the convergence in distribution. Recently, by using a Malliavin calculus, the convergence on the weighted power variation of a fractional Brownian motion has been studied in Nourdin (2008), Nourdin and Nualart (2008) and Nourdin *et al.* (2007).

For the two-parameter case, a central limit theorem has been obtained in Réveillac (2009a) for the weighted quadratic variations of a standard Brownian sheet. In Réveillac (2009b), the author proved a central limit for the finite-dimensional laws of the weighted quadratic variations of a fractional Brownian sheet. However, the quadratic variation of the ordered rectangle has been only considered in both cases. In this paper we study the asymptotic behavior of the variation of the product of the unordered rectangles of a standard Brownian sheet.

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Recall that the standard Brownian sheet $B = (B_z, z \in [0, 1]^2)$, is a centered Gaussian process with the covariance

$$\mathbb{E}(B_a B_b) = \frac{1}{4} \prod_{i=1}^2 (a_i + b_i - |a_i - b_i|),$$

where $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0, 1]^2$. We may consider a central limit theorem of the 2-power and 1-power bi-variation of the unordered rectangles. To the best of our knowledge, the study on the variation of this type is the first attempt. Hence we consider the non-weighted type in the simplest possible variation. More precisely, we state the main result in the following theorem:

Theorem 1. *Let B be a standard Brownian sheet. Then as $n \rightarrow \infty$, we have*

$$\sqrt{n} \sum_{k,l=0}^{n-1} \left(B_{\frac{k+1}{n}, \frac{l}{n}} - B_{\frac{k}{n}, \frac{l}{n}} \right)^2 \left(B_{\frac{k}{n}, \frac{l+1}{n}} - B_{\frac{k}{n}, \frac{l}{n}} \right) \xrightarrow{\mathcal{L}} \mathbb{N}\left(0, \frac{1}{9}\right). \quad (1.1)$$

For the full development of a stochastic calculus with respect to a standard Brownian sheet, the various types of stochastic integrals are necessary. In particular, the mixed integrals with respect to a standard Brownian sheet and Lebesgue measure are needed. We may use our result to deduce a central limit theorem for the bi-power variation process of a two-parameter process $(X_z, z \in [0, 1]^2)$ defined by the mixed integral

$$X_z = \int_{[0,z]^2} \beta(B_{a \star b}) \mathbf{1}_{[a \leq_1 b]} da db,$$

where $a \star b = (a_1, b_2)$, $[a \leq_1 b] = \{(a, b) \in [0, 1]^4 : a_1 \leq b_1, a_2 \geq b_2\}$ and $da = da_1 da_2$. For the proof of Theorem 1, we will use the Malliavin calculus and the result on the convergence of the multiple stochastic integrals worked by Nualart and Ortiz-Latorre (2008).

2. Preliminaries

We briefly recall some basic facts about the Malliavin calculus for the Gaussian processes. For a more detailed reference, see Nualart (2006). Suppose that \mathbb{H} is a real separable Hilbert space with a scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. Let $B = (B(h), h \in \mathbb{H})$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $E(B(h)B(g)) = \langle h, g \rangle_{\mathbb{H}}$. When B is a standard Brownian sheet, the scalar product is given by

$$\langle \mathbf{1}_{[0,a]}, \mathbf{1}_{[0,b]} \rangle_{\mathbb{H}} = \frac{1}{4} \prod_{i=1}^2 (a_i + b_i - |a_i - b_i|). \quad (2.1)$$

For every $n \geq 1$, let \mathcal{H}_n be the n th Wiener chaos of B , i.e. the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_n(B(h)) : h \in \mathbb{H}, \|h\|_{\mathbb{H}} = 1\}$, where H_n is the n th Hermite polynomial

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}} \right).$$

We define a linear isometric mapping $I_n : \mathbb{H}^{\otimes n} \rightarrow \mathcal{H}_n$ by

$$I_n(h^{\otimes n}) = n! H_n(B(h)),$$

where $\mathbb{H}^{\odot n}$ is the symmetric tensor product. Let \mathcal{S} denote the class of smooth random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = f(B(h_1), \dots, B(h_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all of its partial derivative are bounded). We will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables \mathcal{S} with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{l=1}^k \mathbb{E}[\|D^l F\|_{\mathbb{H}^{\odot l}}^p] \right)^{\frac{1}{p}},$$

where D^l denotes the iterated Malliavin derivative. Then the following duality formula holds:

$$\mathbb{E}[FI_n(h)] = \mathbb{E}[\langle D^n F, h \rangle_{\mathbb{H}^{\odot n}}], \quad (2.2)$$

for every $h \in \mathbb{H}^{\odot n}$ and $F \in \mathbb{D}^{n,2}$.

In this paper we will only use multiple stochastic integrals with respect to a standard Brownian sheet $B = (B_z, z \in [0, 1]^2)$, and in this case $\mathbb{H} = L^2([0, 1]^2)$. We will use the notation \mathbb{H} throughout this paper. If $f \in \mathbb{H}^{\odot p}$, the Malliavin derivative of the multiple stochastic integrals is given by

$$D_a I_n(f) = n I_{n-1}(f(\cdot, a)), \quad \text{for } a \in [0, 1]^2.$$

If $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$, the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, is the element of $\mathbb{H}^{\odot(p+q-2r)}$ defined by

$$\begin{aligned} (f \otimes_r g)(a_1, \dots, a_{p+q-2r}) &= \int_{[0,1]^{2r}} f(a_1, \dots, a_{p-r}, b_1, \dots, b_r) \\ &\quad \times g(a_{p+1}, \dots, a_{p+q-r}, b_1, \dots, b_r) db_1 \dots db_r, \end{aligned} \quad (2.3)$$

where $a_i = (a_i^1, a_i^2)$ and $db_i = db_i^1 db_i^2$. Notice that the tensor product $f \otimes g$ and the contraction $f \otimes_r g$, $1 \leq r \leq p \wedge q$, are not necessarily symmetric even though f and g are symmetric. We will denote their symmetrizations by $f \tilde{\otimes} g$ and $f \tilde{\otimes}_r g$, respectively. We will give the formula for the product of the multiple stochastic integrals.

Proposition 1. *Let $f \in \mathbb{H}^{\odot p}$ and $g \in \mathbb{H}^{\odot q}$ be two functions. Then*

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \quad (2.4)$$

From Proposition 1, we have

$$\mathbb{E}[I_p(f)I_q(g)] = \begin{cases} 0, & \text{if } p \neq q, \\ p! \langle \tilde{f}, \tilde{g} \rangle_{\mathbb{H}^{\odot p}}, & \text{if } p = q, \end{cases} \quad (2.5)$$

where \tilde{f} denotes the symmetrization of f .

3. The Proof of Theorem 1

For simplicity, we write

$$\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} = \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}] \times [0, \frac{l}{n}]} \quad \text{and} \quad \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} = \mathbf{1}_{[0, \frac{k}{n}] \times [\frac{l}{n}, \frac{l+1}{n}]}.$$

Then the direct computation gives, by (2.1),

$$\left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} = \begin{cases} \frac{1}{2n^2}(j+l-|j-l|), & \text{if } i=k, \\ 0, & \text{if } i \neq k, \end{cases} \quad (3.1)$$

$$\left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}} = \begin{cases} \frac{1}{2n^2}(i+k-|i-k|), & \text{if } j=l, \\ 0, & \text{if } j \neq l, \end{cases} \quad (3.2)$$

$$\left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}} = \begin{cases} \frac{1}{n^2}, & \text{if } i < k \text{ and } j > l, \\ 0, & \text{if } i \geq k \text{ or } j \leq l, \end{cases} \quad (3.3)$$

$$\left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} = \begin{cases} \frac{1}{n^2}, & \text{if } k < i \text{ and } l > j, \\ 0, & \text{if } k \geq i \text{ or } l \leq j. \end{cases} \quad (3.4)$$

By using the multiplication formula (2.4) in Proposition 1, we have

$$\left(B_{\frac{k+1}{n}, \frac{l}{n}} - B_{\frac{k}{n}, \frac{l}{n}}\right)^2 \left(B_{\frac{k}{n}, \frac{l+1}{n}} - B_{\frac{k}{n}, \frac{l}{n}}\right) = I_3 \left(\left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right)^{\otimes 2} \otimes \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) + \left\langle \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} I_1 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right). \quad (3.5)$$

Let us set

$$G_n = \sum_{k,l=1}^n \left(B_{\frac{k+1}{n}, \frac{l}{n}} - B_{\frac{k}{n}, \frac{l}{n}} \right)^2 \left(B_{\frac{k}{n}, \frac{l+1}{n}} - B_{\frac{k}{n}, \frac{l}{n}} \right).$$

Then we write $G_n = \ell_1(n) + \ell_2(n)$, where

$$\ell_1(n) = \sum_{k,l=1}^n I_3 \left(\left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right)^{\otimes 2} \otimes \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) \quad \text{and} \quad \ell_2(n) = \sum_{k,l=1}^n \frac{l}{n^2} I_1 \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right).$$

First we compute the expectation of the square of $\ell_i(n)$, $i = 1, 2$, separately:

(I) Computation of $\lim_{n \rightarrow \infty} n^2 \mathbb{E}[\ell_1^2(n)]$:

The isometry formula (2.5) for multiple stochastic integrals gives

$$\begin{aligned} & n^2 \mathbb{E}[\ell_1^2(n)] \\ &= n^2 3! \sum_{i,j,k,l=1}^n \left\langle \left(\epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)} \right)^{\otimes 2} \otimes \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right)^{\otimes 2} \otimes \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}^{\otimes 3}} \\ &= \frac{2n^2}{3} \sum_{i,j,k,l=1}^n \left[3 \left(\left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} \right)^2 \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}} + 6 \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(1)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}} \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right\rangle_{\mathbb{H}} \right] \\ &= \tilde{\ell}_{11}(n) + \tilde{\ell}_{12}(n). \end{aligned}$$

By the equalities (3.1) and (3.2), we get

$$\tilde{\ell}_{11}(n) = \left(\sum_{i=0}^{n-1} \left(\frac{i}{n} \right) \left(\frac{1}{n} \right) \right)^2.$$

Using the limit of Riemann sums, we obtain

$$\lim_{n \rightarrow \infty} \tilde{\ell}_{11}(n) = \left(\int_0^1 x dx \right)^2 = \frac{1}{4}. \quad (3.6)$$

From (3.1)–(3.4), it follows that $\tilde{\ell}_{12}(n) = 0$ for all $n = 1, 2, \dots$. Combining the above results, we get

$$n^2 \mathbb{E} \left[\left(\sum_{k,l=1}^n I_3 \left(\left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right)^{\otimes 2} \otimes \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) \right)^2 \right] \rightarrow \frac{1}{4}. \quad (3.7)$$

(II) Computation of $\lim_{n \rightarrow \infty} \mathbb{E}[\ell_2^2(n)]$:

By (2.5), the expectation of $\ell_2^2(n)$ becomes

$$\mathbb{E}[\ell_2^2(n)] = n^{-4} \sum_{i,j,k,l=1}^n jl \left\langle \epsilon_{\frac{i}{n}, \frac{j}{n}}^{(2)}, \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right\rangle_{\mathbb{H}}.$$

From (3.2), it follows that

$$\mathbb{E}[\ell_2^2(n)] = \frac{1}{2} \sum_{j=0}^{n-1} \left(\frac{j}{n} \right)^2 \left(\frac{1}{n} \right) \sum_{i,k=0}^{n-1} \left(\left(\frac{i}{n} \right) + \left(\frac{k}{n} \right) - \left| \frac{i-k}{n} \right| \right) \left(\frac{1}{n^2} \right).$$

By the limit of the Riemann sums, we get

$$\lim_{n \rightarrow \infty} \mathbb{E}[\ell_2^2(n)] = \frac{1}{2} \int_0^1 x^2 dx \times \int_0^1 \int_0^1 (x+y-|x-y|) dx dy = \frac{1}{9}. \quad (3.8)$$

From (3.8), it follows that as $n \rightarrow \infty$,

$$\mathbb{E} \left[I_1 \left(\sum_{k,l=0}^{n-1} \frac{l}{n^2} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right)^2 \right] \rightarrow \frac{1}{9}. \quad (3.9)$$

(III) Convergence in distribution:

Now we prove the convergence in distribution. First we give the theorem describing the convergence in distribution of a sequence of multiple stochastic integrals to a normal distribution (see Theorem 4 in Nualart and Ortiz-Latorre (2008) or Nualart and Peccati (2005)).

Theorem 2. Let $\{F_n = I_k(f_n), n \geq 1\}$, $f_n \in \mathbb{H}^{\odot k}$ for every $n \geq 1$, be a sequence of square integrable random variables in the k th Wiener chaos such that

$$\mathbb{E}[F_n^2] = \|f_n\|_{\mathbb{H}^{\odot k}}^2 \rightarrow 1, \quad \text{as } n \rightarrow \infty.$$

Then the followings are equivalent:

- (i) The sequence $\{F_n, n \geq 1\}$ converges to a normal distribution $\mathbb{N}(0, 1)$.
- (ii) $\lim_{n \rightarrow \infty} \mathbb{E}[F_n^4] = 3$.
- (iii) For all $1 \leq l \leq k-1$, $\lim_{n \rightarrow \infty} \|f_n \otimes_l f_n\|_{\mathbb{H}^{\otimes 2(k-l)}} = 0$.
- (iv) $\|DF_n\|_{\mathbb{H}}^2 \rightarrow k$ in $L^2(\Omega)$, where D is the Malliavin derivative with respect to a Brownian sheet $B = (B_z, z \in [0, 1]^2)$.

The sequence $\{G_n, n \geq 1\}$ can be written as

$$\begin{aligned} G_n &= I_3 \left(\sum_{k,l=1}^n \left(\epsilon_{\frac{k}{n}, \frac{l}{n}}^{(1)} \right)^{\otimes 2} \otimes \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) + I_1 \left(\sum_{k,l=1}^n \frac{l}{n^2} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)} \right) \\ &:= G_{n,1} + G_{n,2}. \end{aligned}$$

From (3.7) and (3.9), we have that $\mathbb{E}[G_{n,1}^2] \rightarrow 0$ and $\mathbb{E}[G_{n,2}^2] \rightarrow 1/9$ as $n \rightarrow \infty$. By Theorem 2, it is sufficient to show that $\|DG_{n,2}\|_{\mathbb{H}}^2 \rightarrow 1/9$ in $\mathbb{L}^2(\Omega)$ as $n \rightarrow \infty$. Let us set $F_n = 3G_{n,2}$. Then $\mathbb{E}[F_n^2] \rightarrow 1$ and $\|DF_n\|_{\mathbb{H}}^2 \rightarrow k = 1$. From Theorem 2, it follows that the sequence $\{3G_{n,2}, n \geq 1\}$ converges to a normal distribution $\mathbb{N}(0, 1)$. The property of derivative for the multiple stochastic integrals gives

$$D_a G_{n,2} = \sum_{k,l=1}^n \frac{l}{n^2} \epsilon_{\frac{k}{n}, \frac{l}{n}}^{(2)}(a),$$

and its norm is $\|DG_{n,2}\|_{\mathbb{H}}^2 = \mathbb{E}[\ell_2^2(n)]$. So we get $\|DG_{n,2}\|_{\mathbb{H}}^2 \rightarrow 1/9$ in $\mathbb{L}^2(\Omega)$. Hence we complete the proof of Theorem 1. \square

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