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The Mixing Properties of Subdiagonal Bilinear Models

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Abstract

We consider a subdiagonal bilinear model and give sufficient conditions for the associated Markov chain defined by Pham (1985) to be uniformly ergodic and then obtain the β-mixing property for the given process. To derive the desired properties, we employ the results of generalized random coefficient autoregressive models generated by a matrix-valued polynomial function and vector-valued polynomial function.

Keywords: Subdiagonal Bilinear model, geometric ergodicity, β-mixing, stationarity, generalized random coefficient autoregressive model.

1. Introduction

Among many nonlinear time series models, we are interested in bilinear models introduced by Granger and Anderson (1978) and Subba Rao (1981). As shown by Subba Rao and Gabr (1984), the bilinear model is particularly attractive in modelling processes with sample paths of occasional sharp spikes and when interactions between $\{X_t\}$ and the error process $\{\epsilon_t\}$ are significant. Bilinear models are studied by Bhaskara Rao *et al.* (1983), Pham (1985, 1986), Liu and Brockwell (1988), Chanda (1992), Liu (1992), Terdik (1999), Giraitis and Surgailis (2002), Lee (2006) and Kristensen (2009, 2010). In those papers, probabilistic as well as statistical properties such as stationarity, invertibility, ergodicity, mixing property, existence of higher order moments, central limit theorem, estimation problems including model identification and finding suitable white noise are examined.

General bilinear process of order p, q, m, l is defined by

$$y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=0}^{q} b_i \epsilon_{t-i} + \sum_{i=1}^{l} \sum_{j=1}^{m} c_{ij} y_{t-j} \epsilon_{t-i},$$

where ϵ_t is a sequence of independent and identically distributed(iid) random variables and a_i, b_i and c_{ij} are real constants.

In this paper, we restrict ourselves to a type of subdiagonal bilinear models given by

$$y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=0}^{q} b_i \epsilon_{t-i} + \sum_{i=1}^{P} \sum_{j=i}^{Q+i} c_{ij} y_{t-j} \epsilon_{t-i} \tag{1.1}$$

and give some simple easy-to-verify conditions that simultaneously imply stationarity and exponential β-mixing. We begin by showing that a subdiagonal bilinear model can be rewritten as a case of a

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generalized polynomial random coefficient autoregressive model(GRCA). One of the advantages of such a technique is that it enables the use of the results on GRCA models (see, Feigin and Tweedie, 1985; Doukhan, 1994; Carassco and Chen, 2002). β-mixing with a geometric convergence rate of the bilinear models can be derived from the V-uniform ergodicity of the auxiliary Markov process.

Let $\{X_t : t = 0, 1, 2, \ldots\}$ be a discrete time Markov chain defined on R^k, $k \geq 1$ with time homogeneous n-step transition probabilities

$$P^{(n)}(x, A) = P(X_n \in A | X_0 = x), \quad x \in R^k,\ A \in \mathcal{B}(R^k),$$

where $\mathcal{B}(R^k)$ is a Borel σ-field on R^k.

A Markov chain $\{X_t\}$ is called V-uniformly ergodic if there exists some probability measure π on $\mathcal{B}(R^k)$ and positive real numbers $r < 1$ and $c > 0$ such that , for all $n \in N$,

$$|||P^{(n)}(x, \cdot) - \pi(\cdot)|||_V \leq cr^n,$$

where $||| \cdot |||_V$ denotes the V-norm distance between two probability measures.

Note that every V-uniformly ergodic process is geometric ergodic and exponential β-mixing and geometric ergodicity are equivalent for Markov processes. Since β-mixing is stronger than strong mixing, a geometrically ergodic Markov process accommodates limiting theorems such as a functional central limit theorem and the law of iterated logarithm for β-mixing process and/or strong mixing process.

For further terminologies and results in the Markov chain theory, we refer to Meyn and Tweedie (1993).

2. Main Results

Consider the following subdiagonal bilinear process;

$$y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=0}^{q} b_i \epsilon_{t-i} + \sum_{i=1}^{P} \sum_{j=0}^{Q} c_{i,j} y_{t-i-j} \epsilon_{t-i}, \tag{2.1}$$

which is the same type of (1.1), where $\{\epsilon_t\}$ is a sequence of iid random variables and a_i, b_i and $c_{i,j}$ are real constants. (2.1) can be represented as

$$y_t = Z_{1,t-1} + b_0 \epsilon_t, \quad Z_t = A(\epsilon_t) Z_{t-1} + B(\epsilon_t), \tag{2.2}$$

where $Z_t = (Z_{1,t}, \ldots, Z_{n,t})' \in R^n$, $n = \max\{p, P+q, P+Q\}$, $A(\epsilon_t) \in R^{n \times n}$ are matrix valued polynomial function and $B(\epsilon_t) \in R^n$ are a vector-valued polynomial function. The precise forms of $Z_{i,t}, A(\epsilon_t)$ and $B(\epsilon_t)$ are given in Section 3.

We make an additional assumption on the error process ϵ_t.

Condition C1: The probability distribution of ϵ_t is absolutely continuous with respect to the Lebesgue measure. The support of ϵ_t is defined by its strictly positive density and contains an open set and zero. Assume that $E[|\epsilon_t|^{2s}] < \infty$ for some $0 < s \leq 1$.

For simplicity of notation, $A(\epsilon_t)$ and $B(\epsilon_t)$ are denoted by A_t and B_t, respectively. Let $\log^+ x = \max\{\log x, 0\}$ and $|| \cdot ||$ be any matrix norm.

For a sequence of iid random matrices $\{A_t\}$ with $E[\log^+ ||A_t||] < \infty$, top Lyapunov exponent γ is defined by

$$\gamma = \lim_{t\to\infty} \frac{1}{t} E\left[\log ||A_t A_{t-1} \cdots A_1||\right]. \tag{2.3}$$

Note that γ does not depend on the matrix norm $|| \cdot ||$.

The following theorem is given in Kistensen (2009).

Theorem 1. *Suppose $\gamma < 0$. Then (2.1) have a unique, strictly stationary ergodic solution given by*

$$y_t^* = Z_{1,t-1}^* + b_0 \epsilon_t,$$
$$Z_t^* = B_t + \sum_{i=1}^{\infty} A_t \cdots A_{t+1-i} B_{t-i}.$$

For any given value Z_0, Z_t converges almost surely to Z_t^* and y_t converges almost surely to y_t^* as $t \to \infty$.

Lemma 1. *Let $\gamma < 0$. Then (1) $\sum_{i=1}^{\infty}(\Pi_{k=0}^{i-1} A_{t-k}) B_{t-i}$ converges a.s. (2) $\Pi_{j=1}^{k} A_j x \to 0, \forall x$ a.s as $k \to \infty$.*

A Markov process $\{Z_t\}_{t=0}^{\infty}$ is said to hold the Foster-Lyapunov drift condition if there exists a positive function V on R^n, a compact set $K \subset R^n$ and real constants $\delta < \infty, \nu > 0$ and $0 < \rho < 1$ such that

$$E[V(Z_{t+1})|Z_t = z] \leq \rho V(z) - \nu, \quad z \in K^c,$$
$$E[V(Z_{t+1})|Z_t = z] \leq \delta, \quad z \in K.$$

Lemma 2. *If $\gamma < 0$, then the Foster-Lyapunov drift condition holds for $\{Z_{t_0 t}\}_{t=0}^{\infty}$ with some integer $t_0 > 0$.*

Let $\rho(A)$ denote the spectral radius of a matrix A. Define $\Phi(z) = z^n - \sum_{i=1}^{n} a_i z^{n-i}$, then $\Phi(z)$ is the characteristic polynomial of $A(0)$. Our main theorem is given as follows:

Theorem 2. *In addition to the condition C1 and $\gamma < 0$, suppose that $\rho(A(0)) < 1$. Then $\{Z_{t_0 t}\}_{t=0}^{\infty}$ in (2.2) is uniformly ergodic and β-mixing with exponential decay. Also, y_t in (2.1) is β-mixing with exponential decay rates and $E|y_t|^r < \infty$ for some $0 < r < s$.*

Under the Assumption *C1* and some rank condition on C_n which is given by $C_n = [A^{n-1}B|A^{n-2}B| \cdots |AB|A]$ for properly defined A and B, geometric ergodicity can be obtained. However, it is not easy to determine whether the rank condition holds or not. (see, *e.g.* Bougerol and Picard, 1992; Kristensen, 2010). In general, it is easier to check $\rho(A(0)) < 1$ than the rank condition.

Lemma 3. *Sufficient conditions for $\gamma < 0$ are :*

(1) $E[\log ||A_t||] < 0$.

(2) $E[||A_t \cdots A_1||^r] < 1,$ *for some* $0 < r < s$.

(3) $\rho(E[A_t A_t'']) < 1$.

Corollary 1. *Let the Condition C1, $\Phi(z) \neq 0$ if $|z| > 1$ and one of (1)~(3) in Lemma 3 hold, then the conclusion in Theorem 2 holds.*

Corollary 2. *If C1 and $\sum |a_i| + E|\epsilon_t| \sum |c_{i,j}| < 1$ hold, then the result of Theorem 2 holds.*

Remark 1. As shown in Kristensen (2009), ARMA model or various GARCH-type processes such as APGARCH, NGARCH, VGARCH, LGARCH and EGARCH are special cases of the subdiagonal bilinear model given in (2.1) and hence Theorem 2 can be applied to obtain conditions for the geometric ergodicity and β-mixing properties for those types of the processes.

We now consider a autoregressive model with bilinear errors defined by

$$X_t = \sum_{i=1}^{r} \phi_i X_{t-i} + \eta_t, \tag{2.4}$$

$$\eta_t = a_0 + \sum_{i=1}^{p} a_i \eta_{t-i} + \sum_{i=0}^{q} b_i \epsilon_{t-i} + \sum_{i=1}^{P} \sum_{j=0}^{Q} c_{i,j} \eta_{t-i-j} \epsilon_{t-i}. \tag{2.5}$$

To make a Markovian representation of the model (2.4)~(2.5), define $Y_t = (X_t, X_{t-1}, \ldots, X_{t-r+1}, Z_{1,t}, \ldots, Z_{n,t})'$, where $Z_t = (Z_{1,t}, \ldots, Z_{n,t})' \in R^n$, $n = \max\{p, P+q, P+Q\}$, $\eta_t = Z_{1,t-1} + b_0\epsilon_t$ and $Z_t = A(\epsilon_t)Z_{t-1} + B(\epsilon_t)$. Then $Y_t = C(\epsilon_t)Y_{t-1} + D(\epsilon_t)$, with matrix valued polynomial function $C(\epsilon_t)$ and vector valued polynomial function $D(\epsilon_t)$ which are given as follows:

$$C(\epsilon_t) = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{r-1} & \phi_r & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & & & & \\ 0 & 1 & \cdots & 0 & 0 & & O_{(r-1)\times n} & & \\ & & \ddots & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & & & & \\ & & O_{n\times r} & & & & A_{n\times n}(\epsilon_t) & & \end{pmatrix}$$

and

$$D(\epsilon_t) = (b_0\epsilon_t, 0, \ldots, 0, B(\epsilon_t))'.$$

Let γ^* be the top Lyapunov exponent of $C(\epsilon_t)$ and let $\Psi(z) = z^r - \sum_{i=1}^{r} \phi_i z^{r-i}$.

Theorem 3. *Under the Conditions C_1, $\gamma^* < 0$ and $\Psi(z)\Phi(z) \neq 0$ if $|z| > 1$, $\{X_t\}$ in (2.4) is β-mixing with exponential decay rates.*

Corollary 3. *If C_1, $\gamma^* < 0$, $\sum |\phi_i| < 1$ and $\sum |a_i| < 1$ are satisfied, then the result in Theorem 3 holds.*

Corollary 4. *Under C_1, $\sum |\phi_i| < 1$ and $\sum |a_i| + E|\epsilon_t| \sum |c_{ij}| < 1$, the β-mixing property of X_t can be derived.*

3. Proofs

To obtain a Markovian representation (2.2) for the process generated by the Equation (2.1), define $\bar{Q} = \max\{q, Q\}$, $\bar{P} = \max\{p - \bar{Q}, P\}$, and

$$Z_{i,t} = (a_i + c_{i,0}\epsilon_t)Z_{1,t-1} + Z_{i+1,t-1} + \sum_{j=1}^{\bar{P}} c_{i,j}\epsilon_t Z_{\bar{Q}+j,t-1} + \left\{(a_i b_0 + b_i)\epsilon_t + b_0 c_{i,0}\epsilon_t^2\right\}, \quad 1 \le i \le \bar{Q} - 1,$$

$$Z_{\bar{Q},t} = \left(a_{\bar{Q}} + c_{\bar{Q},0}\epsilon_t\right)Z_{1,t-1} + \sum_{j=1}^{\bar{P}} \left(a_{\bar{Q}+j} + c_{\bar{Q},j}\epsilon_t\right)Z_{\bar{Q}+j,t-1} + \left\{a_0 + \left(a_{\bar{Q}}b_0 + b_{\bar{Q}}\right)\epsilon_t + b_0 c_{\bar{Q},0}\epsilon_t^2\right\},$$

$$Z_{\bar{Q}+1,t} = Z_{1,t-1} + b_0\epsilon_t,$$
$$Z_{\bar{Q}+i,t} = Z_{\bar{Q}+i-1,t-1}, \quad i = 2, 3, \ldots, n - \bar{Q}.$$

For more details, we may consult Pham (1985, 1986) and Kristensen (2009, Appendix A).

For notational simplicity, we assume that $p = q = P = Q$ by taking a_i, b_i, c_{ij} zero whenever i exceeds p, q Q respectively or j exceeds P. In this case $n = 2p$. The explicit expressions of $A(\epsilon_t)$ and $B(\epsilon_t)$ are given by:

$$A(\epsilon_t)_{2p\times 2p} = \begin{pmatrix} A_{1t} & A_{2t} \\ A_{3t} & A_{4t} \end{pmatrix},$$

where

$$A_{1t} = \begin{pmatrix} a_1 + c_{1,0}\epsilon_t & 1 & 0 & 0 & \cdots & 0 \\ a_2 + c_{2,0}\epsilon_t & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p-1} + c_{p-1,0}\epsilon_t & 0 & 0 & 0 & \cdots & 1 \\ a_p + c_{p,0}\epsilon_t & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A_{2t} = \epsilon_t \cdot \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,p} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p-1,1} & c_{p-1,2} & \cdots & c_{p-1,p} \\ c_{p,1} & c_{p,2} & \cdots & c_{p,p} \end{pmatrix},$$

$$A_{3t} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad A_{4t} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$B(\epsilon_t)_{2p\times 1} = B_0 + B_1\epsilon_t + B_2\epsilon_t^2,$$

where

$$B_0 = (0, \ldots, 0, a_0, 0, \ldots, 0)',$$
$$B_1 = \left(a_1 b_0 + b_1, a_2 b_0 + b_2, \ldots, a_p b_0 + b_p, b_0, 0, \ldots, 0\right)',$$
$$B_2 = \left(b_0 c_{1,0}, b_0 c_{2,0}, \ldots, b_0 c_{p,0}, 0, \ldots, 0\right)'.$$

Proof: *Proof of Theorem 1* See Theorem 1.1 and Theorem 2.5 of Bougerol and Picard (1992) or Theorem 1 of Kristensen (2009). □

Proof: *Proof of Lemma 1* Desired results follow directly from Theorem 1 by taking $Z_0 = x$. □

Proof: *Proof of Lemma 2* From definition of γ in (2.3), $\gamma < 0$ implies the existence of some integer t_0 such that $E[\log \|A_{t_0} \cdots A_1\|] < 0$, and hence $E\|A_{t_0} \cdots A_1\|^r < \rho$ for some $\rho < 1$ and r, $0 < r < s$ (see, Hardy *et al.*, 1952). Now take a test function $V : R^{2p} \to R^+$ as $V(x) = \|x\|^r + 1$. Then by the Minkowski's inequality for integrals, we have that

$$\begin{aligned} E[V(Z_{t_0})|Z_0 = z] &= E\left[\left\|A_{t_0} \cdots A_1 z + \sum_{i=0}^{n_0-1} \left(\Pi_{k=0}^{i-1} A_{t_0 t-k}\right) B_{t_0-i}\right\|^r\right] + 1 \\ &\le E\|A_{t_0} \cdots A_1\|^r \|z\|^r + E\left\|\sum_{i=0}^{t_0-1} \left(\Pi_{k=0}^{i-1} A_{t_0-k}\right) B_{t_0-i}\right\|^r + 1 \\ &\le \rho V(z) + M, \end{aligned} \tag{3.1}$$

where $M = E\| \sum_{i=0}^{t_0-1} (\Pi_{k=0}^{i-1} A_{t_0-k}) B_{t_0-i}\|^r + 1 < \infty$. Clearly,

$$\sup_{z \in K} E[V(Z_{t_0})|V_0 = z] < \infty, \tag{3.2}$$

for any compact set $K \subset R^n$. Hence (3.1) and (3.2) and $V(z) \to \infty$ as $\|z\| \to \infty$ yield the conclusion. □

Proof: *Proof of Theorem 2* Let $Z_0 = z$. Then we have that

$$Z_{t_0} = A_{t_0} \cdots A_1 z + \sum_{i=0}^{t_0-1} \left(\Pi_{k=0}^{i-1} A_{t_0-k}\right) B_{t_0-i}.$$

$\rho(A(0)^t) < 1$ follows from the assumption $\rho(A(0)) < 1$. Therefore combining Lemma 2.1, Lemma 2.2 and Theorem 1 of Carrasco and Chen (2002) (see also, Doukhan (p.97)), we obtain that $\{Z_{t_0 t}\}_{t=0}^{\infty}$ is V-uniformly ergodic and hence $\{Z_t\}$ is β-mixing with exponential decay.

The conditional distribution of (y_t, Z_t) in (2.2) given (y_s, Z_s), $s < t$ depends only on Z_{t-1}. Therefore β-mixing of y_t process follows from that of Z_t with at least the same convergence rate. The existence of moments is obtained by Meyn and Tweedie (1993). □

In the following two corollaries, we provide easy-to-check conditions.

Proof: *Proof of Corollary 1* From assumption on $\Phi(z)$, we have that $\rho(A(0)) < 1$. Applying Lemma 3 and Theorem 2, the desired result is obtained. □

Proof: *Proof of Corollary 2* For $n \times n$ matrix $A = [a_{i,j}]$, denote $|A| = [|a_{i,j}|]$. From assumption $\sum |a_i| + E|\epsilon_t| \sum |c_{i,j}| < 1$ and some simple but tedious calculation, we can show that $\rho(A(0)) < 1$ and $\rho(E|A_1|) < 1$. Moreover $\rho(E|A_1|) < 1$ ensures $\gamma < 0$. Apply Theorem 2 to get the results. □

Proof: *Proof of Theorem 3* The conclusion follows from the fact that $\Psi(z)\Phi(z) \neq 0$ for $|z| > 1$ implies $\rho(C(0)) < 1$. □

Proof: *Proof of Corollary 3* If $\sum |\phi_i| < 1$ and $\sum |a_i| < 1$, then $\rho(C(0)) < 1$. □

Proof: *Proof of Corollary 4* Note that from $\sum |a_i| + E|\epsilon_t| \sum |c_{ij}| < 1$, we have that $\gamma^* < 0$ and $\rho(C(0)) < 1$. Then apply Theorem 3. □

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