

Generalized Multi-Phase Multivariate Ratio Estimators for Partial Information Case Using Multi-Auxiliary Variables

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Abstract

In this paper we propose generalized multi-phase multivariate ratio estimators in the presence of multi-auxiliary variables for estimating population mean vector of variables of interest. Some special cases have been deduced from the suggested estimator in the form of remarks. The expressions for mean square errors of proposed estimators have also been derived. The suggested estimators are theoretically compared and an empirical study has also been conducted.

Keywords: Multi-phase sampling, multivariate ratio estimator, multi-auxiliary variables.

1. Introduction

The estimation of the population mean is constant issue in sampling theory and several efforts have been made to improve the precision of the estimators in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas of ratio, regression and product estimators.

Olkin (1958) was the first author to deal with the problem of estimating the mean of a survey variable when auxiliary variables are made available. John (1969) proposed two multivariate generalizations of ratio and product estimators which actually reduce to the Olkin's (1958) and Singh's (1967a) estimators. Srivastava (1971) proposed a general ratio-type estimator which generates a large class of estimators including most of the estimators up to that time proposed. Sen (1972) developed a multivariate ratio estimator under two-phase sampling using multi-auxiliary variables. Singh and Namjoshi (1988) discussed a class of multivariate regression estimators of population mean of study variable in two-phase sampling.

Ceccon and Diana (1996) provided a multivariate extension of the Naik and Gupta (1991) univariate class of estimators. Ahmed (2003) put forward chain based general estimators using multivariate auxiliary information under multiphase sampling. In the same situation, Perri (2005) recommended some new estimators obtained from Singh's (1965, 1967b) estimators.

In multipurpose surveys, the problem is to estimate population means of several variables simultaneously (Swain, 2000). Tripathi and Khattree (1989) estimated means of several variables of interest, using multi-auxiliary variables, under simple random sampling. Further Tripathi (1989) extended the results to the case of two phase sampling.

We suggest general classes of ratio estimators for estimating the population mean of study variable for two-phase and multi-phase sampling using multi-auxiliary variables when information on all

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multi-auxiliary variables (Full Information Case) or not on all auxiliary variables (No Information Case) is available for population (see Samiuddin and Hanif, 2007).

Before suggesting the estimators we provide Multi-phase sampling scheme and some useful notations and results in the following section.

2. Multi-Phase Sampling Using Multi-Auxiliary Variables

Consider a population of N units. Let Y be the variable of interest and X_1, X_2, \dots, X_q are q auxiliary variables. For multi-phase sampling design let n_h and n_k ($n_h < n_k$) be sample sizes for h^{th} and k^{th} phase respectively. $x_{(h)i}$ and $x_{(k)i}$ denote the i^{th} auxiliary variables from h^{th} and k^{th} phase samples respectively and y_k denote the variable of interest from the k^{th} phase. Let, \bar{X}_i , C_{x_i} and ρ_{y,x_i} denote the population mean, coefficient of variation of i^{th} auxiliary variables respectively and the population correlation coefficient of Y and X_i . Further let $\theta_h = 1/n_h - 1/N$, $\theta_k = 1/n_k - 1/N$. Also $y_{(k)i} = Y + e_{y_{(k)i}}$, $x_{(h)i} = X_i + e_{x_{(h)i}}$ and $x_{(k)i} = X_i + e_{x_{(k)i}}$ ($i = 1, 2, \dots, k$), where $e_{y_{(k)i}}$, $e_{x_{(h)i}}$ and $e_{x_{(k)i}}$ are sampling errors. We assume that $E_k(e_{y_{(k)i}}) = E_h(e_{x_{(h)i}}) = E_k(e_{x_{(k)i}}) = 0$ where E_h and E_k denote the expectations of errors of h^{th} and k^{th} phase sampling respectively. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

$$\begin{aligned} E_k(e_{y_{(k)i}})^2 &= \left(1 - \frac{n_k}{N}\right) \sigma_{y_i}^2 \\ E_k(\bar{e}_{y_{(k)i}})^2 &= \left(1 - \frac{n_k}{N}\right) \frac{\sigma_y^2}{n_k} = \theta_k \bar{Y}_i^2 C_{y_i}^2 \\ E_k(e_{y_{(k)i}} e_{x_{(k)i}}) &= \left(1 - \frac{n_k}{N}\right) \sigma_{y_i x_i} = \theta_k \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_i x_i} \\ E_k(e_{y_{(k)i}} \bar{e}_{x_{(k)i}}) &= \left(1 - \frac{n_k}{N}\right) \frac{\sigma_{y_i x_i}}{n_k} = \theta_k \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_i x_i} \\ E_h E_{k|h} [e_{y_k} (e_{x_{(h)i}} - e_{x_{(k)i}})] &= E_h E_{k|h} (e_{y_{(k)i}} e_{x_{(h)i}}) - E_k (e_{y_{(k)i}} e_{x_{(k)i}}) \\ &= \frac{1}{N} (n_k - n_h) \sigma_{y_i x_i} \\ E_h E_{k|h} [e_{y_{(k)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}})] &= \left(1 - \frac{n_h}{N}\right) \frac{\sigma_{y_i x_i}}{n_h} - \left(1 - \frac{n_k}{N}\right) \frac{\sigma_{y_i x_i}}{n_k} \\ &= (\theta_h - \theta_k) \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_i x_i}. \end{aligned}$$

Similarly

$$\begin{aligned} E_h E_{k|h} [\bar{e}_{x_{(k)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}})] &= (\theta_h - \theta_k) \sigma_{x_i}^2 = (\theta_h - \theta_k) \bar{X}_i^2 C_{x_i}^2 \\ E_h E_{k|h} [\bar{e}_{x_{(h)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}})] &= 0 \\ E_h E_{k|h} [\bar{e}_{x_{(h)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}})]^2 &= (\theta_k - \theta_h) \sigma_{x_i}^2 = (\theta_k - \theta_h) \bar{X}_i^2 C_{x_i}^2 \\ E_h E_{k|h} [(\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}})(\bar{e}_{x_{(h)j}} - \bar{e}_{x_{(k)j}})] &= (\theta_k - \theta_h) \sigma_{x_i x_j} \\ &= (\theta_k - \theta_h) \bar{X}_i \bar{X}_j C_{x_i} C_{x_j} \rho_{x_i x_j}, \quad (i \neq j) \end{aligned}$$

and

$$\begin{aligned} E_h E_{k|h} [(\bar{e}_{x_{(k)i}})(\bar{e}_{x_{(h)j}} - \bar{e}_{x_{(k)j}})] &= (\theta_h - \theta_k) \sigma_{x_i x_j} \\ &= (\theta_h - \theta_k) \bar{X}_i \bar{X}_j C_{x_i} C_{x_j} \rho_{x_i x_j}, \quad (i \neq j). \end{aligned}$$

The following notations in the following paragraph will be used in deriving the mean square errors of proposed estimators.

$|R|_{y_i x_q}$ denotes the determinant of population correlation matrix of variables $y_i, x_1, x_2, \dots, x_{q-1}$ and x_q . $|R|_{y_i x_i} |_{y_i x_q}$ denotes the determinant of i^{th} minor of $|R|_{y_i x_q}$ corresponding to the i^{th} element of $\rho_{y_i x_i}$. $\rho_{y_i x_r}^2$ denotes the multiple coefficient of determination of y on x_1, x_2, \dots, x_{r-1} and x_r . $\rho_{y_i x_q}^2$ denotes the multiple coefficient of determination of y on x_1, x_2, \dots, x_{q-1} and x_q . $|R|_{x_s}$ denotes the determinant of population correlation matrix of variables $x_{r+1}, x_{r+2}, \dots, x_{r+s-1}$ and x_r . $|R|_{x_q}$ denotes the determinant of population correlation matrix of variables x_1, x_2, \dots, x_{q-1} and x_q . $|R|_{y_i x_r}$ denotes the determinant of the correlation matrix of $y_i, x_1, x_2, \dots, x_{r-1}$ and x_r . $|R|_{y_i x_q}$ denotes the determinant of the correlation matrix of $y_i, x_1, x_2, \dots, x_{q-1}$ and x_q . $|R|_{y_i y_j x_r}$ denotes the determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i, y_j, x_1, x_2, \dots, x_{r-1}$ and x_r , for $(i \neq j)$. $|R|_{y_i y_j x_q}$ denotes the determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i, y_j, x_1, x_2, \dots, x_{q-1}$ and x_q , for $(i \neq j)$.

2.1. Result: 1

The following result will help in deriving the mean square errors of suggested estimators

$$\frac{|R|_{y_i x_q}}{|R|_{x_q}} = (1 - \rho_{y_i x_q}^2), \quad (\text{Arora and Lal, 1989}).$$

3. Generalized Multi-Phase Multivariate Ratio Estimator for Partial Information Case

Let we have q auxiliary variable X_1, X_2, \dots, X_q and population means for first r auxiliary variables are not known and for the rest $q - r = s$ auxiliary variables are known. For estimating the mean vector of variables of interest Y_1, Y_2, \dots, Y_p , let $\bar{y}_{(k)i}$ denotes the sample mean of i^{th} study variable for k^{th} phase and $\bar{x}_{(h)i}$ and $\bar{x}_{(k)i}$ denotes the sample mean of i^{th} auxiliary variable for h^{th} and k^{th} phase respectively. The generalized multi-phase multivariate ratio estimator in the presence of, multi-auxiliary variables for partial information case can be suggested as:

$$T_{hk(1 \times p)} = \left[\bar{y}_{(k)1} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(h)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(k)i}} \right)^{\gamma_{i1}} \right. \\ \bar{y}_{(k)2} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(h)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(k)i}} \right)^{\gamma_{i1}} \\ \cdots \bar{y}_{(k)p} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(h)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(k)i}} \right)^{\gamma_{i1}} \left. \right] \quad (3.1)$$

or

$$T_{hk(1 \times p)} = \left[(\bar{Y}_1 + \bar{e}_{y_{(k)1}}) \left(1 + \sum_{i=1}^r \frac{\alpha_{i1}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) \right) \right. \\ \left. \left(1 - \sum_{i=r+1}^{r+s=q} \frac{\beta_{i1}}{\bar{X}_i} \bar{e}_{x_{(h)i}} \right) \left(1 - \sum_{i=r+1}^{r+s=q} \frac{\gamma_{i1}}{\bar{X}_i} \bar{e}_{x_{(k)i}} \right) \right]$$

$$\begin{aligned} & \left(\bar{Y}_p + \bar{e}_{y_{(k)p}} \right) \left(1 + \sum_{i=1}^r \frac{\alpha_{ip}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) \right) \\ & \cdots \left(1 - \sum_{i=r+1}^{r+s=q} \frac{\beta_{ip}}{\bar{X}_i} \bar{e}_{x_{(h)i}} \right) \left(1 - \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ip}}{\bar{X}_i} \bar{e}_{x_{(k)i}} \right) \end{aligned}$$

or

$$\begin{aligned} T_{hk(1 \times p)} = & \left[\begin{array}{cccc} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) & (\bar{Y}_2 + \bar{e}_{y_{(k)2}}) & \cdots & (\bar{Y}_p + \bar{e}_{y_{(k)p}}) \end{array} \right] \\ & + \left[\begin{array}{cccc} \sum_{i=1}^r \frac{\alpha_{i1}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s=q} \frac{\beta_{i1}}{\bar{X}_i} \bar{e}_{x_{(h)i}} - \sum_{i=r+1}^{r+s=q} \frac{\gamma_{i1}}{\bar{X}_i} \bar{e}_{x_{(k)i}} \\ \sum_{i=1}^r \frac{\alpha_{i2}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s=q} \frac{\beta_{i2}}{\bar{X}_i} \bar{e}_{x_{(h)i}} - \sum_{i=r+1}^{r+s=q} \frac{\gamma_{i2}}{\bar{X}_i} \bar{e}_{x_{(k)i}} \\ \vdots \\ - \sum_{i=1}^r \frac{\alpha_{ip}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s=q} \frac{\beta_{ip}}{\bar{X}_i} \bar{e}_{x_{(h)i}} + \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ip}}{\bar{X}_i} \bar{e}_{x_{(k)i}} \end{array} \right] \end{aligned}$$

or

$$\begin{aligned} T_{hk(1 \times p)} = & \left[\begin{array}{cccc} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) & (\bar{Y}_2 + \bar{e}_{y_{(k)2}}) & \cdots & (\bar{Y}_p + \bar{e}_{y_{(k)p}}) \end{array} \right] \\ & + \left[\begin{array}{cccc} \sum_{i=1}^r \frac{\alpha_{i1}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) & \sum_{i=1}^r \frac{\alpha_{i2}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) & \cdots & \sum_{i=1}^r \frac{\alpha_{ip}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) \\ \sum_{i=r+1}^{r+s=q} \frac{\beta_{i1}}{\bar{X}_i} \bar{e}_{x_{(h)i}} & \sum_{i=r+1}^{r+s=q} \frac{\beta_{i2}}{\bar{X}_i} \bar{e}_{x_{(h)i}} & \cdots & \sum_{i=r+1}^{r+s=q} \frac{\beta_{ip}}{\bar{X}_i} \bar{e}_{x_{(h)i}} \\ \sum_{i=r+1}^{r+s=q} \frac{\gamma_{i1}}{\bar{X}_i} \bar{e}_{x_{(k)i}} & \sum_{i=r+1}^{r+s=q} \frac{\gamma_{i2}}{\bar{X}_i} \bar{e}_{x_{(k)i}} & \cdots & \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ip}}{\bar{X}_i} \bar{e}_{x_{(k)i}} \end{array} \right] \end{aligned}$$

or

$$\begin{aligned} T_{hk(1 \times p)} = & \left[\begin{array}{cccc} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) & (\bar{Y}_2 + \bar{e}_{y_{(k)2}}) & \cdots & (\bar{Y}_p + \bar{e}_{y_{(k)p}}) \end{array} \right]_{1 \times p} \\ & + \left[\begin{array}{cccc} (\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}) & (\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}) & \cdots & (\bar{e}_{x_{(h)r}} - \bar{e}_{x_{(k)r}}) \end{array} \right]_{1 \times r} \\ & \left[\frac{\bar{Y}_j}{\bar{X}_i} \alpha_{ij} \right]_{(r \times p)} - \left[\begin{array}{cccc} \bar{e}_{x_{(h)r+1}} & \bar{e}_{x_{(h)r+2}} & \cdots & \bar{e}_{x_{(h)r+s}} \end{array} \right]_{1 \times s} \left[\frac{\bar{Y}_j}{\bar{X}_i} \beta_{ij} \right]_{(s \times p)} \\ & - \left[\begin{array}{cccc} \bar{e}_{x_{(k)r+1}} & \bar{e}_{x_{(k)r+2}} & \cdots & \bar{e}_{x_{(k)r+s}} \end{array} \right]_{1 \times s} \left[\frac{\bar{Y}_j}{\bar{X}_i} \gamma_{ij} \right]_{(s \times p)} \end{aligned}$$

or

$$T_{hk(1 \times p)} = \bar{y}_{1 \times p} + \bar{d}_{x_{hk(1 \times r)}} A_{r \times p} - \bar{d}_{x_{h(1 \times s)}} B_{s \times p} - \bar{d}_{x_{k(1 \times s)}} C_{s \times p}. \quad (3.2)$$

Where

$$\begin{aligned} \bar{d}_{x_{hk}} &= \left[\begin{array}{cccc} (\bar{x}_{(h)1} - \bar{x}_{(k)1}) & (\bar{x}_{(h)2} - \bar{x}_{(k)2}) & \cdots & (\bar{x}_{(h)r} - \bar{x}_{(k)r}) \end{array} \right] \\ &= \left[\begin{array}{cccc} (\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}) & (\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}) & \cdots & (\bar{e}_{x_{(h)r}} - \bar{e}_{x_{(k)r}}) \end{array} \right]_{1 \times r} \end{aligned}$$

$$\begin{aligned}\bar{d}_{x_h} &= \left[\begin{array}{cccc} (\bar{x}_{(h)r+1} - \bar{X}_{r+1}) & (\bar{x}_{(h)r+2} - \bar{X}_{r+2}) & \cdots & (\bar{x}_{(h)r+s} - \bar{X}_{r+s}) \end{array} \right] \\ &= \left[\begin{array}{cccc} \bar{e}_{x_{(h)r+1}} & \bar{e}_{x_{(h)r+2}} & \cdots & \bar{e}_{x_{(h)r+s}} \end{array} \right]_{1 \times s} \\ \bar{d}_{x_k} &= \left[\begin{array}{cccc} (\bar{x}_{(k)r+1} - \bar{X}_{r+1}) & (\bar{x}_{(k)r+2} - \bar{X}_{r+2}) & \cdots & (\bar{x}_{(k)r+s} - \bar{X}_{r+s}) \end{array} \right] \\ &= \left[\begin{array}{cccc} \bar{e}_{x_{(1)2}} & \bar{e}_{x_{(2)2}} & \cdots & \bar{e}_{x_{(2)r}} \end{array} \right]_{1 \times r},\end{aligned}$$

$$\begin{aligned}A &= \left[\frac{\bar{Y}_j}{\bar{X}_i} \alpha_{ij} \right]_{(r \times p)}, \quad \text{for } i = 1, 2, \dots, r, j = 1, 2, \dots, p \\ B &= \left[\frac{\bar{Y}_j}{\bar{X}_i} \beta_{ij} \right]_{(s \times p)}, \quad \text{for } i = r+1, r+2, \dots, r+s, j = 1, 2, \dots, p.\end{aligned}$$

and

$$C = \left[\frac{\bar{Y}_j}{\bar{X}_i} \gamma_{ij} \right]_{(s \times p)}, \quad \text{for } i = r+1, r+2, \dots, r+s, j = 1, 2, \dots, p.$$

Letting, $\bar{y} = \bar{Y} + \bar{d}_y$, where and $\bar{d}_y = [\bar{e}_{y_{(k)1}} \bar{e}_{y_{(k)2}} \cdots \bar{e}_{y_{(k)p}}]$. We can write (3.2) as:

$$T_{hk(1 \times p)} = \bar{Y} + \bar{d}_y + \bar{d}_{x_{hk}} A - \bar{d}_{x_h} B - \bar{d}_{x_k} C.$$

We use information related to auxiliary variables from first and second phase both then the mean square error of $T_{hk(1 \times p)}$ can be written as:

$$\begin{aligned}\Sigma_{T_{hk}(p \times p)} &= E_1 E_{2/1} (T_{hk} - \bar{Y})' (T_{hk} - \bar{Y}) \\ &= E_h E_{k/h} \left[(\bar{d}_y + \bar{d}_{x_{hk}} A - \bar{d}_{x_h} B - \bar{d}_{x_k} C)' (\bar{d}_y + \bar{d}_{x_{hk}} A - \bar{d}_{x_h} B - \bar{d}_{x_k} C) \right].\end{aligned}\quad (3.3)$$

We can write

$$\begin{aligned}E_h E_{k/h} (\bar{d}_y' \bar{d}_y) &= \theta_k \Sigma_y = \theta_k [\sigma_{y_i y_j}]_{(p \times p)}, \quad \text{for } i = j, \sigma_{y_i y_j} = \sigma_{y_i}^2 \\ E_h E_{k/h} (\bar{d}_y' d_{x_h}) &= \theta_h \Sigma_{yx} = \theta_h [\sigma_{y_i x_j}]_{(p \times s)} \\ E_h E_{k/h} (\bar{d}_y' d_{x_k}) &= \theta_k \Sigma_{yx} = \theta_k [\sigma_{y_i x_j}]_{(p \times s)} \\ E_h E_{k/h} (\bar{d}_y' d_{x_{hk}}) &= (\theta_k - \theta_h) \Sigma_{yx} = (\theta_k - \theta_h) [\sigma_{y_i x_j}]_{(p \times r)} \\ E_h E_{k/h} (d_{x_h}' d_{x_h}) &= \theta_h \Sigma_x = \theta_h [\sigma_{x_i x_j}]_{(s \times s)}, \quad \text{for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{k/h} (d_{x_k}' d_{x_k}) &= \theta_k \Sigma_x = \theta_k [\sigma_{x_i x_j}]_{(s \times s)}, \quad \text{for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{k/h} (d_{x_{hk}}' d_{x_{hk}}) &= (\theta_k - \theta_h) \Sigma_x = (\theta_k - \theta_h) [\sigma_{x_i x_j}]_{(r \times r)}, \quad \text{for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{k/h} (d_{x_h}' d_{x_k}) &= \theta_h \Sigma_x = \theta_h [\sigma_{x_i x_j}]_{(s \times s)}, \quad \text{for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{k/h} (d_{x_h}' d_{x_{hk}}) &= 0 \\ E_h E_{k/h} (d_{x_k}' d_{x_{hk}}) &= (\theta_h - \theta_k) \Sigma_x = (\theta_h - \theta_k) [\sigma_{x_i x_j}]_{(s \times r)}, \quad \text{for } i = j, \sigma_{x_i x_j} = \sigma_{x_i}^2.\end{aligned}$$

Using above substitutions in expression of variance covariance matrix given in (3.3), we write:

$$\begin{aligned}
 \Sigma_{T_{hk}(p \times p)} &= \theta_k \Sigma_{y(p \times p)} - \theta_h \Sigma_{yX(p \times r)} A_{(r \times p)} - \theta_k \Sigma_{yX(p \times s)} B_{(s \times p)} \\
 &\quad + (\theta_h - \theta_k) \Sigma_{yX(p \times r)} C_{(r \times p)} - \theta_1 B'_{(p \times s)} \Sigma'_{yX(s \times p)} + \theta_1 B'_{(p \times s)} \Sigma_{X(s \times s)} B_{(s \times p)} \\
 &\quad + \theta_h B'_{(p \times s)} \Sigma_{X(s \times s)} C_{(s \times p)} - \theta_k C'_{(p \times s)} \Sigma'_{yX(s \times p)} + \theta_1 C'_{(p \times s)} \Sigma_{X(s \times s)} B_{(s \times p)} \\
 &\quad + \theta_k C'_{(p \times s)} \Sigma_{X(s \times s)} C_{(s \times p)} - (\theta_h - \theta_k) C_{(p \times s)} \Sigma_{X(s \times r)} A_{(r \times p)} \\
 &\quad + (\theta_h - \theta_k) A_{(p \times r)} \Sigma'_{yX(r \times p)} - (\theta_h - \theta_k) A'_{(p \times r)} \Sigma'_{X(r \times s)} C_{(s \times p)} \\
 &\quad + (\theta_k - \theta_h) A'_{(p \times r)} \Sigma_{X(r \times r)} A_{(r \times p)}. \tag{3.4}
 \end{aligned}$$

Differentiating (3.4) with respect to the matrices on unknown constants A , B and C and equating to zero, solving these equations for optimum values of A , B and C we get:

$$A_{(r \times p)} = W_{X(r \times r)}^{-1} \left(\Sigma'_{yX(r \times p)} - \Sigma_{X(s \times s)} \Sigma_{yX(s \times p)}^{-1} \Sigma'_{yX(s \times p)} \right) \tag{3.5}$$

$$B_{(s \times p)} = \left(\Sigma_{X(s \times s)}^{-1} \Sigma_{X(s \times r)} W_{X(r \times r)}^{-1} \Sigma'_{yX(r \times p)} \right) - \left(\Sigma_{X(s \times s)}^{-1} \Sigma_{X(s \times r)} W_{X(r \times r)}^{-1} \Sigma_{X(r \times s)} \Sigma_{X(s \times s)}^{-1} \Sigma'_{yX(s \times p)} \right) \tag{3.6}$$

and

$$C_{(s \times p)} = \left(\Sigma_{X(s \times s)}^{-1} \Sigma'_{yX(s \times p)} + \Sigma_{X(s \times s)}^{-1} \Sigma_{X(s \times r)} W_{X(r \times r)}^{-1} \Sigma_{X(r \times s)} \Sigma_{X(s \times s)}^{-1} \Sigma'_{yX(s \times p)} \right) - \left(\Sigma_{X(s \times s)}^{-1} \Sigma_{X(s \times r)} W_{X(r \times r)}^{-1} \Sigma'_{yX(r \times p)} \right). \tag{3.7}$$

Using the above expressions of A , B and C in (3.4), we can write the variance covariance matrix of T_{hk} as:

$$\begin{aligned}
 \Sigma_{T_{hk}(p \times p)} &= \theta_k \left(\Sigma_{y(p \times p)} - \Sigma_{yX(p \times s)} \Sigma_{X(s \times s)}^{-1} \Sigma'_{yX(s \times p)} \right) \\
 &\quad - (\theta_k - \theta_h) \left(\Sigma_{yX(p \times r)} - \Sigma_{yX(p \times s)} \Sigma_{X(s \times s)}^{-1} \Sigma_{X(s \times r)} \right) \\
 &\quad W_{X(r \times r)}^{-1} \left(\Sigma'_{yX(r \times p)} - \Sigma_{X(r \times s)} \Sigma_{X(s \times s)}^{-1} \Sigma'_{yX(s \times p)} \right), \tag{3.8}
 \end{aligned}$$

where $W_{X(r \times r)}^{-1} = (\Sigma'_{yX(r \times r)} - \Sigma_{X(r \times s)} \Sigma_{X(s \times s)}^{-1} \Sigma'_{X(s \times r)})^{-1}$ provided $\Sigma_{X(r \times r)}^{-1}$, $W_{X(s \times s)}^{-1}$ and $\Sigma_{X(s \times s)}^{-1}$ exist.

The variance covariance matrix in the form of variance of y_i , covariances and correlation coefficients of x_i and y_i can be written as:

$$\Sigma_{T_{hk}(p \times p)} = \left[\sigma_{y_i} \sigma_{y_j} \left\{ \theta_k (\rho_{y_i y_j} - \rho_{y_i y_j, \bar{x}_q}) + \theta_h (\rho_{y_i y_j, \bar{x}_q} - \rho_{y_i y_j, \bar{x}_s}) \right\} \right]_{p \times p}; \quad (i, j = 1, 2, \dots, p) \tag{3.9}$$

for $i = j$, $\sigma_{y_i} \sigma_{y_j} = \sigma_i^2$, $\rho_{y_i y_j} = 1$, $\rho_{y_i y_j, \bar{x}_q} = \rho_{y_i y_j, \bar{x}_s}^2$ and $\rho_{y_i y_j, \bar{x}_s} = \rho_{y_i, \bar{x}_s}^2$.

In determinants of correlation matrices for $|R|_{\bar{x}_r} \neq 0$ and $|R|_{\bar{x}_q} \neq 0$, (3.9) can be written as:

$$\Sigma_{T_{hk}(p \times p)} = \left[\sigma_{y_i} \sigma_{y_j} \left\{ \theta_k \frac{|R|_{y_i y_j, \bar{x}_q}}{|R|_{\bar{x}_q}} + \theta_h \left(\frac{|R|_{y_i y_j, \bar{x}_s}}{|R|_{\bar{x}_s}} - \frac{|R|_{y_i y_j, \bar{x}_q}}{|R|_{\bar{x}_q}} \right) \right\} \right]; \quad (i, j = 1, 2, \dots, p), \tag{3.10}$$

for $i = j$, $\sigma_{y_i} \sigma_{y_j} = \sigma_i^2$, $|R|_{y_i y_j, \bar{x}_q} = |R|_{y_i, \bar{x}_q}$ and $|R|_{y_i y_j, \bar{x}_s} = |R|_{y_i, \bar{x}_s}$.

3.1. Remark: 1

To develop generalized multivariate ratio estimator for two-phase sampling using multi-auxiliary variables for Partial Information Case, replace h by 1 and k by 2 in (3.1), we get the following estimator

$$T_{12(1 \times p)} = \left[\bar{y}_{(2)1} \prod_{i=1}^r \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(1)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\gamma_{i1}} \right]$$

$$\bar{y}_{(2)2} \prod_{i=1}^r \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(1)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\gamma_{i1}} \\ \cdots \bar{y}_{(2)p} \prod_{i=1}^r \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(1)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\gamma_{i1}} \Big]. \quad (3.11)$$

The expressions of unknown matrices for which the mean square error of above estimator will be minimum are same as given in (3.5), (3.6) and (3.7). The expression for variance covariance matrix can be directly written from (3.8) just replacing h by 1 and k by 2 as:

$$\Sigma_{T_{12}(p \times p)} = \theta_2 \left(\Sigma_{y_{(p \times p)}} - \Sigma_{y_{(p \times s)}} \Sigma_{x_{(s \times s)}}^{-1} \Sigma'_{y_{(s \times p)}} \right) \\ - (\theta_2 - \theta_1) \left(\Sigma_{y_{(p \times r)}} - \Sigma_{y_{(p \times s)}} \Sigma_{x_{(s \times s)}}^{-1} \Sigma_{x_{(s \times r)}} \right) \\ W_{x_{(r \times r)}}^{-1} \left(\Sigma'_{y_{(r \times p)}} - \Sigma_{x_{(r \times s)}} \Sigma_{x_{(s \times s)}}^{-1} \Sigma'_{y_{(s \times p)}} \right). \quad (3.12)$$

The variance covariance matrix in the form of variance of y_i , covariances and correlation coefficients of x_i and y_i is written as:

$$\Sigma_{T_{12}(p \times p)} = [\sigma_{y_i} \sigma_{y_j} \{ \theta_2 (\rho_{y_i y_j} - \rho_{y_i y_j, x_q}) + \theta_1 (\rho_{y_i y_j, x_q} - \rho_{y_i y_j, x_s}) \}]_{p \times p}; \quad (i, j = 1, 2, \dots, p), \quad (3.13)$$

for $i = j$, $\sigma_{y_i} \sigma_{y_j} = \sigma_i^2$, $\rho_{y_i y_j} = 1$, $\rho_{y_i y_j, x_q} = \rho_{y_i, x_q}^2$ and $\rho_{y_i y_j, x_s} = \rho_{y_i, x_s}^2$.

In determinants of correlation matrices for $|R|_{\tilde{x}_r} \neq 0$ and $|R|_{\tilde{x}_q} \neq 0$, (3.13) can be written as:

$$\Sigma_{T_{12}(p \times p)} = \left[\sigma_{y_i} \sigma_{y_j} \left\{ \theta_2 \frac{|R|_{y_i y_j x_q}}{|R|_{\tilde{x}_q}} + \theta_1 \left(\frac{|R|_{y_i y_j x_s}}{|R|_{\tilde{x}_s}} - \frac{|R|_{y_i y_j x_q}}{|R|_{\tilde{x}_q}} \right) \right\} \right]; \quad (i, j = 1, 2, \dots, p), \quad (3.14)$$

for $i = j$, $\sigma_{y_i} \sigma_{y_j} = \sigma_i^2$, $|R|_{y_i y_j x_q} = |R|_{y_i x_q}$ and $|R|_{y_i y_j x_s} = |R|_{y_i x_s}$.

3.2. Remark: 2

We can develop a univariate generalized ratio estimator for multiphase sampling using multi auxiliary variable for Partial Information Case if we put $p = 1$ in (3.1) as:

$$T_{hk} = \bar{y}_{(k)} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(h)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s=q} \left(\frac{\bar{X}_i}{\bar{x}_{(k)i}} \right)^{\gamma_{i1}}. \quad (3.15)$$

The expression for vectors of unknown constants for which the mean square error will be minimum can be written from (3.5), (3.6) and (3.7) as

$$A_{(r \times 1)} = W_{x_{(r \times r)}}^{-1} \left(\Sigma'_{y_{(r \times 1)}} - \Sigma_{x_{(r \times s)}} \Sigma_{x_{(s \times s)}}^{-1} \Sigma'_{y_{(s \times 1)}} \right) \quad (3.16)$$

$$B_{(s \times 1)} = \left(\Sigma_{x_{(s \times s)}}^{-1} \Sigma_{x_{(s \times r)}} W_{x_{(r \times r)}}^{-1} \Sigma'_{y_{(r \times 1)}} \right) - \left(\Sigma_{x_{(s \times s)}}^{-1} \Sigma_{x_{(r \times s)}} W_{x_{(r \times r)}}^{-1} \Sigma_{x_{(s \times r)}} \Sigma_{x_{(s \times s)}}^{-1} \Sigma'_{y_{(s \times 1)}} \right) \quad (3.17)$$

and

$$C_{(s \times 1)} = \left(\Sigma_{x_{(s \times s)}}^{-1} \Sigma'_{y_{(s \times 1)}} + \Sigma_{x_{(s \times s)}}^{-1} \Sigma_{x_{(s \times r)}} W_{x_{(r \times r)}}^{-1} \Sigma_{x_{(r \times s)}} \Sigma_{x_{(s \times s)}}^{-1} \Sigma'_{y_{(s \times 1)}} \right) - \left(\Sigma_{x_{(s \times s)}}^{-1} \Sigma_{x_{(s \times r)}} W_{x_{(r \times r)}}^{-1} \Sigma'_{y_{(r \times 1)}} \right). \quad (3.18)$$

The above expressions for unknown matrices can be written in determinants form as:

$$\alpha_i = (-1)^{i+1} \frac{\bar{Y}}{\bar{X}_i} \frac{C_y}{C_{x_i}} \frac{|R_{yx_i}|_{yx_q}}{|R|_{x_q}} = (-1)^{i+1} \beta_{yx_i \cdot x_q}, \quad (i = 1, 2, \dots, r) \quad (3.19)$$

$$\beta_i = (-1)^{i+1} \frac{\bar{Y}}{\bar{X}_i} \frac{C_y}{C_{x_i}} \left\{ \frac{|R_{yx_i}|_{yx_s}}{|R|_{x_s}} - \frac{|R_{yx_i}|_{yx_q}}{|R|_{x_q}} \right\} \quad (3.20)$$

$$= (-1)^{i+1} (\beta_{yx_i \cdot x_s} - \beta_{yx_i \cdot x_q}), \quad (i = r+1, r+2, \dots, r+s)$$

$$\gamma_i = (-1)^{i+1} \frac{\bar{Y}}{\bar{X}_i} \frac{C_y}{C_{x_i}} \frac{|R_{yx_i}|_{yx_q}}{|R|_{x_q}} = (-1)^{i+1} \beta_{yx_i \cdot x_q}, \quad (i = r+1, r+2, \dots, r+s). \quad (3.21)$$

The expression for mean square error can be directly written from (3.8) as:

$$\begin{aligned} \text{MSE}(T_{hk}) &= \theta_k \left(\sigma_y^2 - \Sigma_{yX_{(1 \times s)}} \Sigma_{X_{(s \times s)}}^{-1} \Sigma'_{yX_{(s \times 1)}} \right) \\ &\quad - (\theta_k - \theta_h) \left(\Sigma_{yX_{(1 \times r)}} - \Sigma_{yX_{(1 \times s)}} \Sigma_{X_{(s \times s)}}^{-1} \Sigma_{X_{(s \times r)}} \right) \\ &\quad W_{X_{(r \times r)}}^{-1} \left(\Sigma'_{yX_{(r \times 1)}} - \Sigma_{X_{(r \times s)}} \Sigma_{X_{(s \times s)}}^{-1} \Sigma'_{yX_{(s \times 1)}} \right). \end{aligned} \quad (3.22)$$

It can be written the form of multiple coefficient of determination as:

$$\text{MSE}(T_{hk}) = \bar{Y}^2 C_y^2 \left[\theta_k \left(1 - \rho_{y \cdot x_q}^2 \right) + \theta_h \left(\rho_{y \cdot x_q}^2 - \rho_{y \cdot x_s}^2 \right) \right]. \quad (3.23)$$

3.3. Remark: 3

To develop a generalized univariate ratio estimator for two phase sampling using multi-auxiliary variables for Partial Information Case we put $h = 1$ and $k = 2$ in (3.15). The required estimator becomes

$$T_{12} = \bar{y}_{(2)} \prod_{i=1}^r \left(\frac{\bar{x}_{(1)i}}{\bar{x}_{(2)i}} \right)^{\alpha_{i1}} \prod_{i=r+1}^{r+s} \left(\frac{\bar{X}_i}{\bar{x}_{(1)i}} \right)^{\beta_{i1}} \prod_{i=r+1}^{r+s} \left(\frac{\bar{X}_i}{\bar{x}_{(2)i}} \right)^{\gamma_{i1}}. \quad (3.24)$$

The expression for vectors of unknown constants for which the mean square error will be minimum are same as given in (3.16), (3.17) and (3.18), and these expressions are also given in determinants of correlation matrices in (3.19), (3.20) and (3.21). The expression for mean square error can be written from (3.22) just by replacing $h = 1$ and $k = 2$ as:

$$\begin{aligned} \text{MSE}(T_{12}) &= \theta_2 \left(\sigma_y^2 - \Sigma_{yX_{(1 \times s)}} \Sigma_{X_{(s \times s)}}^{-1} \Sigma'_{yX_{(s \times 1)}} \right) \\ &\quad - (\theta_2 - \theta_1) \left(\Sigma_{yX_{(1 \times r)}} - \Sigma_{yX_{(1 \times s)}} \Sigma_{X_{(s \times s)}}^{-1} \Sigma_{X_{(s \times r)}} \right) \\ &\quad W_{X_{(r \times r)}}^{-1} \left(\Sigma'_{yX_{(r \times 1)}} - \Sigma_{X_{(r \times s)}} \Sigma_{X_{(s \times s)}}^{-1} \Sigma'_{yX_{(s \times 1)}} \right). \end{aligned} \quad (3.25)$$

It can be written the form of multiple coefficient of determination as:

$$\text{MSE}(T_{12}) = \bar{Y}^2 C_y^2 \left[\theta_2 \left(1 - \rho_{y \cdot x_q}^2 \right) + \theta_1 \left(\rho_{y \cdot x_q}^2 - \rho_{y \cdot x_s}^2 \right) \right]. \quad (3.26)$$

3.4. Remark: 4

Generalized multivariate ratio estimator as suggested by Hanif *et al.* (2009) for multi-phase sampling using multi-auxiliary variables when information on all auxiliary variables is not available for population (No Information Case) can be developed by putting β'_i 's and γ'_i 's equals to zero in (3.1) as:

$$T'_{hk(1 \times p)} = \left[\bar{y}_{(k)1} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i1}} \quad \bar{y}_{(k)2} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{i2}} \quad \dots \quad \bar{y}_{(k)p} \prod_{i=1}^r \left(\frac{\bar{x}_{(h)i}}{\bar{x}_{(k)i}} \right)^{\alpha_{ip}} \right]. \quad (3.27)$$

The expression of unknown matrix for which the mean square error will be minimum can be directly obtained by considering only those matrices from (3.5), (3.6) and (3.7) those includes only order $p \times r$ and $r \times r$ than we get the required matrix that is $\Sigma_{x(r \times r)}^{-1} \Sigma'_{y,x(r \times p)}$. The variance covariance matrix can be obtained from (3.8) just by considering those matrices having order $p \times r$ and $r \times r$. The variance covariance matrix is

$$\Sigma_{T'_{hk}(p \times p)} = \theta_k \Sigma_{y(p \times p)} - (\theta_k - \theta_h) \Sigma'_{yx(p \times r)} \Sigma_{x(r \times r)}^{-1} \Sigma_{yx(r \times p)}. \quad (3.28)$$

All special cases of estimator given in (3.27), in the case on no information and full information, have been discussed by Hanif *et al.* (2009).

4. Empirical Study of Newly Developed Estimators

It can be empirically investigated that the estimator constructed for full information case will be more efficient than the estimator developed for partial information case and the estimator for partial information case will be more efficient than the estimator for no information case. In the case of multiphase, the estimator will be less efficient by increasing the phases but cost will be reduced.

For empirical investigation we use determinants of variance covariance matrices/MSE. We use the data of 1998 census reports of province Punjab, Pakistan for four districts Jhang, Gujrat, Kasur and Sialkot. The detail of populations and variables description is given in Table 1 and 2 respectively of Appendix. We consider three variables of interests denoted by Y 's and five auxiliary variables denoted by X 's for computing the determinants of variance covariance matrices multivariate ratio estimators. For univariate ratio estimators we consider Y_2 as study variable and the same five auxiliary variables as considered in multivariate case. The necessary population parameters for computing variance covariance matrices/MSE's are given in Tables 3, 4, 5, 6 and 7. We calculate pair-wise determinant of variance covariance matrices/MSE for no information case because in this case two phases at a time can be used. For full information case we calculate variance covariance matrices /MSE's for each five phase separately as it is admissible. The determinants of variance covariance matrices of multivariate ratio estimators for multiphase sampling using pair-wise phases for no information case are given in Tables 8 and 9, for partial information case, in Tables 10 and 11 and using each phase for full information case in Table 12. The mean square errors of univariate estimators for multiphase sampling using pair-wise phases for no information case are given in Tables 13 and 14 and for partial information case in Tables 15 and 16 and for full information case using each phase in Table 17.

From Tables 8, 9, 10, 11, and 12, we can say that the multivariate ratio estimators for full information case are more efficient than partial information case and estimators for partial information case are more efficient than no information case for each phase *e.g.* T2 is more efficient than T12, T3 is more efficient than T13 & T23 etc. and the same is true for univariate ratio estimators (see Tables 13, 14, 15, 16 and 17). Furthermore we can say for no information case and partial information case from Tables 8, 9, 10 and 11 that as we increase phases the efficiency decreases *e.g.* T12, is more efficient

than all others, T13 is more efficient than all others except T12, T34 is more efficient than T35, T45 but less efficient than all others and so on, similarly the same argument can be made for univariate case from Tables Tables 13 and 14. Also for full information case the estimators become less efficient as we increase phases because the sample size decreases by increasing phases, it can be seen from Tables 10, 11, 15 and 16 for multivariate and univariate estimators respectively.

Appendix

Table 1: Detail of Populations

Sr. No.	Source of Populations
1	Population census report of Jhang district (1998), Pakistan
2	Population census report of Gujrat district (1998), Pakistan
3	Population census report of Kasur (1998), Pakistan
4	Population census report of Sialkot district (1998), Pakistan

Table 2: Description of variables (Each variables is taken from Rural Locality)

Y_1	Literacy ratio	X_2	Population of primary but below matric
Y_2	Population of currently married	X_3	Population of matric and above
Y_3	Total household	X_4	Population of 18 years old and above
X_1	Population of both sexes	X_5	Population of women 15–49 years old

Table 3: Parameters of populations

District	N	n_1	n_2	n_3	n_4	n_5	\bar{Y}_1	\bar{Y}_2	\bar{Y}_3	C_{y_1}	C_{y_2}	C_{y_3}
Jhang	368	184	92	46	23	12	29.70	860.11	897.71	0.27	0.59	0.51
Gujrat	204	102	51	26	13	6	57.53	1101.28	1102.54	0.14	0.48	0.49
Kasur	181	91	45	23	11	6	31.89	1393.20	1449.02	0.75	0.55	0.53
Sailkot	269	135	67	34	17	8	52.06	1058.74	998.22	0.15	0.65	0.65

Table 4: Parameters of populations (Cont...)

District	σ_{y_1}	σ_{y_2}	σ_{y_3}	σ_{x_1}	σ_{x_2}	σ_{x_3}	σ_{x_4}	σ_{x_5}	$\rho_{y_1 y_2}$
Jhang	8.02	511.91	459.84	5626.45	455.06	170.67	2455.17	1064.48	.182
Gujrat	8.36	533.04	537.24	3507.16	940.48	381.69	8139.68	830.01	.055
Kasur	23.82	767.64	767.80	5515.42	1095.69	357.89	2719.21	1355.64	.295
Sailkot	7.64	685.02	644.89	4787.25	1172.71	603.22	2461.59	1151.32	.324

Table 5: Parameters of populations (Cont...)

District	$\rho_{y_1 y_3}$	$\rho_{y_2 y_3}$	$\rho_{y_1 x_1}$	$\rho_{y_1 x_2}$	$\rho_{y_1 x_3}$	$\rho_{y_1 x_4}$	$\rho_{y_1 x_5}$	$\rho_{y_2 x_1}$	$\rho_{y_2 x_2}$	$\rho_{y_2 x_3}$	$\rho_{y_2 x_4}$	$\rho_{y_2 x_5}$
Jhang	.164	.733	.131	.460	.548	.185	.129	.428	.912	.659	.484	.425
Gujrat	.056	.988	.092	.334	.543	.069	.103	.995	.941	.764	.490	.996
Kasur	.301	.989	.299	.255	.352	.301	.250	.998	.758	.879	.989	.799
Sailkot	.316	.997	.323	.426	.461	.338	.313	.999	.983	.931	.996	.939

Table 6: Parameters of populations (Cont...)

District	$\rho_{y_3 x_1}$	$\rho_{y_3 x_2}$	$\rho_{y_3 x_3}$	$\rho_{y_3 x_4}$	$\rho_{y_3 x_5}$	$\rho_{x_1 x_2}$	$\rho_{x_1 x_3}$	$\rho_{x_1 x_4}$
Jhang	.474	.732	.748	.559	.489	.416	.421	.317
Gujrat	.984	.933	.749	.487	.986	.954	.796	.509
Kasur	.991	.752	.878	.988	.792	.764	.889	.993
Sailkot	.996	.980	.933	.994	.938	.983	.931	.997

Table 7: Parameters of populations (Cont...)

District	$\rho_{x_1x_5}$	$\rho_{x_2x_3}$	$\rho_{x_2x_4}$	$\rho_{x_2x_5}$	$\rho_{x_3x_4}$	$\rho_{x_3x_5}$	$\rho_{x_4x_5}$	$\rho_{y,x_1\dots x_5}^2$
Jhang	.275	.824	.475	.432	.590	.464	.325	0.885
Gujrat	.996	.892	.500	.958	.420	.797	.505	0.996
Kasur	.802	.798	.764	.614	.896	.719	.797	0.995
Sialkot	.939	.959	.985	.928	.939	.887	.938	0.997

Table 8: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (No Information Case)

District	T12 ($h = 1, k = 2$)	T13 ($h = 1, k = 3$)	T14 ($h = 1, k = 4$)	T15 ($h = 1, k = 5$)	T23 ($h = 2, k = 3$)
Jhang	212279.97	1223241.60	7221747.65	46128460.86	3572866.63
Gujrat	95363.18	312081.54	1102996.85	4211396.91	1123210.67
Kasur	203091.03	901801.71	3838230.04	16192403.14	2176260.22
Sialkot	9555.27	41464.31	173806.26	723929.58	126175.45

Table 9: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (No Information Case)

District	T24 ($h = 2, k = 4$)	T25 ($h = 2, k = 5$)	T34 ($h = 3, k = 4$)	T35 ($h = 3, k = 5$)	T45 ($h = 4, k = 5$)
Jhang	16011134.82	79961159.68	38868782.02	158093587.70	2.60491E+11
Gujrat	3372979.29	11251537.17	10702183.50	30944428.16	93069748.63
Kasur	8973735.68	36805031.49	19922737.86	79389873.19	170075469.89
Sialkot	482925.70	1897366.87	1256890.23	4554991.24	11150157.88

Table 10: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (Partial Information Case)

District	T12 ($h = 1, k = 2$)	T13 ($h = 1, k = 3$)	T14 ($h = 1, k = 4$)	T15 ($h = 1, k = 5$)	T23 ($h = 2, k = 3$)
Jhang	138163.6	890963.2	5820643.1	40379695.2	2250013.1
Gujrat	1683.1	10844.4	73847.7	534079.5	18218.9
Kasur	247034.4	1500728.6	9605459.6	66188757.3	2573608.2
Sialkot	322.1	2207.0	15843.0	118825.5	3970.3

Table 11: Determinants of variance covariance matrices of multivariate ratio estimators for pair-wise phases (Partial Information Case)

District	T24 ($h = 2, k = 4$)	T25 ($h = 2, k = 5$)	T34 ($h = 3, k = 4$)	T35 ($h = 3, k = 5$)	T45 ($h = 4, k = 5$)
Jhang	11146370.3	61382546.8	24203288.5	108298411.4	222062545.3
Gujrat	103581.9	653434.5	167622.1	901358.8	1434473.5
Kasur	14114833.3	84669409.0	23285985.9	121940138.0	197720243.3
Sialkot	22574.0	145036.8	38559.8	202554.0	338300.7

Table 12: Determinants of variance covariance matrices of multivariate ratio estimators for each phase (Full Information Case)

District	T1 ($k = 1$)	T2 ($k = 2$)	T3 ($k = 3$)	T4 ($k = 4$)	T5 ($k = 5$)
Jhang	1023.378901	27631.23032	351018.963	3453903.79	30487480.83
Gujrat	27.15981853	367.7474988	3710.595857	33124.8694	279522.8025
Kasur	103.7803005	1306.215104	12804.97189	112839.0944	946342.2579
Sialkot	2.156156072	36.85754751	405.2293669	3755.836241	32256.39965

Table 13: MSE's of univariate ratio estimators for pair-wise phases (No Information Case)

District	T12 ($h = 1, k = 2$)	T13 ($h = 1, k = 3$)	T14 ($h = 1, k = 4$)	T15 ($h = 1, k = 5$)	T23 ($h = 2, k = 3$)
Jhang	212279.9	1223241.6	7221747.6	46128460.8	3572866.6
Gujrat	95363.1	312081.5	1102996.8	4211396.9	1123210.6
Kasur	203091.0	901801.7	3838230.0	16192403.1	2176260.2
Sialkot	9555.27	41464.3	173806.26	723929.5	126175.4

Table 14: MSE's of univariate ratio estimators for pair-wise phases (No Information Case)

District	T24 ($h = 2, k = 4$)	T25 ($h = 2, k = 5$)	T34 ($h = 3, k = 4$)	T35 ($h = 3, k = 5$)	T45 ($h = 4, k = 5$)
Jhang	16011134.8	79961159.68	38868782.02	158093587.7	2.60491E+11
Gujrat	3372979.2	11251537.17	10702183.50	30944428.16	93069748.63
Kasur	8973735.6	36805031.49	19922737.86	79389873.19	170075469.89
Sialkot	482925.7	1897366.87	1256890.23	4554991.24	11150157.88

Table 15: MSE's of univariate ratio estimators for pair-wise phases (No Information Case)

District	T12 ($h = 1, k = 2$)	T13 ($h = 1, k = 3$)	T14 ($h = 1, k = 4$)	T15 ($h = 1, k = 5$)	T23 ($h = 2, k = 3$)
Jhang	652.99	980.5526	1635.678	2945.928	1795.189
Gujrat	309.4727	900.3699	2082.164	4445.753	624.3232
Kasur	378.5487	1044.532	2376.499	5040.433	771.9548
Sialkot	234.4814	682.5967	1578.827	3371.289	474.9672

Table 16: MSE's of univariate ratio estimators for pair-wise phases (No Information Case)

District	T24 ($h = 2, k = 4$)	T25 ($h = 2, k = 5$)	T34 ($h = 3, k = 4$)	T35 ($h = 3, k = 5$)	T45 ($h = 4, k = 5$)
Jhang	2450.314	4079.586	5389.837	8648.382	4079.586
Gujrat	1806.117	1254.024	3617.613	2513.426	1254.024
Kasur	2103.922	1558.767	4222.701	3132.391	1558.767
Sialkot	1371.198	955.9386	2748.4	1917.882	955.9386

Table 17: MSE's of univariate ratio estimators for each-wise phase (Full Information Case)

District	T1 ($k = 1$)	T2 ($k = 2$)	T3 ($k = 3$)	T4 ($k = 4$)	T5 ($k = 5$)
Jhang	81.89064	245.6719	573.2344	1228.36	2538.61
Gujrat	213.5579	509.0065	1099.904	2281.698	4645.286
Kasur	251.1011	584.0929	1250.076	2582.043	5245.977
Sialkot	142.167	366.2247	814.34	1710.571	3503.032

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