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A NOVEL FILLED FUNCTION METHOD FOR GLOBAL OPTIMIZATION

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Abstract. This paper considers the unconstrained global optimization with the revised filled function methods. The minimization sequence could leave from a local minimizer to a better minimizer of the objective function through minimizing an auxiliary function constructed at the local minimizer. Some promising numerical results are also included.

1. Introduction

Consider the following unconstrained programming problem:

$$\min\{f(x) : x \in \mathbb{R}^n\}, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$.

Many results devoted to global optimization are available in the literature. See, for example [1]-[8]. In order to ensure the ability to escape from local minimum, many global optimization algorithms would include in their consideration a subproblem of transcending local optimality, namely: given a local minimizer x^*, find a better local minimizer, or showing that x^* is a global minimizer upon termination. Among all the different types of global optimization algorithms available in the literature, one popular approach is called the auxiliary function approach. In this approach, the resolution to the subproblem under concerned is to replace the original cost function with an auxiliary function. This replacement procedure, in principle, should ensure any local search applied to the auxiliary function starting from x^*, would lead to a lower minimum of the original cost function, if there exists one. Thus, global minimizer can be obtained just by implementing local search methods to the auxiliary function and the original function. However, it is rather difficult to construct such an auxiliary function.

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Filled function method is a typical auxiliary function method. The primary filled function was proposed by Ge [2]. The definition of the filled function is as follows:

Definition 1.1. Let x_1^* be a current minimizer of $f(x)$. A function $P(x)$ is called a filled function of $f(x)$ at x_1^* if $P(x)$ has the following properties:

(1) x_1^* is a maximizer of $P(x)$ and the whole basin B_1^* of $f(x)$ at x_1^* becomes a part of a hill of $P(x)$;
(2) $P(x)$ has no minimizers or saddle points in any higher basin of $f(x)$ than B_1^*;
(3) if $f(x)$ has a lower basin than B_1^*, then there is a point $x^{'}$ in such a basin that minimizes $P(x)$ on the line through x and x_1^*.

For the definitions of basin and hill, refer to [2].

The filled function given at x_1^* in [2] has the following form:

$$P(x, x_1^*, r, \rho) = \frac{1}{r + f(x)} \exp\left(-\frac{\|x - x_1^*\|^2}{\rho^2}\right), \tag{1.2}$$

where the parameters r and ρ need to be chosen appropriately.

However, the filled function algorithm described in [2] still has some unexpected features. Such as: The efficiency of the filled function algorithm strongly depend on two parameters r and ρ; When the domain is large or ρ is small, the factor $\exp(\frac{-\|x-x^*\|^2}{\rho^2})$ will be approximately zero, This smoothing increases with this factor, the filled function (1.2) will become very flat. This makes the efficiency of the filled function algorithm decrease. Although some other filled functions were proposed later, all of them are still not satisfactory for global optimization due to the above features.

In paper [8], a new definition of the filled function is given as following:

Definition 1.2. $P(x, x_1^*)$ is called a filled function of $f(x)$ at a local minimizer x_1^* if $P(x, x_1^*)$ has the following properties:

(1) x_1^* is a local maximizer of $P(x, x_1^*)$;
(2) $P(x, x_1^*)$ has no stationary point in the region
$$S_1 = \{x : f(x) \geq f(x_1^*), x \in \Omega \setminus \{x_1^*\}\}.$$
(3) If x_1^* is not a global minimizer of $f(x)$, then $P(x, x_1^*)$ does have a minimizer in the region
$$S_2 = \{x : f(x) < f(x_1^*), x \in \Omega\}.$$

The new filled function algorithm overcomes the disadvantages mentioned above in a certain extent.

In this paper, we construct a novel filled function satisfying the new definition. Numerical results indicate the filled function is very efficient and reliable.

The paper is organized as follows: In Section 2, we state the problem under some assumptions and give a novel filled function which has two adjustable parameters is proposed and its properties are investigated. In Section 3, we give a new filled function algorithm. In Section 4, test functions and numerical experiments are reported. Finally, in Section 5, we give some concluding remarks.

2. Some assumptions and a new filled function

Consider the following unconstrained programming problem:

$$\min f(x) \text{ such that } x \in \mathbb{R}^n.$$

Throughout this paper we make the following assumptions:

Assumpution 1. $f(x)$ is Lipschitz continuous on $\mathbb{R}^n$, i.e., there exists a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$ holds for all $x, y \in \mathbb{R}^n$.

Assumpution 2. $f(x)$ is coercive, i.e., $f(x) \to +\infty$, as $\|x\| \to +\infty$.

Notice that Assumption 2 implies the existence of a robust compact set $\Omega \subset \mathbb{R}^n$ whose interior contains all minimizers of $f(x)$. We assume that the value of $f(x)$ for x on the boundary of Ω is greater than the value of $f(x)$ for any x inside Ω. Then the original problem is equivalent to the following problem:

$$\text{(P)} \qquad \min f(x) \text{ such that } x \in \Omega.$$

Assumpution 3. $f(x)$ has only a finite number of minimums in Ω.

Let $L(P)$ stand for the set of local minimizers of $f(x)$ and the function

$$\varphi_q(t) = \begin{cases} \arctan\left(-\frac{q^2}{t^2}\right) + \frac{\pi}{2}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0 \end{cases} \tag{2.1}$$

is given. It is easy to prove that $\varphi_q(t)$ is a continuously differentiable function.

The new filled function given at x_1^* has the following form:

$$F(x, x^*, q, r) = \frac{1}{q + \|x - x^*\|}\varphi_q\left(f(x) - f(x^*) + r\right), \tag{2.2}$$

where $q > 0$ and r satisfies

$$0 < r < \max_{\substack{x^*, x_1^* \in L(P) \\ f(x^*) < f(x_1^*)}} (f(x_1^*) - f(x^*)).$$

The following theorems show that $F(x, x^*, q, r)$ is a filled function satisfying Definition 1.2.

Theorem 2.1. *Suppose that $f(x)$ holds Assumption 1. Further suppose that x_1^* is a local minimizer of $f(x)$. For any $r > 0$, when $q > 0$ is satisfactorily small, x_1^* is a local maximizer of $F(x, x_1^*, q, r)$.*

Proof. Since x_1^* is a local minimizer of $f(x)$, there exists a neighborhood $N(x_1^*,\delta)$ of x_1^* with $\delta > 0$ such that $f(x) \geq f(x^*)$ for all $x \in N(x_1^*,\delta)$. Then,

$$
\begin{aligned}
& F(x, x_1^*, q, r) - F(x_1^*, x_1^*, q, r) \\
= & \frac{1}{q + \|x - x_1^*\|} \left(\arctan \left(-\frac{q^2}{(f(x) - f(x_1^*) + r)^2} \right) + \frac{\pi}{2} \right) \\
& - \frac{1}{q} \left(\arctan \left(-\frac{q^2}{r^2} \right) + \frac{\pi}{2} \right) \\
= & \frac{1}{q + \|x - x_1^*\|} \arctan \left(-\frac{q^2}{(f(x) - f(x_1^*) + r)^2} \right) - \frac{1}{q} \arctan \left(-\frac{q^2}{r^2} \right) \\
& + \frac{-\|x - x_1^*\|}{q(q + \|x - x_1^*\|)} \frac{\pi}{2} \\
= & \frac{1}{q + \|x - x_1^*\|} \times \left(\arctan \left(\frac{q^2}{r^2} \right) - \arctan \left(\frac{q^2}{(f(x) - f(x_1^*) + r)^2} \right) \right) \\
& + \frac{1}{q \times (q + \|x - x_1^*\|)} \times \|x - x_1^*\| \left(\arctan \left(\frac{q^2}{r^2} \right) - \frac{\pi}{2} \right) \\
< & \frac{1}{q \times (q + \|x - x_1^*\|)} \\
& \times \left(q \times \left(\frac{q^2}{r^2} - \frac{q^2}{(f(x) - f(x_1^*) + r)^2} \right) + \|x - x_1^*\| \left(\arctan 1 - \frac{\pi}{2} \right) \right) \\
& (\text{when } x \geq y,\ \arctan(x) - \arctan(y) \leq x - y;\ \text{and let } q^2 < r^2) \\
= & \frac{1}{q \times (q + \|x - x_1^*\|)} \\
& \times \left(q^3 \times \frac{(f(x) - f(x_1^*)) \times (f(x) - f(x_1^*) + 2r)}{r^2 \times (f(x) - f(x_1^*) + r)^2} - \|x - x_1^*\| \frac{\pi}{4} \right).
\end{aligned}
$$

Since any $x \in N(x_1^*, \delta)$ have $f(x) \geq f(x_1^*)$, thus $f(x) - f(x_1^*) + r \geq r > 0$, therefore

$$
\begin{aligned}
& \frac{(f(x) - f(x_1^*)) \times (f(x) - f(x_1^*) + 2r)}{r^2 \times (f(x) - f(x_1^*) + r)^2} \\
= & \frac{(f(x) - f(x_1^*))^2 + 2r(f(x) - f(x_1^*))}{r^2 \times (f(x) - f(x_1^*) + r)^2} \\
= & \frac{(f(x) - f(x_1^*))^2}{r^2 \times (f(x) - f(x_1^*) + r)^2} + \frac{2r(f(x) - f(x_1^*))}{r^2 \times (f(x) - f(x_1^*) + r)^2} \\
\leq & \frac{f(x) - f(x_1^*)}{r^2 \times (f(x) - f(x_1^*) + r)} + \frac{2r(f(x) - f(x_1^*))}{r^4} \\
\leq & \frac{L\|x - x_1^*\|}{r^3} + \frac{2L\|x - x_1^*\|}{r^3} \\
= & \frac{3L\|x - x_1^*\|}{r^3}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& F(x, x_1^*, q, r) - F(x_1^*, x_1^*, q, r) \\
< & \frac{1}{q \times (q + \|x - x_1^*\|)} \left(q^3 \times \frac{3L\|x - x_1^*\|}{r^3} - \|x - x_1^*\| \frac{\pi}{4} \right) \\
\leq & \frac{1}{q \times (q + \|x - x_1^*\|)} \left(q \times \frac{3L\|x - x_1^*\|}{r} - \|x - x_1^*\| \frac{\pi}{4} \right) \\
& (\text{because } \ q^2 < r^2) \\
= & \frac{\|x - x_1^*\|}{q \times (q + \|x - x_1^*\|)} \left(q \times \frac{3L}{r} - \frac{\pi}{4} \right) \\
\leq & \, 0 \quad (\text{when } q < \frac{\pi r}{12L}).
\end{aligned}
$$

Above of all, when $0 < q < \min\{r, \frac{\pi r}{12L}\}$ and $x \in N(x_1^*, \delta)$, $x \neq x_1^*$, we have,

$$F(x, x_1^*, q, r) - F(x_1^*, x_1^*, q, r) < 0,$$

x_1^* is a local maximizer of $F(x, x_1^*, q, r)$. □

Theorem 2.2. *Suppose that $f(x)$ is continuously differentiable. If x_1^* is a local minimizer of $f(x)$, then the function $F(x, x_1^*, q, r)$ has no stationary points in the region $S_1 = \{x : f(x) \geq f(x_1^*), x \in \Omega/\{x_1^*\}\}$ when $r > 0$ and $q > 0$ is satisfactorily small.*

Proof. Let $x \in S_1$, i.e., $f(x) \geq f(x_1^*)$ and $x \neq x_1^*$, and let $L = \sup\{\|\nabla f(x)\| : x \in \Omega\}$. We have

$$
\begin{aligned}
& \nabla F(x, x_1^*, q, r)^T \frac{x - x^*}{\|x - x^*\|} \\
= & -\frac{1}{(q + \|x - x_1^*\|)^2} \left(\arctan\left(-\frac{q^2}{(f(x) - f(x_1^*) + r)^2} \right) + \frac{\pi}{2} \right) \\
& + \frac{2q^2 \nabla f(x)^T (x - x_1^*)}{(\|x - x_1^*\|)(q + \|x - x_1^*\|)((f(x) - f(x_1^*) + r)^3)} \\
& \times \frac{1}{1 + \left(-\frac{q^2}{(f(x) - f(x_1^*) + r)^2} \right)^2} \\
\leq & -\frac{1}{(q + \|x - x_1^*\|)^2} \left(\arctan\left(-\frac{q^2}{r^2} \right) + \frac{\pi}{2} \right) + \frac{2L}{q + \|x - x_1^*\|} \frac{q^2}{r^3} \\
\leq & -\frac{1}{(q + \max_{x \in \Omega} \|x - x_1^*\|)^2} \left(\arctan\left(-\frac{q^2}{r^2} \right) + \frac{\pi}{2} \right) + 2L \frac{q}{r^3}.
\end{aligned}
$$

Let $M = \max_{x \in X} \|x - x_1^*\|$, and $0 < q < \min\{1, \frac{r^3}{4L(1+M)^2}\}$. We have

$$\nabla F(x, x_1^*, q, r)^T \frac{x - x_1^*}{\|x - x_1^*\|} < -\frac{1}{(1+M)^2} \left(\arctan\left(-\frac{q}{r^2} \right) + \frac{\pi}{2} \right) + \frac{1}{2(1+M)^2}.$$

Thus, when $0 < q < \min\{\frac{r^3}{4L(1+M)^2}, r^2 \tan(\frac{\pi-1}{2}), 1\}$, we have

$$\nabla F(x, x_1^*, q, r)^T \frac{x - x_1^*}{\|x - x_1^*\|} < 0.$$

It implies that the function $F(x, x_1^*, q, r)$ has no stationary points in the region $S_1 = \{x : f(x) \geq f(x_1^*), x \in \Omega \setminus \{x_1^*\}\}$ when $r > 0$ and $q > 0$ is satisfactorily small. □

Theorem 2.3. *If $x_1^* \in L(P)$ and it is not a global minimizer of $f(x)$ in Ω, then there exists a minimizer $\overline{x_1^*}$ of $F(x, x_1^*, q, r)$ in the region $S_2 = \{x : f(x) < f(x_1^*), x \in \Omega\}$.*

Proof. Since $f(x)$ is continuous and x_1^* is not its global minimizer, there exists a point $\overline{x_1^*}$, such that

$$f(\overline{x_1^*}) - f(x_1^*) + r = 0,$$

namely, $F(\overline{x_1^*}, x_1^*, q, r) = 0$. On the other hand $F(x, x_1^*, q, r) \geq 0$ from the form of the filled function. Therefore, $F(x, x_1^*, q, r) \geq F(\overline{x_1^*}, x_1^*, q, r)$. Thus $\overline{x_1^*}$ is a minimizer of $F(x, x_1^*, q, r)$. □

Theorem 2.4. *Suppose that Assumption 1 is satisfied. If $x_1, x_2 \in \Omega$ and satisfy the following conditions*:

(1) $f(x_1) \geq f(x_1^*)$ *and* $f(x_2) \geq f(x_1^*)$,
(2) $\|x_2 - x_1^*\| > \|x_1 - x_1^*\|$.

Then, when $r > 0$ and q is positive and satisfactorily small, $F(x_1, x_1^*, q, r) > F(x_2, x_1^*, q, r)$.

Proof. Consider the following two cases:

(1) If $f(x_1^*) \leq f(x_2) \leq f(x_1)$, then it is obvious that the result follows.
(2) If $f(x_1^*) \leq f(x_1) \leq f(x_2)$, we will show $F(x_1, x_1^*, q, r) > F(x_2, x_1^*, q, r)$ also holds.

When $f(x_1^*) \leq f(x_1) \leq f(x_2)$, we have

$$\begin{aligned}
&F(x_2, x_1^*, q, r) - F(x_1, x_1^*, q, r) \\
= &\frac{1}{q + \|x_2 - x_1^*\|}\left(\arctan\left(-\frac{q^2}{(f(x_2) - f(x_1^*) + r)^2}\right) + \frac{\pi}{2}\right) \\
&- \frac{1}{q + \|x_1 - x_1^*\|}\left(\arctan\left(-\frac{q^2}{(f(x_1) - f(x_1^*) + r)^2}\right) + \frac{\pi}{2}\right) \\
= &\frac{-1}{q + \|x_2 - x_1^*\|}\arctan\left(\frac{q^2}{(f(x_2) - f(x_1^*) + r)^2}\right) \\
&+ \frac{1}{q + \|x_1 - x_1^*\|}\arctan\left(\frac{q^2}{(f(x_1) - f(x_1^*) + r)^2}\right) \\
&- \frac{\|x_2 - x_1^*\| - \|x_1 - x_1^*\|}{(q + \|x_2 - x_1^*\|)(q + \|x_1 - x_1^*\|)}\frac{\pi}{2}
\end{aligned}$$

and

$$\lim_{q\to 0}\left(F(x_2, x_1^*, q, r) - F(x_1, x_1^*, q, r)\right) = -\frac{\|x_2 - x_1^*\| - \|x_1 - x_1^*\|}{\|x_2 - x_1^*\|\|x_1 - x_1^*\|}\frac{\pi}{2} < 0.$$

Then, there must exist a constant $q_0 > 0$ such that

$$F(x_1, x_1^*, q, r) > F(x_2, x_1^*, q, r)$$

while $q < q_0$, and q_0 is not related to the values of $f(x)$ at x_1 and x_2. □

Theorem 2.5. *If $x_1, x_2 \in \Omega$ and satisfy the following conditions*:

(1) $\|x_2 - x_1^*\| > \|x_1 - x_1^*\|$,

(2) $f(x_1) \geq f(x_1^*) > f(x_2)$ *and* $f(x_2) - f(x_1^*) + r > 0$.

Then, we have $F(x_2, x_1^*, q, r) < F(x_1, x_1^*, q, r)$.

Proof. By Conditions 1 and 2, we have

$$\frac{1}{q + \|x_2 - x_1^*\|} < \frac{1}{q + \|x_1 - x_1^*\|}$$

and

$$0 < f(x_2) - f(x_1^*) + r < f(x_1) - f(x_1^*) + r.$$

Hence

$$F(x_2, x_1^*, q, r) < F(x_1, x_1^*, q, r).$$ □

Now we make some remarks. First, in the phase of minimizing the new filled function, Theorems 2.4 and 2.5 guarantee that the present local minimizer x_1^* of the objective function is escaped and the minimum of the new filled function will be always achieved at a point where the objective function value is not larger than the objective function value of the current minimum. Second, the parameters q and r are easier to be appropriately chosen than those of the original filled function (1.2). When the parameter q is small and $f(x) \geq f(x_1^*)$, the factor $\arctan\left(\frac{-q^2}{(f(x)-f(x_1^*)+r)^2}\right) + \frac{\pi}{2}$ will be approximately $\frac{\pi}{2}$, therefore, the new filled function does not become flat. In the next section, a new filled function algorithm is given, it has a simple termination criteria.

3. The filled function algorithm

In the above section, we discussed some properties of the filled function. Now, we present an algorithm in the following:

Algorithm

1. Initial Step

Choose $r = 1$, and $0 < r_0 < 1$ as the tolerance parameters for terminating the minimization process of problem (P).

Choose $0 < q_0 < 1$ and $M > 0$.

Choose direction e_i, $i = 1, 2, \ldots, k_0$ with integer $k_0 > 2n$, where n is the number of variable.

Choose an initial point $x_1^0 \in \Omega$.

Let $k = 1$.

2. Main Step

1^0. Obtain a local minimizer of prime problem (P) by implementing a local downhill search procedure staring from the x_k^0. Let x_k^* be the local minimizer obtained. Let $i = 1$, $r = 1$, $q = r \ln 2$.

2^0. If $i \leq k_0$, then goto 5^0, otherwise goto 3^0.

3^0. If $r \leq r_0$, then terminate the iteration, the x_k^* is the global minimizer of problem (P), otherwise, goto 4^0.

4^0. If $q \leq q_0$, then let $r = r/2$, $q = r \ln 2$, $i = 1$, goto 5^0, otherwise, let $q = q/10$, $i = 1$, goto 5^0.

5^0. $\bar{x}_k^* = x_k^* + \sigma e_i$ (where σ is a very small positive number), if $f(\bar{x}_k^*) < f(x_k^*)$ then let $k = k + 1$, $x_k^0 = \bar{x}_k^*$ and goto 1^0; otherwise, goto 6^0.

6^0. Let

$$F(x, x_k^*, q, r) = \frac{1}{q + \|x - x_k^*\|} \varphi_q(f(x) - f(x_k^*) + r)$$

and $y_0 = \bar{x}_k^*$. Turn to inner loop.

3. Inner Loop

1^0. Let $m = 0$.

2^0. $y_{m+1} = \varphi(y_m)$, where φ is an iteration function. It denotes a local downhill search method for the following problem:

$$\min F(x, x_k^*, q, r) \quad \text{such that } \ x \in \Omega.$$

Such as F-R method, BFGS method, etc.

3^0. If $y_{m+1} \notin \Omega$, then let $i = i + 1$, goto main step 2^0, otherwise goto 4^0.

4^0. If $f(y_{m+1}) \leq f(x_k^*)$, then let $k = k + 1$, $x_k^0 = y_{m+1}$ and goto main step 1^0, otherwise let $m = m + 1$ and goto 2^0.

The idea and mechanism of algorithm are explained as follows:

There are two phrases in the algorithm. One is that of minimizing the original function f, the other is that of minimizing the new filled function $F(x, x_1^*, q, r)$ in the inner loop. We let $r = 1$ and $q = r \ln 2$ in the initialization, afterwards, r and q are gradually reduced via the two-phase cycle until they are less than sufficiently small positive scales. If the parameters r and q are sufficiently small, we cannot find the point x with $f(x) < f(x_1^*)$ yet, then we believe that there does not exist a better local minimizer of $f(x)$. The algorithm is terminated.

4. Numerical examples

In this section, we apply the filled algorithm to several test examples. The proposed algorithm is programmed in Fortran 95 for working on the windows XP system with Intel cl.7G CPU and 256M RAM. Numerical results prove that the method is efficient.

The computational results are summarized in tables for each example problem. The symbols used in the tables are given as follows:

k**:** The iteration number in finding the k-th local minimizer;
x_k^0**:** The starting point in the k-th iteration;
q,r**:** The parameters used for finding the k-th local minimizer;
x_k^***:** The k-th local minimizer;
$f(x_k^*)$**:** The function value of the k-th local minimizer.

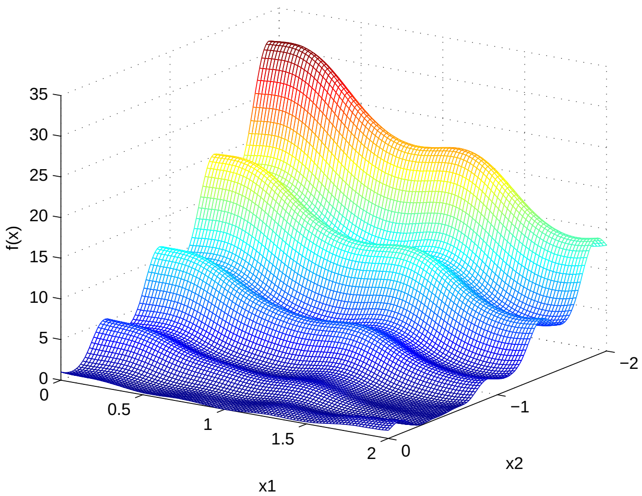
Example 4.1 (Two-dimensional ($n = 2$) function in [9])**.**

$$\min f(x) = [1 - 2x_2 + c\sin(4\pi x_2) - x_1]^2 + [x_2 - 0.5\sin(2\pi x_1)]^2$$
$$\text{such that } 0 \leq x_1 \leq 10, -10 \leq x_2 \leq 0,$$

where $c = 0.2, 0.5, 0.05$.

The proposed filled function approach succeeds in identifying the global minimum solutions: $f(x^*) = 0$ for all c. The computational results are summarized in Tables 1-3, for $c = 0.2$, 0.5, 0.05, respectively. An illustration with $c = 0.5$ is given in Fig.1.

Fig. 1. Two−dimensional function with c=0.5 .



Example 4.2 (Six-hump back camel ($n = 2$) function in [2])**.**

$$\min f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4$$
$$\text{such that } -3 \leq x_1 \leq 3, -3 \leq x_2 \leq 3.$$

Three initial points $x = (-2, 1)$, $(2, -1)$, and $(-2, -1)$ are used. The proposed filled function approach succeeds in identifying the global minimum solutions: $x^* = (0.0898420131, 0.712656403)$ or $(-0.0898420131, -0.712656403)$, where $f(x^*) = -1.03162845349$. The computational results are summarized in Table 4-6, respectively. An illustration is given in Fig.2.

Fig. 2. Six−hump back camel function.

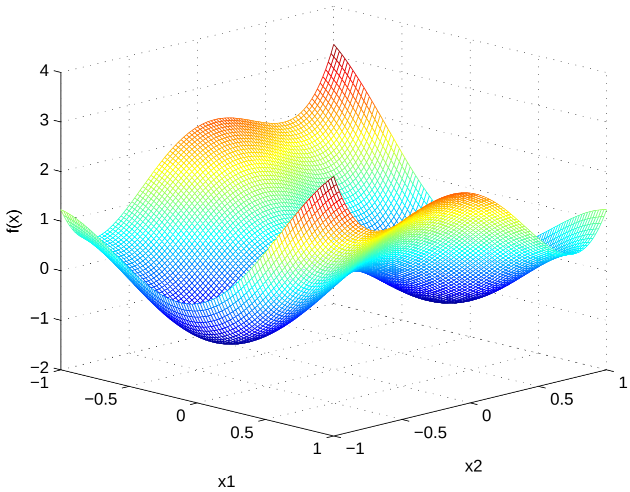


TABLE 4.1. numerical results for example 4.1 with $c = 0.2$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 6.00000 \\ -2.00000 \end{pmatrix}$	$\begin{pmatrix} 5.72207 \\ -1.88059 \end{pmatrix}$	2.50700	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 4.82207 \\ -1.88001 \end{pmatrix}$	$\begin{pmatrix} 4.73873 \\ -1.74168 \end{pmatrix}$	1.62119	$\begin{pmatrix} 0.0693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 4.64379 \\ -1.46565 \end{pmatrix}$	$\begin{pmatrix} 4.70961 \\ -1.39852 \end{pmatrix}$	1.35656	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
3	$\begin{pmatrix} 3.80962 \\ -1.397570 \end{pmatrix}$	$\begin{pmatrix} 3.73865 \\ -1.26494 \end{pmatrix}$	0.616471	$\begin{pmatrix} 0.0693147 \\ 1.00000 \end{pmatrix}$
4	$\begin{pmatrix} 2.87168 \\ -1.04045 \end{pmatrix}$	$\begin{pmatrix} 2.7380 \\ -0.788362 \end{pmatrix}$	8.86734×10^{-2}	$\begin{pmatrix} 0.173287 \\ 0.250000 \end{pmatrix}$
5	$\begin{pmatrix} 1.55470 \\ -0.267043 \end{pmatrix}$	$\begin{pmatrix} 1.59089 \\ -0.270259 \end{pmatrix}$	2.08413×10^{-12}	$\begin{pmatrix} 0.346574 \\ 0.500000 \end{pmatrix}$

TABLE 4.2. Numerical results for Example 4.1 with $c = 0.5$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 0.00000 \\ 0.00000 \end{pmatrix}$	$\begin{pmatrix} 0.0420240 \\ -0.0947718 \end{pmatrix}$	0.517454	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 0.478431 \\ -0.147737 \end{pmatrix}$	$\begin{pmatrix} 0.552444 \\ -0.103676 \end{pmatrix}$	0.0332208	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 1.63533 \\ -0.255225 \end{pmatrix}$	$\begin{pmatrix} 1.58724 \\ -0.260556 \end{pmatrix}$	3.78979×10^{-12}	$\begin{pmatrix} 0.0216608 \\ 0.0312500 \end{pmatrix}$

Example 4.3 (The Rastrigin ($n = 2$) function in [10]).

$$\begin{aligned} &\min f(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2) \\ &\text{such that } -1 \le x_1 \le 1, -1 \le x_2 \le 1. \end{aligned}$$

The initial points $x = (1, 1)$ are used. The proposed filled function approach succeeds in identifying the global minimum solutions $x^* = (0.0, 0.0)$, where $f(x^*) = -2$. The computational results are summarized in Table 7, respectively. An illustration is given in Fig.3.

Fig. 3. The Rastrigin function.

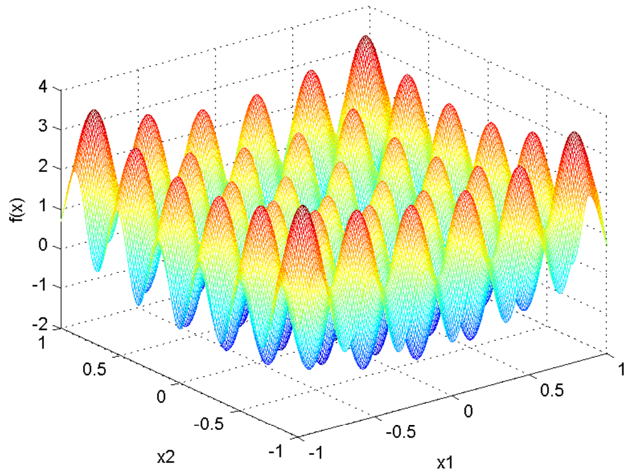


TABLE 4.3. Numerical results for Example 4.1 with $c = 0.05$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 10.00000 \\ -10.00000 \end{pmatrix}$	$\begin{pmatrix} 9.73068 \\ -3.74754 \end{pmatrix}$	12.1010	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 9.67124 \\ -3.45658 \end{pmatrix}$	$\begin{pmatrix} 9.71836 \\ -3.42803 \end{pmatrix}$	11.9544	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 8.81836 \\ -3.42769 \end{pmatrix}$	$\begin{pmatrix} 8.72988 \\ -3.29645 \end{pmatrix}$	9.07329	$\begin{pmatrix} 0.0693147 \\ 1.00000 \end{pmatrix}$
3	$\begin{pmatrix} 7.82988 \\ -3.29619 \end{pmatrix}$	$\begin{pmatrix} 7.72795 \\ -2.83466 \end{pmatrix}$	6.50309	$\begin{pmatrix} 0.0693147 \\ 1.00000 \end{pmatrix}$
4	$\begin{pmatrix} 6.82795 \\ -2.83433 \end{pmatrix}$	$\begin{pmatrix} 6.72482 \\ -2.37244 \end{pmatrix}$	4.39429	$\begin{pmatrix} 0.0693147 \\ 1.00000 \end{pmatrix}$
5	$\begin{pmatrix} 5.81364 \\ -1.58860 \end{pmatrix}$	$\begin{pmatrix} 5.71982 \\ -1.91624 \end{pmatrix}$	2.74338	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
6	$\begin{pmatrix} 4.58050 \\ -1.26884 \end{pmatrix}$	$\begin{pmatrix} 4.71289 \\ -1.48911 \end{pmatrix}$	1.53505	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
7	$\begin{pmatrix} 3.81289 \\ -1.48826 \end{pmatrix}$	$\begin{pmatrix} 3.73054 \\ -1.23061 \end{pmatrix}$	0.618436	$\begin{pmatrix} 0.0693147 \\ 1.00000 \end{pmatrix}$
8	$\begin{pmatrix} 2.85024 \\ -1.05583 \end{pmatrix}$	$\begin{pmatrix} 2.72998 \\ -0.793414 \end{pmatrix}$	0.102163	$\begin{pmatrix} 0.173287 \\ 0.250000 \end{pmatrix}$
9	$\begin{pmatrix} 1.55240 \\ -0.192514 \end{pmatrix}$	$\begin{pmatrix} 1.59746 \\ -0.287409 \end{pmatrix}$	9.57566×10^{-12}	$\begin{pmatrix} 0.346576 \\ 0.50000 \end{pmatrix}$

TABLE 4.4. Numerical results for Example 4.2 with initial point $(-2, 1)$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} -2.00000 \\ 1.00000 \end{pmatrix}$	$\begin{pmatrix} -1.60710 \\ 0.568653 \end{pmatrix}$	2.10425	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} -1.00792 \\ 0.597201 \end{pmatrix}$	$\begin{pmatrix} 0.0898409 \\ 0.712656 \end{pmatrix}$	-1.03163	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

Example 4.4 (n-dimensional Sine-square ($n = 2, 3, 5, 7, 10$) function in [2]).

$$\min f(x) = \frac{\pi}{n}[10\sin^2(\pi x_1) + g(x) + (x_n - 1)^2]$$

TABLE 4.5. Numerical results for Example 4.2 with initial point $(2, -1)$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 2.00000 \\ -1.00000 \end{pmatrix}$	$\begin{pmatrix} 1.60710 \\ -0.568653 \end{pmatrix}$	2.10425	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 1.64230 \\ 0.301075 \end{pmatrix}$	$\begin{pmatrix} 1.70361 \\ 0.796083 \end{pmatrix}$	-0.215464	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 0.604065 \\ 0.774673 \end{pmatrix}$	$\begin{pmatrix} 0.0898410 \\ 0.712656 \end{pmatrix}$	-1.03163	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

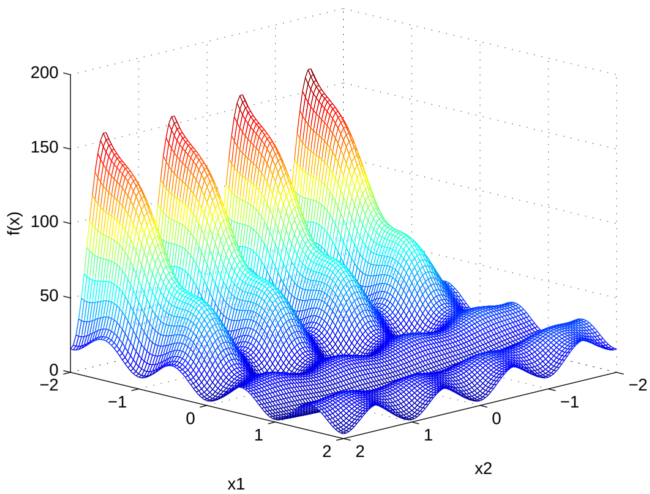
TABLE 4.6. Numerical results for Example 4.2 with initial point $(-2, -1)$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} -2.00000 \\ -1.00000 \end{pmatrix}$	$\begin{pmatrix} -1.70361 \\ -0.796083 \end{pmatrix}$	-0.215464	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} -0.604065 \\ -0.774673 \end{pmatrix}$	$\begin{pmatrix} -0.0898410 \\ -0.712656 \end{pmatrix}$	-1.03163	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

TABLE 4.7. Numerical results for Example 4.3

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 1.0000 \\ 1.0000 \end{pmatrix}$	$\begin{pmatrix} 1.04076 \\ 1.04076 \end{pmatrix}$	0.179775	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 0.729632 \\ 0.729632 \end{pmatrix}$	$\begin{pmatrix} 0.693844 \\ 0.693844 \end{pmatrix}$	-1.03121	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 0.0325776 \\ 0.735064 \end{pmatrix}$	$\begin{pmatrix} -2.69833 \times 10^{-8} \\ 0.693844 \end{pmatrix}$	-1.51560	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
3	$\begin{pmatrix} 1.45534 \times 10^{-4} \\ -0.0531016 \end{pmatrix}$	$\begin{pmatrix} 1.56597 \times 10^{-8} \\ -2.33462 \times 10^{-8} \end{pmatrix}$	-2.00000	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

Fig. 4. n−dimensional Sine−square (n=2) function.



$$\text{such that } -10 \le x_i \le 10,\ i = 1, 2, \ldots, n,$$

TABLE 4.8. Numerical results for Example 4.4 with $n = 2$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} -4.00000 \\ -4.00000 \end{pmatrix}$	$\begin{pmatrix} -3.94897 \\ -3.99793 \end{pmatrix}$	78.1264	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} -3.15386 \\ -2.97141 \end{pmatrix}$	$\begin{pmatrix} -2.95942 \\ -2.99742 \end{pmatrix}$	49.9958	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} -1.37286 \\ -2.08215 \end{pmatrix}$	$\begin{pmatrix} -0.979768 \\ -1.99226 \end{pmatrix}$	20.3208	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
3	$\begin{pmatrix} 0.737593 \\ -1.33529 \end{pmatrix}$	$\begin{pmatrix} 1.00000 \\ 1.00000 \end{pmatrix}$	1.18806×10^{-11}	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

TABLE 4.9. Numerical results for Example 4.4 with $n = 3$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} -3.00000 \\ -3.00000 \\ -3.00000 \end{pmatrix}$	$\begin{pmatrix} -2.95942 \\ -2.99741 \\ -2.99747 \end{pmatrix}$	50.0751	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} -2.15942 \\ -2.99652 \\ -2.99747 \end{pmatrix}$	$\begin{pmatrix} -1.96967 \\ -2.99541 \\ -2.99746 \end{pmatrix}$	42.8104	$\begin{pmatrix} 0.00693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} -1.16967 \\ -2.99529 \\ -2.99746 \end{pmatrix}$	$\begin{pmatrix} -0.979675 \\ -2.98967 \\ -2.99746 \end{pmatrix}$	37.6032	$\begin{pmatrix} 0.00693147 \\ 1.00000 \end{pmatrix}$
3	$\begin{pmatrix} -0.0796835 \\ -2.98832 \\ -2.99746 \end{pmatrix}$	$\begin{pmatrix} 0.0117214 \\ -2.95844 \\ -2.99742 \end{pmatrix}$	34.3635	$\begin{pmatrix} 0.00693147 \\ 1.00000 \end{pmatrix}$
4	$\begin{pmatrix} 0.911721 \\ -2.95815 \\ -2.99742 \end{pmatrix}$	$\begin{pmatrix} 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	6.00157×10^{-12}	$\begin{pmatrix} 0.00693147 \\ 1.00000 \end{pmatrix}$

TABLE 4.10. Numerical results for Example 4.4 with $n = 5$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} -1.00000 \\ -1.00000 \\ -1.00000 \\ -1.00000 \\ -1.00000 \end{pmatrix}$	$\begin{pmatrix} -0.979833 \\ -0.994829 \\ -0.994907 \\ -0.994907 \\ -0.994920 \end{pmatrix}$	12.5155	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} -0.0798346 \\ -0.994270 \\ -0.994907 \\ -0.994907 \\ -0.994920 \end{pmatrix}$	$\begin{pmatrix} -0.0104528 \\ -0.979407 \\ -0.994827 \\ -0.994907 \\ -0.994920 \end{pmatrix}$	10.6301	$\begin{pmatrix} 0.00693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 0.910453 \\ -0.979208 \\ -0.994827 \\ -0.994907 \\ -0.994920 \end{pmatrix}$	$\begin{pmatrix} 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	3.67777×10^{-11}	$\begin{pmatrix} 0.00693147 \\ 1.00000 \end{pmatrix}$

where $g(x) = \sum_{i=1}^{n-1}[(x_i - 1)^2(1 + 10\sin^2(\pi x_{i+1}))]$.

The function is tested for $n = 2, 3, 5, 7, 10$. The global minimum solution is uniformly expressed as: $x^* = (1.0000, 1.0000, \ldots, 1.0000)$ and $f(x^*) = 0.0000$. The computational results are summarized in Tables 8-12, respectively. An illustration with $n = 2$ is given in Fig.4.

5. Conclusions

Based on the new definition of the filled function, we give a novel filled function which satisfying the definition and develop an algorithm for unconstrained

TABLE 4.11. Numerical results for Example 4.4 with $n = 7$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 2.00000 \\ 2.00000 \\ 2.00000 \\ 2.00000 \\ 2.00000 \\ 2.00000 \\ 2.00000 \end{pmatrix}$	$\begin{pmatrix} 1.98986 \\ 1.98965 \\ 1.98965 \\ 1.98965 \\ 1.98965 \\ 1.98965 \\ 1.98975 \end{pmatrix}$	3.10951	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 1.08755 \\ 2.04409 \\ 1.99299 \\ 1.98966 \\ 1.98965 \\ 1.98965 \\ 1.98975 \end{pmatrix}$	$\begin{pmatrix} 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	5.44137×10^{-11}	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

TABLE 4.12. Numerical results for Example 4.4 with $n = 10$

k	x_k	local minimizer x_k^*	$f(x_k^*)$	(q, r)
0	$\begin{pmatrix} 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \\ 6.00000 \end{pmatrix}$	$\begin{pmatrix} 5.94897 \\ 5.99793 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \end{pmatrix}$	78.4316	$\begin{pmatrix} - \\ - \end{pmatrix}$
1	$\begin{pmatrix} 3.26286 \\ 5.09981 \\ 5.99802 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \end{pmatrix}$	$\begin{pmatrix} 2.97968 \\ 4.98967 \\ 5.99682 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \end{pmatrix}$	69.0622	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
2	$\begin{pmatrix} 1.88528 \\ 5.08820 \\ 5.99763 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \end{pmatrix}$	$\begin{pmatrix} 1.98828 \\ 4.95842 \\ 5.99677 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \\ 5.99797 \end{pmatrix}$	68.0902	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$
3	$\begin{pmatrix} 0.981955 \\ 5.08927 \\ 5.99696 \\ 6.00201 \\ 5.99635 \\ 5.99692 \\ 5.99687 \\ 5.99689 \\ 5.99698 \\ 5.99617 \end{pmatrix}$	$\begin{pmatrix} 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$	7.45707×10^{-11}	$\begin{pmatrix} 0.693147 \\ 1.00000 \end{pmatrix}$

global optimization. The computational results show that this filled function is quite efficient and reliable.

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