J. Korean Math. Soc. **47** (2010), No. 6, pp. 1123–1135
DOI 10.4134/JKMS.2010.47.6.1123

EIGENVALUE PROBLEM OF BIHARMONIC EQUATION WITH HARDY POTENTIAL

Yangxin Yao, Shaotong He, and Qingtang Su

Abstract. In this paper, we consider the eigenvalue problem of biharmonic equation with Hardy potential. We improve the results of references by introducing a new Hilbert space.

1. Introduction

In 2006, Adimurthi, M. Grossi, and S. Santra [2] proved that, if $0 \in \Omega \subset B_R(0)$ is a bounded domain in $\mathbb{R}^4$, and $R > 0$, $R_1 > eR$, then $\forall u \in H_0^2(\Omega)$ or $\forall u \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

(1)

$$\int_\Omega |\triangle u|^2 \,\mathrm{d}x - \int_\Omega \frac{u^2}{|x|^4 (\ln R_1/|x|)^2} \,\mathrm{d}x \geq \sum_{i=2}^{\infty} \int_\Omega \frac{u^2}{|x|^4 (\ln R_1/|x|)^2} X_2^2 \cdots X_i^2 \,\mathrm{d}x,$$

where -1 is the best constant and can't be achieved by any nontrival function $u \in H_0^2(\Omega)$ or $\forall u \in H^2(\Omega) \cap H_0^1(\Omega)$, where

$$X_i(x) := Y_i\left(\frac{|x|}{R_1}\right), \ i = 1, 2, 3, \ldots$$

and

$$Y_1(t) := (1 - \ln t)^{-1}, \ t \in (0, 1],$$
$$Y_i(t) := Y_{i-1}(Y_1(t)), \ t \in (0, 1], \ i = 2, 3, 4 \ldots,$$
$$Y_i(0) = 0, \ Y_i(1) = 1, \ 0 \leq Y_i(t) \leq 1.$$

Furthermore, if we define

$$\lambda(\Omega) = \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega |\triangle u|^2 \,\mathrm{d}x - \int_\Omega \frac{u^2}{|x|^4 (\ln R/|x|)^2} \,\mathrm{d}x \;\middle|\; \int_\Omega u^2 \,\mathrm{d}x = 1 \right\},$$

Received December 25, 2008.
2000 *Mathematics Subject Classification.* 35J40, 35J60, 35J85.
Key words and phrases. biharmonic equations, Hardy type inequality, maximum principle.
Supported by NSFC(10771074).

then $\lambda(\Omega)$ can't be achieved by any domain Ω. This means that the following eigenvalue problem

$$\begin{cases} \triangle^2 u - \dfrac{u}{|x|^4(\ln R/|x|)^2} = \lambda u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u \in H_0^2(\Omega) \end{cases} \tag{2}$$

has no solution for $\lambda = \lambda(\Omega)$. Adimurthi, M. Grossi, and S. Santra [2] have considered the following eigenvalue problem

$$\begin{cases} \triangle^2 u - \dfrac{q(x)u}{|x|^4(\ln R/|x|)^2} = \lambda u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u \in H_0^2(\Omega), \end{cases} \tag{3}$$

where $0 \leq q(x) \leq 1$. Define

$$\lambda(q) = \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega |\triangle u|^2 \,\mathrm{d}x - \int_\Omega \frac{q(x)u^2}{|x|^4(\ln R/|x|)^2} \,\mathrm{d}x \;\middle|\; \int_\Omega u^2 \,\mathrm{d}x = 1 \right\}$$

if $N = 4$, and $q(x)$ satisfies the following assumptions, they have the following interesting results:

(i) If $q(x)$ satisfies

$$\liminf_{x \to 0} (\ln \ln R/|x|)^2 (1 - q(x)) > 3,$$

then $\lambda(q)$ is achieved by u, and (3) has solutions for $\lambda = \lambda(q)$. Furthermore, if Ω is a unit ball centered with the origin, we can choose $u > 0$.

(ii) If Ω is a unit ball centered with the origin, then $\lambda(q)$ is not achieved by any non-negative function, provided $q(x)$ satisfies

$$\sup_{0 < |x| \leq R_1} (\ln \ln R/|x|)^2 (1 - q(x)) \leq 3$$

for some $0 < R_1 < 1$.

For the case $N \geq 5$, A. Tertikas and N. Zographopolous [6] have proved the following inequality

$$\int_\Omega \left(|\triangle u|^2 - \frac{N^2(N-4)^2}{16} \frac{|u|^2}{|x|^4} \right) \mathrm{d}x \geq \left(1 + \frac{N(N-4)}{8} \right) \int_\Omega \frac{u^2}{|x|^4 (\ln R/|x|)^2} \,\mathrm{d}x \tag{4}$$

which holds for any $u \in H_0^2(\Omega)$, where $R > \mathrm{e} \sup_{x \in \Omega} |x|$. If we define $\lambda_N(\Omega)$ as

$$\lambda_N(\Omega) = \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega \left(|\triangle u|^2 - \frac{N^2(N-4)^2}{16} \frac{|u|^2}{|x|^4} \right) \mathrm{d}x \;\middle|\; \int_\Omega u^2 \,\mathrm{d}x = 1 \right\},$$

then $\lambda_N(\Omega)$ is not achieved by any domain Ω [6]. This means that the following eigenvalue problem

$$\begin{cases} \triangle^2 u - \dfrac{N^2(N-4)^2}{16}\dfrac{u}{|x|^4} = \lambda u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u \in H_0^2(\Omega) \end{cases} \tag{5}$$

has no solution for $\lambda = \lambda_N(\Omega)$. Adimurthi, M. Grossi, and S. Santra [2] considered the following problem

$$\begin{cases} \triangle^2 u - \dfrac{N^2(N-4)^2}{16}\dfrac{q(x)u}{|x|^4} = \lambda u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u \in H_0^2(\Omega), \end{cases} \tag{6}$$

where $q \in C^0(\overline{\Omega})$, $0 \le q(x) \le 1$. Let

$$\lambda_N(q) = \inf_{u \in H_0^2(\Omega)} \left\{ \int_\Omega |\triangle u|^2 \,\mathrm{d}x - \frac{N^2(N-4)^2}{16} \int_\Omega \frac{q(x)u^2}{|x|^4} \,\mathrm{d}x \;\middle|\; \int_\Omega u^2 \,\mathrm{d}x = 1 \right\}.$$

They get the following interesting results:

(i) $\lambda_N(q)$ is achieved for some function u in $H_0^2(\Omega)$, and (6) has solutions for $\lambda = \lambda_N(q)$ if $q(x)$ satisfies

$$\liminf_{x \to 0} (\ln 1/|x|)^2 (1 - q(x)) > \frac{6(N^2 - 4N + 8)}{N^2(N-4)^2}. \tag{7}$$

Furthermore, if Ω is a unit ball centered with the origin, then we can choose $u > 0$.

(ii) If Ω is a unit ball centered with the origin, then $\lambda_N(q)$ can't be achieved if $q(x)$ satisfies

$$\sup_{0 < |x| \le R_2} (\ln 1/|x|)^2 (1 - q(x)) \le \frac{6(N^2 - 4N + 8)}{N^2(N-4)^2} \tag{8}$$

for some $0 < R_2 < 1$.

It seems that (7) and (8) can not be improved since they have given an almost sufficient and necessary condition. Observe that if $q(x) \equiv 1$, the eigenvalue problems (3) and (6) have no non-trivial solution in $H_0^2(\Omega)$. So our first consideration is to weaken the assumption of $q(x)$ so that the result of Adimurthi in [2] can be improved.

Actually, we can achieve this. We find that, if we consider the above problems in a new Hilbert space, whose norm is not equivalent to that of $H_0^2(\Omega)$, the assumption of $q(x)$ can be weaken.

Furthermore, we pay more attention to the eigenvalue problems with two Hardy potential.

(1) Let $N \geq 5$. We consider the following problem:

$$\begin{cases} \triangle^2 u - \frac{N^2(N-4)^2}{16}\frac{u}{|x|^4} - \mu_1 \frac{q(x)u}{|x|^4(\ln R/|x|)^2} = \lambda\eta(x)u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u = \frac{\partial u}{\partial \gamma} = 0 & x \in \partial\Omega, \end{cases} \tag{9}$$

where $0 \leq \mu_1 \leq 1 + N(N-4)/8$.

(2) Let $N = 4$. We consider the weighted eigenvalue problem with two Hardy potential as follow:

$$\begin{cases} \triangle^2 u - \frac{u}{|x|^4(\ln R/|x|)^2} - \mu_2 \frac{q(x)u}{|x|^4(\ln R/|x|)^2(\ln\ln R/|x|)^2} = \lambda\eta(x)u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u = \frac{\partial u}{\partial \gamma} = 0 & x \in \partial\Omega, \end{cases} \tag{10}$$

where $0 \leq \mu_2 \leq 1$.

For (9), $\mu_1 = 1 + N(N-4)/8$ is the best constant of inequality (4) in the right hand side. In this case, the singular term $1/(|x|^4(\ln R/|x|)^2)$ is called the critical potential.

For the case $N = 4$, no paper has proved that $\mu_2 = 1$ is the best constant of inequality (1) in the right hand side. In this paper, we will give a positive answer that 1 is the best constant. As a result, we are able to identify the critical potential case with the non-critical case.

2. Main results

In order to state our main results, we construct a new Hilbert space as follows.

We define $H_{0,1}^{2,N}(\Omega)$ as the completion of $H_0^2(\Omega)$ with respect to the norm $||\cdot||_{H_{0,1}^{2,N}(\Omega)}$, where $\Omega \in R^N$, $N \geq 4$. And the norm $||\cdot||_{H_{0,1}^{2,N}(\Omega)}$ be defined as

$$\|u\|^2_{H_{0,1}^{2,N}(\Omega)} = \begin{cases} \int_\Omega \left(|\triangle u|^2 - \frac{u^2}{|x|^4(\ln R/|x|)^2} \right) \mathrm{d}x, & N = 4 \\ \int_\Omega \left(|\triangle u|^2 - \frac{N^2(N-4)^2}{16}\frac{u^2}{|x|^4} \right) \mathrm{d}x, & N \geq 5 \end{cases}$$

associated with the inner product

$$a(u,v) = \begin{cases} \int_\Omega \left(\triangle u \triangle v - \frac{uv}{|x|^4(\ln R/|x|)^2} \right) \mathrm{d}x, & N = 4 \\ \int_\Omega \left(\triangle u \triangle v - \frac{N^2(N-4)^2}{16}\frac{uv}{|x|^4} \right) \mathrm{d}x, & N \geq 5. \end{cases}$$

Obviously, the norm $||\cdot||_{H_{0,1}^{2,N}(\Omega)}$ is not equivalent to the norm $||\cdot||_{H_0^2} = (\int_\Omega |\triangle u|^2 \,\mathrm{d}x)^{\frac{1}{2}}$. If $1 \le p < 2$, by the $W^{1,p}$ estimation in [2], we have

$$H_0^2(\Omega) \subset H_{0,1}^{2,N}(\Omega) \subset W_0^{1,p}(\Omega).$$

In order to see this, when $N = 4$, we give some examples to show this. Consider the function $u(x) = u(|x|)$ defined on $B_1(0)$, where

$$u(r) = (\ln 1/r)^a (\ln\ln 1/r)^\delta$$

in $B_{R_0}(0)$ with $0 < R_0 < e^{-1}$, and smooth up to the boundary on $B_1(0)\backslash B_{R_0}(0)$. It's easy to check that $u \in H_0^2(\Omega)$ if and only if $a < 1/2$, or $a = 1/2$ and $\delta < -1/2$, while $u \in H_{0,1}^{2,N}(\Omega)$ if and only if $a < 1/2$, or $a = 1/2$ and $\delta < 0$.

If $N \ge 5$, we observe the function $u(x) = u(|x|)$ defined on $B_1(0)$, where

$$u(r) = r^{-\frac{N-4}{2}} (\ln 1/r)^a (\ln\ln 1/r)^\delta$$

in $B_{R_0}(0)$ with $0 < R_0 < e^{-1}$and smooth up to the boundary on $B_1(0)\backslash B_{R_0}(0)$. It's easy to check that $u \in H_0^2(\Omega)$ if and only if $a < -1/2$, or $a = -1/2$ and $\delta < -1/2$, while $u \in H_{0,1}^{2,N}(\Omega)$ if and only if $a < 0$, or $a = 0$ and $\delta < 0$.

Define $L_\eta^2(\Omega) = \{u \mid \int_\Omega \eta u^2 \,\mathrm{d}x < \infty\}$ with the norm $||u||_{L_\eta^2} = (\int_\Omega \eta u^2 \,\mathrm{d}x)^{1/2}$, where $\eta \ge 0$, and for $N \ge 5$,

$$\limsup_{|x|\to 0} |x|^4 (\ln R/|x|)^2 \eta(x) = 0 \tag{11}$$

for $N = 4$,

$$\limsup_{|x|\to 0} |x|^4 (\ln R/|x|)^2 (\ln\ln R/|x|)^2 \eta(x) = 0. \tag{12}$$

Obviously, $\eta \equiv 1$ satisfies the above conditions of η, and $L_1^2(\Omega) = L^2(\Omega)$.

we mainly deal with the following problems:

• Some related theorems about the new Hilbert space $H_{0,1}^{2,N}(\Omega)$, including the embedding theorem, maximum principle, etc.

• As an application of $H_{0,1}^{2,N}(\Omega)$, we consider the eigenvalue problem (9) as well as (10), and find the existence of solutions and positive solutions.

(1) For $N \ge 5$, we consider the eigenvalue problem with two singular terms as problem (9), where $\eta \ge 0$, $\eta \in L^\infty(\Omega\backslash B_r(0))$, $\forall r > 0$, and η satisfies (11). Define

$$\lambda_{\mu_1}(q) = \inf_{u \in H_{0,1}^{2,N}(\Omega)} \left\{ I_{\mu_1}(u) \,\middle|\, \int_\Omega \eta(x) u^2 \,\mathrm{d}x = 1 \right\},$$

where

$$I_{\mu_1}(u) = \int_\Omega \left(|\triangle u|^2 - \frac{N^2(N-4)^2}{16} \frac{u^2}{|x|^4} - \mu_1 \frac{q(x) u^2}{|x|^4 (\ln R/|x|)^2} \right) \mathrm{d}x.$$

(2) Similarly, for the case of $N = 4$, we discuss the eigenvalue problem (10), where $\eta \geq 0$, $\eta \in L^\infty(\Omega \backslash B_r(0))$, $\forall r \geq 0$, and η satisfies (12). We define

$$\tau_{\mu_2}(q) = \inf_{u \in H^{2,N}_{0,1}(\Omega)} \left\{ J_{\mu_2}(u) \;\middle|\; \int_\Omega \eta(x) u^2 \,\mathrm{d}x = 1 \right\},$$

where

$$J_{\mu_2}(u) = \int_\Omega \left(|\triangle u|^2 - \frac{u^2}{|x|^4 (\ln R/|x|)^2} - \mu_2 \frac{q(x) u^2}{|x|^4 (\ln R/|x|)^2 (\ln\ln R/|x|)^2} \right) \mathrm{d}x.$$

Remark 2.1. It's easy to check that the functionals I_{μ_1},J_{μ_2}($\mu_1 < 1 + N(N-4)/8, \mu_2 < 1$) are coercive on $H^{2,N}_{0,1}(\Omega)$. It's also easy to find that J_{μ_1},I_{μ_2} are weak lower semicontinuous and lower bounded. However, we should be aware that when $\mu_1 = 1+N(N-4)/8, \mu_2 = 1$, the functionals I_{μ_1},J_{μ_2} are not coercive on $H^{2,N}_{0,1}(\Omega)$.

The main result of this paper is as follows:

Theorem 2.1. *Let $N \geq 5$, $0 \leq \mu_1 \leq 1 + N(N-4)/8$, $q \in C^0(\overline{\Omega})$, $0 \leq q(x) \leq 1$, $\eta(x) \geq 0$, $\eta(x) \in L^\infty(\Omega \setminus B_r(0))$, $\forall r > 0$, and η satisfies* (11). *Then*

(1) *If $0 \leq \mu_1 < 1 + N(N-4)/8$, $\lambda_{\mu_1}(q)$ can be achieved and problem* (9) *has a nontrivial solution $u \in H^{2,N}_{0,1}(\Omega)$. Furthermore, if Ω is a unit ball centered with the origin, then we can choose $u > 0$ on Ω.*

(2) *If $\mu_1 = 1 + N(N-4)/8$, and $q(x)$ satisfies the extra condition*

$$\limsup_{|x| \to 0} q(x) = 0, \tag{13}$$

then $\lambda_{\mu_1}(q)$ can be achieved and problem (9) *has a nontrivial solution $u \in H^{2,N}_{0,1}(\Omega)$. Furthermore, if Ω is a unit ball centered with the origin, then we can choose $u > 0$ on Ω.*

Similar to Theorem 2.1, for the case of $N = 4$, we have the following theorem:

Theorem 2.2. *Suppose that $N = 4$, $0 \leq \mu_2 \leq 1$, $q \in C^0(\overline{\Omega})$, $0 \leq q(x) \leq 1$, $\eta(x) \geq 0$, $\eta(x) \in L^\infty(\Omega \setminus B_r(0))$ for any $r > 0$, and η satisfies* (12). *Then*

(1) *If $0 \leq \mu_2 < 1$, $\tau_{\mu_2}(q)$ can be achieved and problem* (10) *has nontrivial solutions $u \in H^{2,N}_{0,1}(\Omega)$.*

(2) *If $\mu_2 = 1$, and $q(x)$ satisfies the extra condition*

$$\limsup_{|x| \to 0} q(x) = 0, \tag{14}$$

then $\tau_{\mu_2}(q)$ is achieved and problem (10) *has nontrivial solutions $u \in H^{2,N}_{0,1}(\Omega)$. Furthermore, if Ω is a unit ball centered with the origin, we can choose $u > 0$ on Ω.*

3. Preliminary lemmas

Lemma 3.1. *The Hilbert space $H^{2,N}_{0,1}(\Omega)$ is embedded into $L^2_\eta(\Omega)$ and the embedding is compact, where $\eta \geq 0$, if $N \geq 5$, then η satisfies* (11)*, while $N = 4$ η satisfies* (12)*.*

Proof. We'll divided the proof into two steps. The first step is to prove that $H^{2,N}_{0,1}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, while the second step is to prove $H^{2,N}_{0,1}(\Omega) \hookrightarrow\hookrightarrow L^2_\eta(\Omega)$.

Step one: Prove $H^{2,N}_{0,1}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$.

From Theorem A.2 of [2], there exist $R_0 > 0, C_1 > 0, C_2 > 0$ such that $\forall\ R \geq R_0,\ \forall u \in H^2_0(\Omega)$

$$\begin{cases} \displaystyle\int_\Omega \left(|\Delta u|^2 - \frac{u^2}{|x|^4(\ln R/|x|)^2} \right) \mathrm{d}x \geq C_1 ||u||^2_{W^{1,p}_0(\Omega)}, & N = 4 \\ \displaystyle\int_\Omega \left(|\Delta u|^2 - \frac{N^2(N-4)^2}{16} \frac{u^2}{|x|^4} \right) \mathrm{d}x \geq C_2 ||u||^2_{W^{1,p}_0(\Omega)}, & N \geq 5, \end{cases}$$

where $1 \leq p < 2$. Since $H^2_0(\Omega)$ is dense in $H^{2,N}_{0,1}(\Omega)$, then the above inequalities are hold for any $u \in H^{2,N}_{0,1}(\Omega)$. It's easy to check that $H^{2,N}_{0,1}(\Omega) \subset W^{1,p}_0(\Omega)$, so $H^{2,N}_{0,1}(\Omega) \hookrightarrow W^{1,p}_0(\Omega)$. Furthermore, if $p > \frac{2N}{N+2}$, by Sobolev embedding theorem, the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By [1], $H^{2,N}_{0,1}(\Omega) \hookrightarrow L^2(\Omega)$ and the embedding is compact, i.e., $H^{2,N}_{0,1}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$.

Step two: Prove $H^{2,N}_{0,1}(\Omega) \hookrightarrow\hookrightarrow L^2_\eta(\Omega)$.

Since $H^{2,N}_{0,1}(\Omega)$ is a Hilbert space, it's reflexive, and it's separable since $H^2_0(\Omega)$ is separable and $H^{2,N}_{0,1}(\Omega)$ is dense in $H^2_0(\Omega)$. By [3], the bounded set of $H^{2,N}_{0,1}(\Omega)$ is weakly compact. Therefore, for any bounded sequence $\{u_n\} \in H^{2,N}_{0,1}(\Omega)$, up to a subsequence, we can assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H^{2,N}_{0,1}(\Omega) \\ u_n \to u, & \text{in } L^2(\Omega). \end{cases}$$

Since for $N \geq 5$, η satisfies (11), so $\forall\ \epsilon > 0$ small enough, there exists $r > 0$, such that $\forall |x| < r$, $|x|^4(\ln R/|x|)^2 \eta(x) < \epsilon$. Observe that

$$\int_\Omega \eta |u_n - u|^2 \,\mathrm{d}x = \int_{B_r(0)} |x|^4(\ln R/|x|)^2 \eta \frac{|u_n - u|^2}{|x|^4(\ln R/|x|)^2} \,\mathrm{d}x + \int_{\Omega \setminus B_r(0)} \eta |u_n - u|^2 \,\mathrm{d}x.$$

Applying (4), $\forall \epsilon > 0$, by the above discussion, there exists $r = r(\epsilon) > 0$, such that

$$\int_{B_r(0)} |x|^4(\ln R/|x|)^2 \eta \frac{|u_n - u|^2}{|x|^4(\ln R/|x|)^2} \,\mathrm{d}x$$

$$< \epsilon \int_{B_r(0)} \frac{|u_n - u|^2}{|x|^4(\ln R/|x|)^2} \, \mathrm{d}x < C\epsilon ||u_n - u||^2_{H^{2,N}_{0,1}(\Omega)}.$$

Since $\{u_n\}$ is bounded in $H^{2,N}_{0,1}(\Omega)$, letting $\epsilon \to 0$, we have $\int_{B_r(0)} \eta |u_n - u|^2 \, \mathrm{d}x \to 0$. Moreover,

$$\int_{\Omega \setminus B_r(0)} \eta |u_n - u|^2 \, \mathrm{d}x \le ||\eta||_{L^\infty(\Omega \setminus B_r(0))} ||u_n - u||^2_{L^2(\Omega)} \to 0, \; n \to \infty$$

therefore $\int_\Omega \eta |u_n - u|^2 \, \mathrm{d}x \to 0$, i.e., $u_n \to u$ in $L^2_\eta(\Omega)$. If $N = 4$, the proof is similar to that of $N \ge 5$. This completes the proof. □

Lemma 3.2. *Let $N = 4$. Then we have*

$$\inf_{u \in H^2_0(\Omega)} \frac{\displaystyle\int_\Omega \left(|\Delta u|^2 - \frac{u^2}{|x|^4(\ln R/|x|)^2} \right) \mathrm{d}x}{\displaystyle\int_\Omega \frac{u^2}{|x|^4(\ln R/|x|)^2(\ln \ln R/|x|)^2} \, \mathrm{d}x} = 1.$$

Proof. For any $\epsilon > 0$, fix $\delta > 0$ and let

$$u_\epsilon(x) = \begin{cases} (\ln R/|x|)^{1/2}(\ln \ln R/|x|)^{1/2+\epsilon}, & \delta \le |x| \le R_1 < 1 \\ a|x| + b, & |x| \le \delta \end{cases} \tag{15}$$

and u_ϵ is smooth up to the boundary. To guarantee u_ϵ has a continuous first order derivative on $|x| = \delta$, we require

$$a = -\frac{1}{2\delta}(\ln R/\delta)^{-1/2}(\ln \ln R/\delta)^{1/2+\epsilon} - \frac{1+2\epsilon}{2\delta}(\ln R/\delta)^{-1/2}(\ln \ln R/\delta)^{-1/2+\epsilon}$$

and

$$b = (\ln R/\delta)^{1/2}(\ln \ln R/\delta)^{1/2+\epsilon} + 1/2(\ln R/\delta)^{-1/2}(\ln \ln R/\delta)^{1/2+\epsilon} + \frac{1+2\epsilon}{2}(\ln R/\delta)^{-1/2}(\ln \ln R/\delta)^{-1/2+\epsilon}.$$

Observe that

$$\begin{aligned} &\int_{B_{R_1}(0)} \frac{u_\epsilon^2}{|x|^4(\ln R/|x|)^2(\ln \ln R/|x|)^2} \, \mathrm{d}x \\ &= 4\omega_4 \int_0^{R_1} \frac{u_\epsilon^2}{r(\ln R/r)^2(\ln \ln R/r)^2} \, \mathrm{d}r \\ &= 4\omega_4 \int_0^\delta \frac{(ar+b)^2}{r(\ln R/r)^2(\ln \ln R/r)^2} \, \mathrm{d}r + 4\omega_4 \int_\delta^{R_1} r^{-1}(\ln R/r)^{-1}(\ln \ln R/r)^{-1+2\epsilon} \, \mathrm{d}r \\ &\triangleq A + B. \end{aligned}$$

For A, we have

$$\begin{aligned} A &= 4\omega_4 \int_0^\delta \left(\frac{a^2 r}{(\ln R/r)^2(\ln \ln R/r)^2} + \frac{2ab}{(\ln R/r)^2(\ln \ln R/r)^2} + \frac{b^2}{r(\ln R/r)^2(\ln \ln R/r)^2} \right) \mathrm{d}r \\ &\triangleq A_1 + A_2 + A_3. \end{aligned}$$

For any $0 \le \delta < 1$, it's easy to check that A_1,A_2,A_3 converge to finite limit as $\epsilon \to 0$.
For B, we have

$$B = 4\omega_4 \int_\delta^{R_1} r^{-1}(\ln R/r)^{-1}(\ln\ln R/r)^{-1+2\epsilon}\,\mathrm{d}r \to \infty\ (\epsilon \to 0)$$

so as $\epsilon \to 0$, we obtain

$$\int_{B_{R_1}(0)} \frac{u_\epsilon^2}{|x|^4(\ln R/|x|)^2(\ln\ln R/|x|)^2}\,\mathrm{d}x \sim B = 4\omega_4 \int_\delta^{R_1} \frac{1}{r\ln R/r(\ln\ln R/r)^{1-2\epsilon}}\,\mathrm{d}r.$$

By direct calculating, if $0 \le r \le \delta$, then

$$\Delta u_\epsilon = 3ar^{-1}$$

while if $\delta < r \le R_1$, then

$$\begin{aligned}\Delta u_\epsilon = & -r^{-2}(\ln R/r)^{-1/2}(\ln\ln R/r)^{1/2+\epsilon}\\ & - \frac{1+2\epsilon}{r^2}(\ln R/r)^{-1/2}(\ln\ln R/r)^{-1/2+\epsilon}\\ & - \frac{1}{4r^2}(\ln R/r)^{-3/2}(\ln\ln R/r)^{1/2+\epsilon}\\ & + \frac{-1/4+\epsilon^2}{r^2}(\ln R/r)^{-3/2}(\ln\ln R/r)^{-3/2+\epsilon}\end{aligned}$$

and therefore,

$$\begin{aligned}&\int_{B_{R_1}(0)} \left(|\Delta u_\epsilon|^2 - \frac{u_\epsilon^2}{|x|^4(\ln R/|x|)^2}\right)\mathrm{d}x\\ &= 4\omega_4 \int_0^{R_1} \left(|\Delta u_\epsilon|^2 r^3 - \frac{u_\epsilon^2}{r(\ln R/r)^2}\right)\mathrm{d}r\\ &= 36\omega_4 \int_0^\delta a^2 r\,\mathrm{d}r - 4\omega_4 \int_0^\delta \frac{(ar+b)^2}{r(\ln R/r)^2}\,\mathrm{d}r\\ &\quad + 4\omega_4 \int_\delta^{R_1} \left(|\Delta u_\epsilon|^2 - r^{-1}(\ln R/r)^{-1}(\ln R/r)^{1+2\epsilon}\right)\mathrm{d}r\\ &\triangleq D_1 + D_2 + D_3,\end{aligned}$$

where

$$\begin{cases} D_1 = 36\omega_4 \displaystyle\int_0^\delta a^2 r\,\mathrm{d}r\\ D_2 = -4\omega_4 \displaystyle\int_0^\delta \frac{(ar+b)^2}{r(\ln R/r)^2}\,\mathrm{d}r\\ D_3 = 4\omega_4 \displaystyle\int_\delta^{R_1} \left(|\Delta u_\epsilon|^2 - r^{-1}(\ln R/r)^{-1}(\ln R/r)^{1+2\epsilon}\right)\mathrm{d}r. \end{cases}$$

Obviously, D_1,D_2 converge to finite limit as $\epsilon \to 0$. For D_3,

$$D_3 \sim 4\omega_4 \int_\delta^{R_1} \frac{1}{r \ln R/r (\ln R/r)^{1-2\epsilon}} \, \mathrm{d}r \to \infty$$

and therefore

$$\int_{B_{R_1}(0)} (|\Delta u_\epsilon|^2 - \frac{u_\epsilon^2}{|x|^4 (\ln R/|x|)^2}) \, \mathrm{d}x \backsim 4\omega_4 \int_\delta^{R_1} \frac{1}{r \ln R/r (\ln R/r)^{1-2\epsilon}} \, \mathrm{d}r.$$

Hence, letting $\epsilon \to 0$, we obtain

$$\int_\Omega (|\Delta u_\epsilon|^2 - \frac{u_\epsilon^2}{|x|^4 (\ln R/|x|)^2}) \, \mathrm{d}x \Big/ \int_\Omega \frac{u_\epsilon^2}{|x|^4 (\ln R/|x|)^2 (\ln \ln R/|x|)^2} \, \mathrm{d}x \to 1.$$

By inequality (1), the proof is completed. □

Remark 3.1. By Lemma 3.2, if $\mu_2 = 1$, the singular term

$$1/|x|^4 (\ln R/|x|)^2 (\ln R/|x|)^2$$

in Theorem 2.2 is called critical potential.

Lemma 3.3. *Consider the problem* (9)*, where* $0 \le \mu_1 \le 1 + N(N-4)/8$*,* $\Omega = B$ *is a unit ball in* $\mathbb{R}^N$ $(N \ge 5)$ *centered with the origin. If* (9) *admits a nontrivial solution* u *for* $\lambda = \lambda_{\mu_1}(q)$*, then* u *doesn't change sign in* B*.*

Proof. The proof is similar to that of Theorem 5.1 of [2]. We will prove it by contradiction. Assume that a solution u of (9) changes sign in B, define

$$K := \left\{ v \in H_{0,1}^{2,N}(B) : v \ge 0 \text{ a.e.}, v = \frac{\partial v}{\partial \gamma} = 0 \text{ on } \partial B \right\}.$$

Then K is a close convex cone and K is not empty. So there exists a projection $P : H_{0,1}^{2,N}(B) \to K$ such that $\forall\ u \in H_{0,1}^{2,N}(B)$,$\forall\ v \in K$

$$a(u - P(u), v - P(u)) \le 0. \tag{16}$$

Since K is a cone, we can replace v with tv in (16), where $t > 0$. Letting $t \to \infty$, we have

$$a(u - P(u), v) \le \lim_{t \to \infty} \frac{1}{t} a(u - P(u), P(u)).$$

Hence we have $\Delta^2(u - P(u)) \le 0$, by Boggio's principle, $u - P(u) \le 0$. Meanwhile, if we replace v with $tP(u)$ in (16), where $t > 0$, then we have

$$(t-1) a(u - P(u), P(u)) \le 0$$

so we have $a(u - P(u), P(u)) = 0$. Therefore u can be divided into $u = u_1 + u_2$, where $u_1 = P(u) \in K$, $u_2 = u - P(u)$, with $u_2 \le 0$. It's not hard to check that

$$\frac{I_{\mu_1}(u_1 - u_2)}{\int_B \eta |u_1 - u_2|^2 \, \mathrm{d}x} < \frac{I_{\mu_1}(u_1 + u_2)}{\int_B \eta |u_1 + u_2|^2 \, \mathrm{d}x},$$

it contradict with the definition of $\lambda_{\mu_1}(q)$. Hence u doesn't change sign in B. Since the Green function is strictly positive, so u is strictly positive or negative in B. $\square$

Similarly we can prove the following theorem.

Lemma 3.4. *Consider the problem* (10)*, with* $0 \le \mu_2 \le 1$ *and* $\Omega = B$ *is a unit ball in* $\mathbb{R}^4$ *centered with the origin. If* (10) *admits a nontrivial solution* u *for* $\tau = \tau_{\mu_2}(q)$*, then* u *doesn't change sign in* B.

4. The proofs of Theorems 2.1, 2.2

Proof of Theorem 2.1. (1) If $0 \le \mu_1 < 1 + N(N-4)/8$, then it's easy to check that $I_{\mu_1}(u)$ is coercive and weak lower semicontinuous in $H^{2,N}_{0,1}(\Omega)$. Define the manifold

$$M := \left\{ u \in H^{2,N}_{0,1}(\Omega) \,\middle|\, \int_\Omega \eta u^2 \, \mathrm{d}x = 1 \right\}.$$

Then M is a weakly closed subset of $H^{2,N}_{0,1}(\Omega)$. Obviously M is not empty. By [8], $I_{\mu_1}(u)$ admits its minimum by a minimizer $u \in M$. So $\lambda_{\mu_1}(q)$ is achieved and also the problem (9) has a nontrivial solution. By Lemma 3.3, we can choose their solution $u > 0$.

(2) If $\mu_1 = 1+N(N-4)/8$, the functional $I_{\mu_1}(u)$ is not coercive in $H^{2,N}_{0,1}(\Omega)$, so we can not follow the steps of (1). To conquer the difficulty, we consider the following problem:

(17)

$$\begin{cases} \triangle^2 u - \dfrac{N^2(N-4)^2}{16}\dfrac{u}{|x|^4} - \left(1+\dfrac{N(N-4)}{8}\right)\dfrac{sq(x)u}{|x|^4(\ln R/|x|)^2} = \lambda\eta(x)u & x \in \Omega \\ u \neq 0 & x \in \Omega \\ u = \dfrac{\partial u}{\partial \gamma} = 0 & x \in \partial\Omega, \end{cases}$$

where $0 \le s < 1$, q and η satisfy the assumptions of the theorem. Observe that the operator

$$\Delta^2 - \frac{N^2(N-4)^2}{16}\frac{1}{|x|^4} - \left(1 + \frac{N(N-4)}{8}\right)\frac{sq(x)}{|x|^4(\ln R/|x|)^2}$$

is coercive in $H^{2,N}_{0,1}(\Omega)$. By the first part of the theorem, the above problem admits a nontrivial solution u_s for $\lambda_s(q) = \lambda_{\mu_1}(sq)$. And observe that $\frac{u_s}{||u_s||_{H^{2,N}_{0,1}}}$ is also a nontrivial solution of (17). Hence $\forall\ 0 \le s < 1$, we can find $\{u_s\}$ such that u_s is a solution of (17) and $||u_s||_{H^{2,N}_{0,1}(\Omega)} = 1$. Therefore, by Lemma 3.1, up to a subsequence, we have

$$\begin{cases} u_s \rightharpoonup u_1, & \text{in } H^{2,N}_{0,1}(\Omega) \\ u_s \to u_1, & \text{in } L^2_\eta(\Omega). \end{cases}$$

We will prove that $u_s \to u_1$ in $H_{0,1}^{2,N}(\Omega)$ as $s \to 1$. In the fact, by (17), we have

$$\int_\Omega \left[|\Delta u_s|^2 - \frac{N^2(N-4)^2}{16}\frac{u_s^2}{|x|^4} - \left(1+\frac{N(N-4)}{8}\right)\frac{sq(x)u_s^2}{|x|^4(\ln R/|x|)^2}\right] \mathrm{d}x$$
$$= \lambda_s(q)\int_\Omega \eta u_s^2 \,\mathrm{d}x.$$

We will verify that, if we take $\omega(x) = \frac{q(x)}{|x|^4(\ln R/|x|)^2}$, then ω satisfies the assumption of η in the definition of $L_\eta^2(\Omega)$.

(1) $\forall x \in \Omega$, $\omega(x) \geq 0$ is obviously;

(2) $\forall x \in \Omega\backslash B_r(0)$, we have $\omega(x) \leq r^{-4}$, where $r > 0$ and $B_r(0) \subset \Omega$. Hence $\omega \in L^\infty(\Omega\backslash B_r(0))$;

(3) Observe that $q(x)$ satisfies (13), we have

$$\limsup_{|x|\to 0} |x|^4(\ln R/|x|)^2\omega(x) = \limsup_{|x|\to 0} q(x) = 0$$

therefore, $L_\omega^2(\Omega)$ is well defined. By Lemma 3.1, $H_{0,1}^{2,N}(\Omega) \hookrightarrow\hookrightarrow L_\omega^2(\Omega)$. Hence, we have

$$\begin{cases} \displaystyle\int_\Omega \frac{q(x)u_s^2}{|x|^4(\ln R/|x|)^2}\,\mathrm{d}x \to \int_\Omega \frac{q(x)u_1^2}{|x|^4(\ln R/|x|)^2}\,\mathrm{d}x, \\ \displaystyle\int_\Omega \eta u_s^2\,\mathrm{d}x \to \int_\Omega \eta u_1^2\,\mathrm{d}x. \end{cases} \tag{18}$$

Therefore

$$I_{\mu_1}(u_s) = \int_\Omega \left[|\Delta u_s|^2 - \frac{N^2(N-4)^2}{16}\frac{u_s^2}{|x|^4} - \left(1+\frac{N(N-4)}{8}\right)\frac{sq(x)u_s^2}{|x|^4(\ln R/|x|)^2}\right] \mathrm{d}x$$
$$= \lambda_s(q)\int_\Omega \eta u_s^2\,\mathrm{d}x \to \lambda_{\mu_1}(q)\int_\Omega \eta u_1^2\,\mathrm{d}x.$$

By the weak lower semicontinuous of I_{μ_1} and the fact that $\lambda_s(q) \to \lambda_{\mu_1}(q)$ as $s \to 1$, we have

$$I_{\mu_1}(u_1) \leq \liminf_{s\to 1} I_{\mu_1}(u_s) = \lambda_{\mu_1}(q)\int_\Omega \eta u_1^2\,\mathrm{d}x.$$

By the definition of $\lambda_{\mu_1}(q)$, we have $I_{\mu_1}(u_1) \geq \lambda_{\mu_1}(q)\int_\Omega \eta u_1^2\,\mathrm{d}x$. Therefore, $I_{\mu_1}(u_1) = \lambda_{\mu_1}(q)\int_\Omega \eta u_1^2\,\mathrm{d}x$, and

$$\begin{aligned} ||u_s||^2_{H_{0,1}^{2,N}(\Omega)} &= \int_\Omega \left(|\Delta u_s|^2 - \frac{N^2(N-4)^2}{16}\frac{u_s^2}{|x|^4}\right)\mathrm{d}x \\ &= \left(1+\frac{N(N-4)}{8}\right)\int_\Omega \frac{sq(x)u_s^2}{|x|^4(\ln R/|x|)^2}\,\mathrm{d}x + \lambda_s(q)\int_\Omega \eta u_s^2\,\mathrm{d}x \\ &\to \left(1+\frac{N(N-4)}{8}\right)\int_\Omega \frac{qu_1^2}{|x|^4(\ln R/|x|)^2}\,\mathrm{d}x + \lambda_{\mu_1}(q)\int_\Omega \eta u_1^2\,\mathrm{d}x \end{aligned}$$

$$= \int_\Omega \left(|\Delta u_1|^2 - \frac{N^2(N-4)^2}{16} \frac{q u_1^2}{|x|^4} \right) \mathrm{d}x = ||u_1||^2_{H^{2,N}_{0,1}(\Omega)}.$$

Hence $||u_s||_{H^{2,N}_{0,1}(\Omega)} \to ||u_1||_{H^{2,N}_{0,1}(\Omega)}$, i.e., $u_s \to u_1$ in $H^{2,N}_{0,1}(\Omega)$. So $\lambda_{\mu_1}(q)$ is achieved by u_1, and the problem (9) has a nontrivial solution u_1. By Theorem 3.3, if Ω is a unit ball centered with the origin, we can choose $u > 0$ or $u < 0$. Observe that $-u$ is also a solution of the problem (9), we can choose $u > 0$. This completes the proof. □

Proof of Theorem 2.2. The proof of this theorem is similar to that of Theorem 2.1. □

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York-London, 1975.

[2] Adimurthi, M. Grossi, and S. Santra, *Optimal Hardy-Rellich inequalities, maximum principle and related eigenvalue problem*, J. Funct. Anal. **240** (2006), no. 1, 36–83.

[3] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.

[4] S. Filippas and A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. **192** (2002), no. 1, 186–233.

[5] F. Gazzola, H. C. Grunau, and E. Mitidieri, *Hardy inequalities with optimal constants and remainder terms*, Trans. Amer. Math. Soc. **356** (2004), no. 6, 2149–2168.

[6] A. Tertikas and N. B. Zographopolous, *Best constants in the Hardy-Rellich inequalities and related improvements*, Adv. Math. **209** (2007), no. 2, 407–459.

[7] M. Willem, *Minimax Theorems*, Birkhäuser Boston, Inc., Boston, MA, 1996.

[8] Y. X. Yao, Y. T. Shen, and Z. H. Chen, *Biharmonic equation and an improved Hardy inequality*, Acta Math. Appl. Sin. Engl. Ser. **20** (2004), no. 3, 433–440.

Yangxin Yao
Department of Mathematics
South China University of Technology
Guangzhou 510640, P. R. China
E-mail address: mayxyao@scut.edu.cn

Shaotong He
Department of Mathematics
South China University of Technology
Guangzhou 510640, P. R. China
E-mail address: 379847218@foxmail.com

Qingtang Su
School of Mathematics and Computational Science
Sun Yat-Sen University
Guangzhou 510275, P. R. China
E-mail address: 656548092@foxmail.com