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ON MIXED TWO-TERM EXPONENTIAL SUMS

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Abstract. In this paper, we shall use analytic methods to study the hybrid mean value involving the mixed two-term exponential sums $C(m,n,r,\chi;q)$, and give several sharp asymptotic formulae.

1. Introduction

The classical two-term exponential sums is defined as

$$C(m,n,r;q)=\sum_{a=1}^{q} e\left(\frac{ma^r+na}{q}\right),$$

where $e(y)=e^{2\pi iy}$, q,m,n,r are integers with $q,r\geq 2$.

This summation is very important in additive number theory, because there exists close connections between this summation and Waring's problem. Many scholars have studied its arithmetical properties. For example, Davenport and Heilbronn [7] proved that

$$C(m,n,r;p^\alpha)\ll_r p^{\theta\alpha}(n,p^\alpha), \quad \text{if} \quad p\nmid m,$$

where $\theta=2/3$ if $r=3$, and $\theta=3/4$ if $r\geq 3$. Applying Weil's estimate for exponential sums over finite fields, Hua [8] proved that $\theta=1/2$ for all $r\geq 2$. Improvements have been made by Smith [17], Loxton and Smith [13], Loxton and Vaughan [14], Dabrowski and Fisher [5], etc. Recently, Ye found certain identities between the two-term exponential sums and hyper-Kloosterman sums (see Theorem 3 of reference [20]), these identities are in turn deduced from generalized Davenport-Hasse identities of Gauss sums. Applying the better bounds for hyper-Kloosterman sums for prime power modulus obtained in [5], he established better bounds for it.

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The mixed two-term exponential sum is defined as

$$C(m,n,r,\chi;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{ma^r+na}{q}\right),$$

where χ denotes a Dirichlet character modulo q. If $q=p$, it follows from Weil's work [18] that for all χ modulo p, and $p\nmid m$, we have

$$|C(m,n,r,\chi;p)|\le rp^{\frac{1}{2}}.$$

For $q=p^\alpha$, Cochrane and Zheng [3] showed the following upper bound

$$|C(m,n,r,\chi;p^\alpha)|\le rp^{\frac{2}{3}\alpha}(n,p^\alpha)^{\frac{1}{3}},$$

where $r\ge 2,\alpha\ge 1,p>2$ and m,n are any integers with $p\nmid m$; While if $p=2$, then for all χ modulo 2^α, they showed

$$|C(m,n,r,\chi;2^\alpha)|\le 2r2^{\frac{2}{3}\alpha}(n,2^\alpha)^{\frac{1}{3}}$$

and proved that the exponent $\frac{2}{3}\alpha$ is the best possible. Under the additional assumption that χ is a multiplicative character modulo p^α of conductor p and $p\nmid(m,n)$, they in [4] got the improved upper bound

$$|C(m,n,r,\chi;p^\alpha)|\le rp^{\frac{\alpha}{2}}.$$

In [19] Xu, Zhang, and Zhang studied the asymptotic properties of the $2l$-th power mean

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}\mid C(m,n,\chi;q)\mid^{2l}$$

and gave the following exact formula in the case $l=2$.

Proposition. *Let $q\ge 3$ be an integer and k be any positive integer with $(k,q)=1$. Then for any fixed integer n with $(n,q)=1$, we have the identity*

$$\sum_{m=1}^{q}\sum_{\chi \bmod q}|C(m,n,k,\chi;q)|^4$$
$$=q\phi^3(q)d(q)\prod_{\substack{p^\alpha\|q\\ \alpha\ge 2}}\left(1-\frac{2(k,p-1)-1}{(\alpha+1)(p-1)}\right)\prod_{p\|q}\left(1-\frac{2(k,p-1)-1}{2(p-1)}+\frac{(k,p-1)^2-1}{2(p-1)^2}\right),$$

where $\phi(q)$ is the Euler function, $d(n)$ is the divisor function, $\prod_{p^\alpha\|q}$ denotes the product over all p such that $p^\alpha\mid q$ and $p^{\alpha+1}\nmid q$.

If we assume that $k+1\equiv 0 \pmod{\phi(q)}$, the mixed two-term exponential sum is the general Kloosterman sum $S(m,n,\chi;q)$ which is defined as follows

$$S(m,n,\chi;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{ma+n\bar{a}}{q}\right),$$

where $a\bar{a} \equiv 1 \pmod q$. Chowla [2] and Malyshev [15] proved that for any integers m and n, we have

$$|S(m,n,\chi;p)| \ll (m,n,p)^{1/2}p^{1/2+\epsilon},$$

where ϵ is any positive real number. But about the properties of $S(m,n,\chi;q)$ for general q, we know very little about it so far. In fact, the value of $|S(m,n,\chi;q)|$ is quite irregular if q is not a prime. However, Zhang (see [22] and [23]) showed that the values of $S(m,n,\chi;q)$ enjoy many good distribution properties by proving that

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}|S(m,n,\chi;q)|^4 = q\phi^3(q)d(q)\prod_{p^\alpha \| q}\left(1-\frac{1}{(\alpha+1)(p-1)}\right)$$

and

$$\sum_{m=1}^{p}|S(m,n,\chi;p)|^4 = \begin{cases} p(2p^2-3p-3), & \text{if } \chi \text{ is the principal character } \bmod p; \\ p^2(3p-7), & \text{if } \chi \text{ is the Legendre symbol;} \\ 2p^2(p-3), & \text{if } \chi \text{ is a complex character } \bmod p. \end{cases}$$

He [24] also studied the hybrid mean value between Dirichlet L-functions and the general Kloosterman sums, and obtained that

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} |S(m,n,\chi;q)|^2\,|L(1,\chi)|^{2k}$$
$$= \left(\frac{\pi^2}{6}\right)^{2k-1}\phi^2(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k-1}\prod_{p\dagger q}\left(1-\frac{1-\binom{2k-2}{k-1}}{p^2}\right)+O\left(q^{\frac{3}{2}+\epsilon}\right).$$

Moreover, Liu and Zhang [12] investigated the hybrid mean value involving the general Kloosterman sums further, and obtained that for any positive integer k and integers m and n with $(mn,q)=1$, we have

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} |S(m,n,\chi;q)|^2\left|\frac{L'}{L}(1,\chi)\right|^{2k} = A(k,q)\phi^2(q)+O\left(q^{\frac{3}{2}+\epsilon}\right),$$

where $A(k,q)=\displaystyle\sum_{\substack{n=1\\(n,q)=1}}^{+\infty}\frac{\tau_k^2(n)}{n^2}$ is a constant depending on k and q,

$$\tau_k(n)=\sum_{m_1m_2\cdots m_k=n}\Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_k),$$

and $\Lambda(n)$ is the Mangoldt function.

It might be interesting to consider similar mean values on the mixed two-term exponential sums. In this paper, we shall use analytic methods to study the hybrid mean value involving the mixed two-term exponential sums $C(m,n,$

$r, \chi; q)$, and give several sharp asymptotic formulae. That is, we shall prove the following:

Theorem 1. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integers m and $k \geq 1$, we have*

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 \left| \frac{L'}{L}(1,\chi) \right|^{2k} = A(k,p^\alpha)p^{2\alpha} + O\left(p^{\frac{3\alpha}{2}+\epsilon}\right).$$

Theorem 2. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integers m and $k \geq 2$, we have*

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 |B_{k,\chi}|^2 = \frac{2\zeta(2k)(k!)^2 p^{(2k+1)\alpha}}{(2\pi)^{2k}} + O(p^{(2k+\frac{1}{2})\alpha+\epsilon}),$$

where $\zeta(s)$ is the Riemann zeta function and $B_{k,\chi}$ is a generalized Bernoulli numbers defined by

$$\sum_{a=1}^{q} \chi(a) \frac{te^{at}}{e^{qt}-1} = \sum_{k=0}^{+\infty} \frac{B_{k,\chi}}{k!} t^k.$$

Theorem 3. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integers m and $k \geq 2$, we have*

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 \tau^h(\overline{\chi}) B_{k,\chi}^h = \frac{2^{h-1}(k!)^h p^{(kh+2)\alpha}}{(-1)^{(k-1)h}(2\pi i)^{kh}} + O\left(p^{(kh+\frac{3}{2})\alpha+\epsilon}\right),$$

where $\tau(\chi)$ is Gauss sums.

Remark. In Theorem 1, if we bound the left hand side by estimation of individual terms, we can use the bounds $|C(m,n,r,\chi;p^\alpha)| = O(p^{\frac{2}{3}\alpha+\epsilon})$ and $|\frac{L'}{L}(1,\chi)| = O(p^\epsilon)$. This will give us a bound for the left hand side $O(p^{\frac{7}{3}\alpha+\epsilon})$, which is much worse than the bound of the right hand side. Similar discussions can be made in Theorems 2 and 3.

2. Several lemmas

To complete the proof of the theorems, we need the following several lemmas.

Lemma 1. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integer m, we have the following sharp bound that*

$$|C(m,n,r;p^\alpha)| \leq p^{\frac{\alpha}{2}}.$$

Proof. This is Corollary 1 of [3]. □

Lemma 2. *Let m, n, r, q be integers $r \geq 2$ and $q \geq 3$. Then we have the identity*

$$|C(m,n,r,\chi;q)|^2 = \phi(q) + \sum_{a=2}^{q}{}' \chi(a) \sum_{b=1}^{q}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{q}\right).$$

Proof. From the properties of residue systems, we have

$$\begin{aligned} |C(m,n,r,\chi;q)|^2 &= \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \chi(a\bar{b}) e\left(\frac{m(a^r-b^r)+n(a-b)}{q}\right) \\ &= \sum_{a=1}^{q}{}' \chi(a) \sum_{b=1}^{q}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{q}\right) \\ &= \phi(q) + \sum_{a=2}^{q}{}' \chi(a) \sum_{b=1}^{q}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{q}\right). \end{aligned}$$

This completes the proof. □

Lemma 3. *For any positive integer k, we have the asymptotic formula that*

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left| \frac{L'}{L}(1,\chi) \right|^{2k} = A(k,q)\phi(q) + O\left(q^{\epsilon}\right).$$

Proof. This is Theorem 2.1 of [12]. □

Lemma 4. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integers m and $k \geq 1$, we have the estimate that*

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} \left[\sum_{a=2}^{p^\alpha}{}' \chi(a) \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \right] \left| \frac{L'}{L}(1,\chi) \right|^{2k} \ll p^{\frac{3\alpha}{2}+\epsilon}.$$

Proof. Let χ_1 be the non-principal real character modulo p^α. Then from Siegel's theorem [16] and the properties of L-functions we know that

$$\frac{L'}{L}(1,\chi_1) \ll \frac{p^{\alpha\epsilon}(\log p^\alpha)^2}{C_1(\epsilon)}.$$

For any complex character χ modulo p^α, and $p^\alpha \leq \exp[C'(\log x)^{\frac{1}{2}}]$, where C' is any positive constant, from [6] we have

$$\Psi(x,\chi) = \sum_{n \leq x} \chi(n)\Lambda(n) \ll x \exp\left(-C''(\log x)^{\frac{1}{2}}\right)$$

holds for some positive $C^{''}$ depending only on $C^{'}$. Then for parameter $N \geq \exp\left(\frac{(\log p^\alpha)^2}{(C^{'})^2}\right)$, by Abel's identity we can get

$$\begin{aligned}\frac{L^{'}}{L}(1,\chi) &= \sum_{n=1}^{+\infty}\frac{\chi(n)\Lambda(n)}{n}\\ &= \sum_{1\leq n\leq N}\frac{\chi(n)\Lambda(n)}{n} + \int_N^{+\infty}\frac{\sum\limits_{N<n\leq y}\chi(n)\Lambda(n)}{y^2}\mathrm{d}y\\ &= \sum_{1\leq n\leq N}\frac{\chi(n)\Lambda(n)}{n} + O\left(\frac{\log N}{\exp\left(C''(\log N)^{\frac{1}{2}}\right)}\right).\end{aligned}$$

Therefore we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod p^\alpha\\ \chi\neq\chi_0}}\left[\sideset{}{'}\sum_{a=2}^{p^\alpha}\chi(a)\sideset{}{'}\sum_{b=1}^{p^\alpha}e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\right]\left|\frac{L^{'}}{L}(1,\chi)\right|^{2k}\\ =&\sum_{\substack{\chi \bmod p^\alpha\\ \chi\neq\chi_0\\ \chi\neq\chi_1}}\left[\sideset{}{'}\sum_{a=2}^{p^\alpha}\chi(a)\sideset{}{'}\sum_{b=1}^{p^\alpha}e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\right]\left|\frac{L^{'}}{L}(1,\chi)\right|^{2k}\\ &+O\left(\frac{p^{\frac{3\alpha}{2}}p^{2k\alpha\epsilon}(\log p^\alpha)^{4k}}{C_1^{2k}(\epsilon)}\right)\\ =&\sum_{\substack{\chi \bmod p^\alpha\\ \chi\neq\chi_0\\ \chi\neq\chi_1}}\left[\sideset{}{'}\sum_{a=2}^{p^\alpha}\chi(a)\sideset{}{'}\sum_{b=1}^{p^\alpha}e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\right]\left|\sum_{1\leq l\leq N}\frac{\chi(l)\Lambda(l)}{l}\right|^{2k}\\ &+O\left(\frac{p^{\frac{5\alpha}{2}}\log^{2k}N}{\exp\left(C''(\log N)^{\frac{1}{2}}\right)}\right)+O\left(\frac{p^{\frac{3\alpha}{2}}p^{2k\alpha\epsilon}(\log p^\alpha)^{4k}}{C_1^{2k}(\epsilon)}\right)\\ =&\sum_{\chi \bmod p^\alpha}\left[\sideset{}{'}\sum_{a=2}^{p^\alpha}\chi(a)\sideset{}{'}\sum_{b=1}^{p^\alpha}e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\right]\left|\sum_{1\leq l\leq N}\frac{\chi(l)\Lambda(l)}{l}\right|^{2k}\\ &+O\left(p^{\frac{3\alpha}{2}}\log^{2k}N\right)+O\left(\frac{p^{\frac{5\alpha}{2}}\log^{2k}N}{\exp\left(C''(\log N)^{\frac{1}{2}}\right)}\right)+O\left(\frac{p^{\frac{3\alpha}{2}}p^{2k\alpha\epsilon}(\log p^\alpha)^{4k}}{C_1^{2k}(\epsilon)}\right).\end{aligned}$$

Then from the orthogonality relations for character sums and Lemma 1, we can get

$$\sum_{\chi \bmod p^\alpha} \left[\sum_{a=2}^{p^\alpha}{}' \chi(a) \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right) \right] \left| \sum_{1\le l\le N} \frac{\chi(l)\Lambda(l)}{l} \right|^{2k}$$
$$= \sum_{\chi \bmod p^\alpha} \left[\sum_{a=2}^{p^\alpha}{}' \chi(a) \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right) \right] \left| \sum_{1\le l\le N^k} \frac{\chi(l)\tau_k(l)}{l} \right|^{2}$$
$$= \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right)$$
$$\times \sum_{1\le n_1\le N^k} \sum_{1\le n_2\le N^k} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2} \sum_{\chi \bmod p^\alpha} \chi(an_1)\overline{\chi}(n_2)$$
$$\le \phi(p^\alpha) \sum_{a=2}^{p^\alpha}{}' |C(m,n,r;p^\alpha)| \sum_{\substack{1\le n_1\le N^k \\ an_1\equiv n_2 \bmod p^\alpha}} \sum_{1\le n_2\le N^k} \frac{\tau_k(n_1)\tau_k(n_2)}{n_1n_2}$$
$$\le p^{\frac{\alpha}{2}}\phi(p^\alpha)\log^{2k} N \sum_{a=2}^{p^\alpha}{}' \sum_{1\le n_1\le N^k} \sum_{\substack{\frac{1-an_1}{p^\alpha}\le l\le \frac{N^k-an_1}{p^\alpha} \\ l\ne 0}} \frac{1}{n_1(lp^\alpha+an_1)}$$
$$+ p^{\frac{\alpha}{2}}\phi(p^\alpha)\log^{2k} N \sum_{a=2}^{p^\alpha}{}' \sum_{1\le n_1\le N^k} \frac{1}{an_1^2}$$
$$\ll p^{\frac{\alpha}{2}}\phi(p^\alpha)\log^{2k} N \sum_{a=2}^{p^\alpha}{}' \frac{1}{a} + p^{\frac{\alpha}{2}} \log^{2k+2} N \sum_{a=2}^{p^\alpha}{}' 1$$
$$\ll p^{\frac{\alpha}{2}}\phi(p^\alpha)\log^{2k+2} N.$$

Taking $N = \max\{\exp(\frac{\log^2 p^\alpha}{(C')^2}), \exp(\frac{\log^2 p^\alpha}{(C'')^2})\}$ in the above, we may immediately get

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi\ne\chi_0}} \left[\sum_{a=2}^{p^\alpha}{}' \chi(a) \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right) \right] \left| \frac{L'}{L}(1,\chi) \right|^{2k} \ll p^{\frac{3\alpha}{2}+\epsilon}.$$

This completes the proof. □

Lemma 5. *Let $q \geq 3$ be an integer. Then for any positive integer $k \geq 2$, we have*

$$\sum_{\chi \neq \chi_0} \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{G(r_1,\chi)\overline{G(r_2,\chi)}}{(r_1 r_2)^k} = 2\zeta(2k)\phi^2(q) + O(q^{1+\epsilon}).$$

Proof. This is Lemma 2.1 of [11]. □

Lemma 6. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integers m and $k \geq 2$, we have*

$$\begin{aligned}\Phi &= \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \\ &\quad \times \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} \frac{\chi(a)G(r_1,\chi)\overline{G(r_2,\chi)}}{(r_1 r_2)^k} \\ &\ll p^{\frac{5\alpha}{2}+\epsilon},\end{aligned}$$

where $G(r,\chi) = \sum_{b=1}^{p^\alpha} e(\frac{rb}{p^\alpha})$.

Proof. From the orthogonality relation for character χ mod p^α, we have

$$\begin{aligned}\Phi &= \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \\ &\quad \times \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} \frac{\chi(a)G(r_1,\chi)\overline{G(r_2,\chi)}}{(r_1 r_2)^k} \\ &= \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{1}{(r_1 r_2)^k} \\ &\quad \times \sum_{\chi \bmod p^\alpha} \chi(as\bar{t}) \sum_{s=1}^{p^\alpha}{}' \sum_{t=1}^{p^\alpha}{}' e\left(\frac{sr_1 - tr_2}{p^\alpha}\right) \\ &\quad - \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \left(\sum_{\substack{t=-\infty \\ t \neq 0}}^{+\infty} \frac{C_{p^\alpha}(t)}{t^k}\right)^2 \\ &= \phi(p^\alpha) \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{C_{p^\alpha}(r_1 - ar_2)}{(r_1 r_2)^k}\end{aligned}$$

$$-\sum_{a=2}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\left(\sum_{\substack{t=-\infty\\t\neq 0}}^{+\infty}\frac{C_{p^\alpha}(t)}{t^k}\right)^2,$$

where $C_{p^\alpha}(t)=G(t,\chi_0)=\sum_{a=1}^{\prime p^\alpha} e(\frac{ta}{p^\alpha})$ is Ramanujan sums. Then from the identity that

$$C_{p^\alpha}(t)=\sum_{d|(p^\alpha,t)} d\mu\left(\frac{p^\alpha}{d}\right),$$

we may have

$$\begin{aligned}
&\sum_{a=2}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty}\frac{C_{p^\alpha}(r_1-ar_2)}{(r_1r_2)^k}\\
&=\sum_{a=2}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\\
&\quad\times\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty}\frac{1}{(r_1r_2)^k}\sum_{d|(p^\alpha,r_1-ar_2)} d\mu\left(\frac{p^\alpha}{d}\right)\\
&\le p^{\frac{\alpha}{2}}\sum_{d|p^\alpha} d\sum_{a=2}^{p^\alpha}{}'\sum_{\substack{r_1=-\infty\\r_1\neq 0\\r_1\equiv ar_2 \bmod d}}^{+\infty}\sum_{\substack{r_2=-\infty\\r_2\neq 0}}^{+\infty}\frac{1}{(r_1r_2)^k}\\
&=p^{\frac{\alpha}{2}}\sum_{d|p^\alpha} d\sum_{a=2}^{p^\alpha}{}'\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty}\frac{1}{a^kr_1^{2k}}+p^{\frac{\alpha}{2}}\sum_{d|p^\alpha} d\sum_{a=2}^{p^\alpha}{}'\sum_{\substack{r_1=-\infty\\r_1\neq 0}}^{+\infty}\sum_{\substack{l=-\infty\\l\neq 0\\ld+ar_1\neq 0}}^{+\infty}\frac{1}{r_1^k(ld+ar_1)^k}\\
&\ll p^{\frac{3\alpha}{2}+\epsilon}.
\end{aligned}$$

Similarly, we can obtain

$$\sum_{a=2}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\left(\sum_{\substack{t=-\infty\\t\neq 0}}^{+\infty}\frac{C_{p^\alpha}(t)}{t^k}\right)^2\ll p^{\frac{3\alpha}{2}+\epsilon}.$$

Thus we have

$$\Phi\ll p^{\frac{5\alpha}{2}+\epsilon}.$$

This completes the proof. □

Lemma 7. *Let q and r be integers with $q \geq 2$ and $(r, q) = 1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \mod q} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \mod q}$ denotes the summation over all primitive characters modulo q and $J(q)$ denotes the number of primitive characters modulo q.

Proof. This is Lemma 3 of [21]. □

Lemma 8. *Let p be an odd prime and α be an integer with $\alpha \geq 2$. Then for any non-primitive character χ modulo p^α, we have the identity that*

$$\tau(\chi) = \sum_{a=1}^{p^\alpha} \chi(a) e\left(\frac{a}{p^\alpha}\right) = 0.$$

Proof. This is Theorem 7.4.2 of [9]. □

Lemma 9. *Let p be an odd prime and α be an integer with $\alpha \geq 2$. Then for any positive integers k and h, we have the asymptotic formula that*

$$\sideset{}{^*}\sum_{\chi \bmod p^\alpha} \tau^h(\overline{\chi}) B_{k,\chi}^h = \frac{2^{h-1}(k!)^h p^{(kh+1)\alpha}}{(-1)^{(k-1)h}(2\pi i)^{kh}} + O(p^{kh\alpha+\epsilon}).$$

Proof. This can be easily deduced from the proof of Theorem in [10]. □

Lemma 10. *Let p be an odd prime, n, r, α be integers with $r, \alpha \geq 2, p \nmid r(r-1), p \nmid n$ and $(r-1, p-1) = 1$. Then for any integers m and $k \geq 2$, we have the estimates*

$$\Psi_1 = \sideset{}{'}\sum_{a=2}^{p^\alpha} \sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1) + nb(a-1)}{p^\alpha}\right) \sum_{\substack{\chi \mod p^\alpha \\ \chi(-1)=-1}} \chi(a) L^h(k, \overline{\chi}) \ll p^{\frac{3\alpha}{2}+\epsilon}$$

and

$$\Psi_2 = \sideset{}{'}\sum_{a=2}^{p^\alpha} \sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1) + nb(a-1)}{p^\alpha}\right) \sum_{\substack{\chi \mod p^\alpha \\ \chi(-1)=1}} \chi(a) L^h(k, \overline{\chi}) \ll p^{\frac{3\alpha}{2}+\epsilon}.$$

Proof. We begin with the estimate for Ψ_1. Let $d_h(r)$ be the h-divisor function (i.e., the number of positive integer solutions of the equation $t = t_1 t_2 \cdots t_h$).

Then for any parameter $N \geq p^\alpha$ and non-principal character χ modulo p^α, applying Abel's identity we have

$$L^h(k, \overline{\chi}) = \sum_{t=1}^{\infty} \frac{\overline{\chi}(t) d_h(t)}{t^k} = \sum_{1 \leq r \leq N} \frac{\overline{\chi}(t) d_h(t)}{t^k} + k \int_N^\infty \frac{A(y, \overline{\chi})}{y^{k+1}} \mathrm{d}y,$$

where $A(y, \overline{\chi}) = \sum_{N < t \leq y} \overline{\chi}(t) d_h(t)$. Applying Lemma 4 in [24] and Cauchy inequality, we have

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \mid A(y, \overline{\chi}) \mid \leq \left(\phi(p^\alpha) \sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} \mid A(y, \overline{\chi}) \mid^2 \right)^{1/2} \ll y^{1-(1/2^{h-1})+\epsilon} \phi^{\frac{3}{2}}(p^\alpha).$$

Hence we have

$$\sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right) \sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}}^{*} \chi(a) k \int_N^\infty \frac{A(y, \overline{\chi})}{y^{k+1}} dy$$

$$\ll p^{\frac{3\alpha}{2}} \int_N^\infty \frac{k}{y^{k+1}} \left(\sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}} |A(y, \overline{\chi})| \right) dy \ll p^{\frac{3\alpha}{2}} \int_N^\infty \frac{y^{1-\frac{1}{2^{h-1}}+\epsilon} \phi^{\frac{3}{2}}(p^\alpha)}{y^2} dy$$

$$\ll \frac{p^{3\alpha+\epsilon}}{N^{2^{1-h}}}.$$

Combining the above we can get

$$\Psi_1 = \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right) \sum_{1 \leq t \leq N} \frac{d_h(t)}{t} \sum_{\substack{\chi \bmod p^\alpha \\ \chi(-1)=-1}}^{*} \overline{\chi}(t) \chi(a)$$

$$+ O\left(\frac{p^{3\alpha+\epsilon}}{N^{2^{1-h}}} \right).$$

Note that for any integer a with $(a, q) = 1$, from Lemma 7 we have

$$\begin{aligned}
\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \chi(a) &= \frac{1}{2} \sum_{\chi \bmod q}^{*} (1 - \chi(-1)) \chi(a) \\
&= \frac{1}{2} \sum_{\chi \bmod q}^{*} \chi(a) - \frac{1}{2} \sum_{\chi \bmod q}^{*} \chi(-a) \\
&= \frac{1}{2} \sum_{s|(q,a-1)} \mu\left(\frac{q}{s}\right) \phi(s) - \frac{1}{2} \sum_{s|(q,a+1)} \mu\left(\frac{q}{s}\right) \phi(s).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\Psi_1 \\
&= \frac{1}{2}\sideset{}{'}\sum_{a=2}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\sum_{1\le t\le N}\frac{d_h(t)}{t}\sum_{\substack{s\mid p^\alpha\\ t\equiv a \bmod s}}\mu\left(\frac{p^\alpha}{s}\right)\phi(s) \\
&\quad -\frac{1}{2}\sideset{}{'}\sum_{a=2}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \\
&\quad \times\sum_{1\le t\le N}\frac{d_h(t)}{t}\sum_{\substack{s\mid p^\alpha\\ t\equiv -a \bmod s}}\mu\left(\frac{p^\alpha}{s}\right)\phi(s)+O\left(\frac{p^{3\alpha+\epsilon}}{N^{2^{1-h}}}\right) \\
&= \frac{1}{2}\sideset{}{'}\sum_{a=2}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\frac{d_h(a)}{a}\sum_{s\mid p^\alpha}\mu\left(\frac{p^\alpha}{s}\right)\phi(s) \\
&\quad + O\left(\frac{p^{3\alpha+\epsilon}}{N^{2^{1-h}}}\right) \\
&\quad +O\left(\sideset{}{'}\sum_{a=2}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\sum_{s\mid p^\alpha}\phi(s)\sum_{\frac{1-a}{s}\le l\le\frac{N}{s}}(ls+a)^{\epsilon-1}\right) \\
&\quad +O\left(\sideset{}{'}\sum_{a=2}^{p^\alpha}\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)\sum_{s\mid p^\alpha}\phi(s)\sum_{\frac{1+a}{s}\le l\le\frac{N}{s}}(ls-a)^{\epsilon-1}\right) \\
&= p^{\frac{3\alpha}{2}+\epsilon}+N^\epsilon+O\left(\frac{p^{3\alpha+\epsilon}}{N^{2^{1-h}}}\right),
\end{aligned}$$

where we have used the estimate $d_h(t)\ll t^\epsilon$.

Now taking $N = p^{3\alpha\cdot 2^{h-2}}$ in the above, we may immediately obtain the following

$$\Psi_1 \ll p^{\frac{3\alpha}{2}+\epsilon}.$$

Using the similar method, we can obtain the second estimate. This completes the proof. □

3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First from Lemma 2 we have

$$\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} |C(m,n,r,\chi;q)|^2\left|\frac{L'}{L}(1,\chi)\right|^{2k}$$

$$= \phi(q) \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left| \frac{L'}{L}(1,\chi) \right|^{2k}$$

$$+ \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left[{\sum_{a=2}^{q}}' \chi(a) {\sum_{b=1}^{q}}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{q} \right) \right] \left| \frac{L'}{L}(1,\chi) \right|^{2k}.$$

Then from Lemmas 3 and 4, we may have

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 \left| \frac{L'}{L}(1,\chi) \right|^{2k} = A(k,p^\alpha)p^{2\alpha} + O\left(p^{\frac{3\alpha}{2}+\epsilon} \right).$$

This completes the proof of Theorem 1.

Then we come to prove Theorem 2. Let $q \geq 3$ be an integer. Then for any character χ modulo q, the generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_{k,\chi} = q^{k-1} {\sum_{a=1}^{q}}' \chi(a) B_k\left(\frac{a}{q} \right).$$

From Theorem 12.19 of [1] we know that if $0 < x \leq 1$, then we have

$$B_k(x) = -\frac{k!}{(2\pi i)^k} \sum_{\substack{t=-\infty \\ t \neq 0}}^{+\infty} \frac{e(tx)}{t^k}.$$

So we have

$$B_{k,\chi} = q^{k-1} {\sum_{a=1}^{q}}' \chi(a) \left(-\frac{k!}{(2\pi i)^k} \sum_{\substack{t=-\infty \\ t \neq 0}}^{+\infty} \frac{e\left(\frac{at}{q}\right)}{t^k} \right) = -\frac{k! q^{k-1}}{(2\pi i)^k} \sum_{\substack{t=-\infty \\ t \neq 0}}^{+\infty} \frac{G(t,\chi)}{t^k}.$$

Therefore from Lemmas 2, 5 and 6, we may have

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 |B_{k,\chi}|^2$$

$$= \frac{(k!)^2 p^{2(k-1)^\alpha} \phi(p^\alpha)}{(2\pi)^{2k}} \sum_{\substack{\chi \bmod p^\alpha \\ \chi \neq \chi_0}} \sum_{\substack{r_1=-\infty \\ r_1 \neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2 \neq 0}}^{+\infty} \frac{G(r_1,\chi)\overline{G(r_2,\chi)}}{(r_1 r_2)^k}$$

$$+ \frac{(k!)^2 p^{2(k-1)\alpha}}{(2\pi)^{2k}} {\sum_{a=2}^{p^\alpha}}' {\sum_{b=1}^{p^\alpha}}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha} \right)$$

$$\times \sum_{\substack{\chi \bmod p^\alpha \\ \chi\neq\chi_0}} \chi(a) \sum_{\substack{r_1=-\infty \\ r_1\neq 0}}^{+\infty} \sum_{\substack{r_2=-\infty \\ r_2\neq 0}}^{+\infty} \frac{G(r_1,\chi)\overline{G(r_2,\chi)}}{(r_1 r_2)^k}$$
$$= \frac{2\zeta(2k)(k!)^2 p^{(2k+1)\alpha}}{(2\pi)^{2k}} + O(p^{(2k+\frac{1}{2})\alpha+\epsilon}).$$

This completes the proof of Theorem 2.

Now we come to prove Theorem 3. From Lemmas 2, 8 and 9, we also have

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi\neq\chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 \tau^h(\overline{\chi}) B_{k,\chi}^h$$
$$= \phi(p^\alpha) \sum_{\substack{\chi \bmod p^\alpha \\ \chi\neq\chi_0}} \tau^h(\overline{\chi}) B_{k,\chi}^h$$
$$+ \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) \sum_{\substack{\chi \bmod p^\alpha \\ \chi\neq\chi_0}} \chi(a)\tau^h(\overline{\chi}) B_{k,\chi}^h$$
$$= \frac{(-k!)^h p^{(k-1)h\alpha}}{(2\pi i)^{kh}} \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)$$
$$\times \sideset{}{^*}\sum_{\chi \bmod p^\alpha} \chi(a)\tau^h(\overline{\chi}) \left(\sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \frac{G(t,\chi)}{t^k}\right)^h + \phi(p^\alpha) \sideset{}{^*}\sum_{\substack{\chi \bmod p^\alpha \\ \chi\neq\chi_0}} \tau^h(\overline{\chi}) B_{k,\chi}^h$$
$$=: \frac{2^{h-1}(k!)^h p^{(kh+2)\alpha}}{(-1)^{(k-1)h}(2\pi i)^{kh}} + O\left(p^{(kh+1)\alpha+\epsilon}\right) + E.$$

Then for any primitive character χ, from the property of Gauss sums that $G(t,\chi) = \overline{\chi}(t)\tau(\chi)$ and $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)p^\alpha$, we have

$$E = \frac{(-k!)^h p^{(k-1)h\alpha}}{(2\pi i)^{kh}} \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)$$
$$\times \sideset{}{^*}\sum_{\chi \bmod p^\alpha} \chi(a)\tau^h(\overline{\chi}) \left(\sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \frac{G(t,\chi)}{t^k}\right)^h$$
$$= \frac{(-k!)^h p^{kh\alpha}}{(2\pi i)^{kh}} \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)$$

$$\times \sum_{\chi \bmod p^\alpha}^{*} \chi^h(-1)\chi(a) \left(\sum_{\substack{t=-\infty \\ t\neq 0}}^{+\infty} \frac{\overline{\chi}(t)}{t^k} \right)^h$$

$$= \frac{(-k!)^h p^{kh\alpha}}{(2\pi i)^{kh}} \sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right)$$

$$\times \sum_{\chi \bmod p^\alpha}^{*} \chi^h(-1)\chi(a) \left(1+\overline{\chi}(-1)(-1)^k\right)^h L^h(k,\overline{\chi})$$

$$= \begin{cases} \dfrac{(-2k!)^h p^{kh\alpha}}{(2\pi i)^{kh}} \displaystyle\sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) & \\ \times \sum^{*}_{\chi(-1)=1} \chi(a) L^h(k,\overline{\chi}), & \text{if } 2 \mid k; \\ \dfrac{(2k!)^h p^{kh\alpha}}{(2\pi i)^{kh}} \displaystyle\sum_{a=2}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' e\left(\frac{mb^r(a^r-1)+nb(a-1)}{p^\alpha}\right) & \\ \times \sum^{*}_{\chi(-1)=-1} \chi(a) L^h(k,\overline{\chi}), & \text{if } 2 \nmid k. \end{cases}$$

Therefore from Lemma 10, we may immediately obtain

$$\sum_{\substack{\chi \bmod p^\alpha \\ \chi\neq\chi_0}} |C(m,n,r,\chi;p^\alpha)|^2 \tau^h(\overline{\chi}) B^h_{k,\chi} = \frac{2^{h-1}(k!)^h p^{(kh+2)\alpha}}{(-1)^{(k-1)h}(2\pi i)^{kh}} + O\left(p^{(kh+\frac{3}{2})\alpha+\epsilon}\right).$$

This completes the proof of Theorem 3.

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