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NEW PRIMAL-DUAL INTERIOR POINT METHODS FOR $P_*(\kappa)$ LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. In this paper we propose new primal-dual interior point methods (IPMs) for $P_*(\kappa)$ linear complementarity problems (LCPs) and analyze the iteration complexity of the algorithm. New search directions and proximity measures are defined based on a class of kernel functions, $\psi(t) = \frac{t^2-1}{2} - \int_1^t e^{q\left(\frac{1}{\xi}-1\right)} d\xi$, $q \geq 1$. If a strictly feasible starting point is available and the parameter $q = \log\left(1 + a\sqrt{\frac{2\tau+2\sqrt{2n\tau}+\theta n}{1-\theta}}\right)$, where $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$, then new large-update primal-dual interior point algorithms have $O((1+2\kappa)\sqrt{n}\log n \log\frac{n}{\varepsilon})$ iteration complexity which is the best known result for this method. For small-update methods, we have $O((1+2\kappa)q\sqrt{qn}\log\frac{n}{\varepsilon})$ iteration complexity.

1. Introduction

In this paper we consider linear complementarity problem (LCP) as follows:

$$s = Mx + q,\ xs = 0,\ x \geq 0,\ \ s \geq 0, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix and $x, s, q \in \mathbb{R}^n$, and xs denotes the componentwise product of vectors x and s.

LCPs have many applications, e.g., linear and quadratic programming, finding a Nash-equilibrium in bimatrix games, economies with institutional restrictions upon prices, contact problems with friction, optimal stopping in Markov chains, circuit simulation, free boundary problems, and calculating the interval hull of linear systems of interval equations ([14]).

The primal-dual interior point method (IPM) for linear optimization (LO) problem was first proposed in [6] and [9]. Since then many other algorithms have been developed based on the primal-dual strategy. Subsequently, Kojima

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et al. [7] generalized the algorithm in [6] to monotone linear complementarity problems, that is $P_*(0)$ LCPs. They also proposed an $O(\sqrt{n}L)$ potential reduction algorithm ([8]). Several variants of the Mizuno-Todd-Ye type predictor-corrector interior point algorithm are proposed. First, Miao [10] extended the Mizuno-Todd-Ye predictor-corrector method to $P_*(\kappa)$ LCPs. His algorithm uses the l_2 neighborhood of the central path and has $O((1+\kappa)\sqrt{n}L)$ iteration complexity. Later, Illés et al. [4] give a version of Mizuno-Todd-Ye predictor-corrector interior point algorithm for the $P_*(\kappa)$ LCP and obtained $O((1+\kappa)^{\frac{3}{2}}\sqrt{n}L)$ iteration complexity.

Most of the classical primal-dual IPM for LO are based on the use of the logarithmic kernel function, e.g. see [13]. Peng et al. [12] introduced self-regular kernel functions for primal-dual IPMs for LO and obtained the best complexity result for large-update primal-dual IPMs for LO with a specific self regular kernel function. Recently, Bai et al. [1] proposed a new class of eligible kernel functions and proposed a unified framework for the complexity analysis of the algorithm. They greatly simplified the complexity analysis of IPMs.

In this paper we propose a new primal-dual IPM for $P_*(\kappa)$ LCP based on a new class of kernel functions which generalize the function defined in [1]. For the complexity analysis we follow the scheme presented in [1]. When the parameter $q = \log\left(1 + \left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\sqrt{\frac{2\tau+2\sqrt{2n\tau}+\theta n}{1-\theta}}\right)$, we have $O((1+2\kappa)\sqrt{n}\log n \log\frac{n}{\varepsilon})$ iteration complexity for large-update methods which is so far the best known complexity result. For small-update methods, we have $O((1+2\kappa)q\sqrt{qn}\log\frac{n}{\varepsilon})$ iteration complexity result.

This paper is organized as follows: In Section 2 we recall basic concepts and the notion of the central path. In Section 3 we describe the kernel function and its properties. Finally, in Section 4 we obtain the complexity result of the algorithm.

We use the following notations throughout the paper : $\mathbb{R}^n_+$ denotes the set of n dimensional nonnegative vectors and $\mathbb{R}^n_{++}$, the set of n dimensional positive vectors. For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $x_{\min} = \min\{x_1, x_2, \ldots, x_n\}$, i.e., the minimal component of x, $\|x\|$ is the 2-norm of x, and X is the diagonal matrix from a vector x, i.e., $X = \mathrm{diag}(x)$. xs denotes the componentwise product of vectors x and s. $x^T s$ is the scalar product of the vectors x and s. e is the n-dimensional vector of ones and I is the n-dimensional identity matrix. J is the index set, i.e., $J = \{1, 2, \ldots, n\}$. For $f(t), g(t) : \mathbb{R}_{++} \to \mathbb{R}_{++}$, we write $f(t) = O(g(t))$ if $f(t) \leq kg(t)$ for some positive constant k and $f(t) = \Theta(g(t))$ if $k_1 g(t) \leq f(t) \leq k_2 g(t)$ for some positive constants k_1 and k_2.

2. Preliminaries

We give some basic concepts and introduce the generic IPM.

Definition 2.1. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a kernel function if ψ is twice differentiable and the following conditions are satisfied:

(i) $\psi^{'}(1) = \psi(1) = 0$,
(ii) $\psi^{''}(t) > 0$, $t > 0$,
(iii) $\lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty$.

Definition 2.2. A function $f : D(\subset \mathbb{R}) \to \mathbb{R}$ is exponentially convex if and only if $f(\sqrt{x_1 x_2}) \leq \frac{1}{2}(f(x_1) + f(x_2))$ for all $x_1, x_2 \in D$.

We denote the strictly feasible set of LCP (1) by $\mathcal{F}^o$, i.e.,

$$\mathcal{F}^o := \{(x, s) \in \mathbb{R}^{2n}_{++} : s = Mx + q\}.$$

Definition 2.3. A $(x, s) \in \mathcal{F}^o$ is an ε-approximate solution if and only if $x^T s \leq \varepsilon$ for $\varepsilon > 0$.

$P_*(\kappa)$ matrix is first introduced by Kojima et al. [5] which is the generalization of positive semi-definite matrices.

Definition 2.4. Let $\kappa \geq 0$. A matrix $M \in \mathbb{R}^{n\times n}$ is called a $P_*(\kappa)$ matrix if

$$(1 + 4\kappa) \sum_{i\in J_+(x)} x_i(Mx)_i + \sum_{i\in J_-(x)} x_i(Mx)_i \geq 0$$

for all $x \in R^n$, where

$$J_+(x) = \{i \in J : x_i(Mx)_i \geq 0\} \text{ and } J_-(x) = \{i \in J : x_i(Mx)_i < 0\}.$$

Definition 2.5. A matrix $M \in \mathbb{R}^{n\times n}$ is called a P_* matrix if it is a $P_*(\kappa)$ matrix for some $\kappa \geq 0$, i.e., $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$.

Note that the class P_* contains the class PSD of positive semi-definite matrices, and the class P of matrices with all the principal minors positive.

Proposition 2.6 ([5]). *If $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix, then*

$$M' = \begin{pmatrix} -M & I \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X,\ S\ \in \mathbb{R}^{n\times n}$.

Corollary 2.7. *Let $M \in \mathbb{R}^{n\times n}$ be a $P_*(\kappa)$ matrix and $x,\ s \in \mathbb{R}^n_{++}$. Then for all $a \in \mathbb{R}^n$ the system*

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = a \end{cases}$$

has a unique solution $(\Delta x, \Delta s)$.

In generic IPM, to find an ε-approximate solution for (1) we perturb the complementarity condition, i.e., the second equation in (1), and we get the following parameterized system:

$$s = Mx + q, \quad xs = \mu e, \quad x > 0,\ s > 0, \tag{2}$$

where $\mu > 0$. Without loss of generality, we assume that (1) is strictly feasible, i.e., there exists (x^0, s^0) such that $s^0 = Mx^0 + q$, $x^0 > 0$, $s^0 > 0$, and moreover,

we have an initial strictly feasible point with $\Psi(x^0, s^0, \mu^0) \leq \tau$ for some $\mu^0 > 0$. Indeed, we may not have an available strictly feasible point (x^0, s^0). In order to solve this difficulty, we embed (1) to an artificial LCP which has a strictly feasible point ([5]). For this given strictly feasible point (x^0, s^0) we can always find a $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \leq \tau$. Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, (2) has a unique solution for any $\mu > 0$. We denote the solution of (2) as $(x(\mu), s(\mu))$ for given $\mu > 0$. We also call it *μ-center* for given μ and the solution set $\{(x(\mu), s(\mu)) \mid \mu > 0\}$ the *central path* for system (1). Note that the sequence $(x(\mu), s(\mu))$ approaches to the solution (x, s) of the system (1) as $\mu \to 0$ ([5]). We define the following notations:

$$d = \sqrt{\frac{x}{s}}, \quad v = \sqrt{\frac{xs}{\mu}}, \quad d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s}. \tag{3}$$

Then we have the scaled Newton-system as follows:

$$\begin{cases} -\bar{M}d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{cases} \tag{4}$$

where $\bar{M} = DMD$ and $D = \mathrm{diag}(d)$.

Note that the righthand side of the second equation in (4) is exactly the negative gradient of the logarithmic barrier function $\Psi_l(v)$,

$$\Psi_l(v) = \sum_{i=1}^{n} \psi_l(v_i), \quad \psi_l(t) = \frac{t^2 - 1}{2} - \log t.$$

In this paper we replace the second equation in (4) with

$$d_x + d_s = -\nabla\Psi(v), \tag{5}$$

where

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i), \quad \psi(t) = \frac{t^2 - 1}{2} - \int_1^t e^{q\left(\frac{1}{\xi} - 1\right)} d\xi, \qquad q \geq 1. \tag{6}$$

So we get the following modified Newton system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = -\mu v \nabla\Psi(v). \end{cases} \tag{7}$$

Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, this system uniquely defines a new search direction $(\Delta x, \Delta s)$ by Corollary 2.7. Throughout the paper, we assume that a proximity parameter τ and a barrier update parameter θ are given, $0 < \theta < 1$. The algorithm works as follows. We assume that a strictly feasible point (x, s) is given which is in a τ-neighborhood of the given μ-center. Then after decreasing μ to $\mu_+ = (1 - \theta)\mu$ for some fixed $\theta \in (0, 1)$, we solve the modified Newton system (7) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, s_+) that is in a τ-neighborhood of

the μ_+-center and then we let $\mu := \mu_+$ and $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1-\theta$ and we solve the modified Newton system targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, e.g. $n\mu \leq \varepsilon$.

Algorithm

Input:
A threshold parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta,\ 0 < \theta < 1$;
starting point (x^0, s^0) and $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \leq \tau$;
begin
$x := x^0;\ s := s^0;\ \mu := \mu^0$;
while $n\mu > \varepsilon$ do
begin
$\mu := (1-\theta)\mu$;
while $\Psi(v) > \tau$ do
begin
solve (7) for Δx and Δs;
determine a step size α from (18);
$x := x + \alpha\Delta x$;
$s := s + \alpha\Delta s$;
end
end
end

Remark 2.8. One distinguishes IPMs as large-update methods when $\theta = \Theta(1)$ and small-update methods when $\theta = \Theta(\frac{1}{\sqrt{n}})$.

3. Properties of the kernel function

For $\psi(t)$ in (6), we have

$$\psi^{'}(t) = t - e^{q(\frac{1}{t}-1)}, \psi^{''}(t) = 1 + \frac{q}{t^2}e^{q(\frac{1}{t}-1)}, \psi^{'''}(t) = -\frac{q(q+2t)}{t^4}e^{q(\frac{1}{t}-1)}. \tag{8}$$

Since $\psi^{''}(t) > 0,\ \psi(t)$ is strictly convex. Note that

$$\psi(1) = \psi^{'}(1) = 0,\ \psi^{'''}(t) < 0,\ t > 0.$$

And due to $\psi(1) = \psi^{'}(1) = 0$, $\psi(t)$ is determined by the second derivative:

$$\psi(t) = \int_1^t \int_1^{\xi} \psi^{''}(\varsigma) d\varsigma d\xi. \tag{9}$$

Note that since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$\Psi(v) = 0 \;\Leftrightarrow\; \delta(v) = 0 \;\Leftrightarrow\; v = e.$$

We use $\Psi(v)$ as a proximity function between the current iterate and the μ-center. We also define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) = \frac{1}{2} \parallel \nabla\Psi(v) \parallel = \frac{1}{2} \parallel d_x + d_s \parallel . \tag{10}$$

In the following lemma we provide a lower bound for $\delta(v)$ in terms of the proximity function $\Psi(v)$.

Lemma 3.1. *We have* $\delta(v) \geq \sqrt{\frac{\Psi(v)}{2}}$.

Proof. Using (9) and $\psi^{''}(t) \geq 1$, we have

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^\xi \psi^{''}(\varsigma)d\varsigma d\xi \leq \int_1^t \int_1^\xi \psi^{''}(\xi)\psi^{''}(\varsigma)d\varsigma d\xi \\ &= \int_1^t \psi^{''}(\xi)\psi^{'}(\xi)d\xi = \frac{1}{2}\psi^{'}(t)^2.\end{aligned}$$

Using the definition of Ψ and $\delta(v)$, we have

$$\Psi(v) = \sum_{i=1}^n \psi(v_i) \leq \frac{1}{2}\sum_{i=1}^n \psi^{'}(v_i)^2 = \frac{1}{2}\|\nabla\Psi(v)\|^2 = 2\delta(v)^2.$$

Since $\delta(v) \geq 0$, we get the desired result. □

In the following lemma we give key properties which are important in the analysis of Algorithm.

Lemma 3.2 (Section 3 in [2])**.** *The kernel function* $\psi(t)$ *in* (6) *satisfies the following properties*:

(i) $t\psi^{''}(t) + \psi^{'}(t) > 0,\ t > 0,$
(ii) $\psi^{'''}(t) < 0,\ t > 0,$
(iii) $2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t) > 0,\ 0 < t \leq 1,$
(iv) $\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0,\ t > 1,\ \beta > 1.$

By Lemma 3.2 (i) and Lemma 1 in [12], we get the following result.

Corollary 3.3. *The kernel function* $\psi(t)$ *is exponentially convex, i.e.,* $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)),\ t_1,\ t_2 > 0.$

Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$ and $\rho : [0, \infty) \to (0, 1]$ the inverse function of $-\frac{1}{2}\psi^{'}(t)$ for $t \in (0, 1]$. For the kernel function $\psi(t) = \frac{t^2-1}{2} - \int_1^t e^{q(\frac{1}{\xi}-1)}d\xi,\ q \geq 1$, we denote the barrier term as

$$\psi_b(t) = -\int_1^t e^{q(\frac{1}{\xi}-1)}d\xi,\ q \geq 1.$$

Let $\underline{\rho} : [0, \infty) \to (0, 1]$ be the inverse function of $-\psi_b^{'}(t)$, $t \in (0, 1]$. Then we have the following lemma.

Lemma 3.4. *We have the following*:

(i) $\psi(t) \leq \frac{1+q}{2}(t-1)^2$, $t \geq 1$.
(ii) $\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s}$, $s \geq 0$.
(iii) $\rho(s) \geq \underline{\rho}(1+2s)$, $s > 0$.

Proof. (i) Using Taylor's Theorem, $\psi(1) = \psi^{'}(1) = 0$, $\psi^{''}(1) = 1 + q$ and $\psi^{'''}(t) < 0$, we have

$$\begin{aligned}\psi(t) &= \psi(1) + \psi^{'}(1)(t-1) + \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3 \\ &= \frac{1}{2}(1+q)(t-1)^2 + \frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3 \\ &\leq \frac{1}{2}(1+q)(t-1)^2\end{aligned}$$

for some ξ such that $1 \leq \xi \leq t$.

(ii) Since $s = \psi(t) = \frac{t^2-1}{2} + \psi_b(t) \leq \frac{t^2-1}{2}$, we have $t = \varrho(s) \geq \sqrt{1+2s}$. On the other hand, by (9) and (8), $s = \psi(t) = \int_1^t \int_1^\xi \psi^{''}(\varsigma) d\varsigma d\xi \geq \int_1^t \int_1^\xi d\varsigma d\xi = \frac{1}{2}(t-1)^2$. Thus $t = \varrho(s) \leq 1 + \sqrt{2s}$. Hence we have $\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s}$, $s \geq 0$.

(iii) Let $t = \rho(s)$. By the definition of ρ, $-2s = \psi^{'}(t) = t + \psi_b^{'}(t)$, $t \leq 1$. By $t \leq 1$ and the definition of $\underline{\rho}$, $-\psi_b^{'}(\rho(s)) = -\psi_b^{'}(t) = t + 2s \leq 1 + 2s = -\psi_b^{'}(\underline{\rho}(1+2s))$. Since $-\psi_b^{''}(t) = -\frac{q}{t^2} e^{q\left(\frac{1}{t}-1\right)} < 0$, $-\psi_b^{'}(t)$ is monotonically decreasing. Thus we have $t = \rho(s) \geq \underline{\rho}(1+2s)$. □

Letting $s = -\psi_b^{'}(t)$, we have $\underline{\rho}(s) = \frac{q}{q+\log s}$, $s > 0$. By Lemma 3.4(iii), we have

$$\rho(s) \geq \underline{\rho}(1+2s) = \frac{q}{q + \log(1+2s)}, \ s \geq 0. \tag{11}$$

Note that at the start of outer iteration of the algorithm, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence, during the course of the algorithm the largest values of $\Psi(v)$ occur just after the updates of μ. In the following lemma we give an estimate for the effect of a μ-update on the value of $\Psi(v)$.

Lemma 3.5. *If* $\Psi(v) \leq \tau$, *then we have for* $0 < \theta < 1$,

(i) $\Psi\left(\frac{v}{\sqrt{1-\theta}}\right) \leq \frac{2\tau + 2\sqrt{2n\tau} + \theta n}{2(1-\theta)}$,
(ii) $\Psi\left(\frac{v}{\sqrt{1-\theta}}\right) \leq \frac{(1+q)(\theta\sqrt{n} + \sqrt{2\tau})^2}{2(1-\theta)}$.

Proof. (i) By the definition of ϱ and $\frac{1}{\sqrt{1-\theta}} \geq 1$, $\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$. By Theorem 3.2 in [1], Lemma 3.4(ii), and $\psi(t) \leq \frac{t^2-1}{2}$, $t \geq 1$, we have

$$\begin{aligned}
&\Psi\left(\frac{v}{\sqrt{1-\theta}}\right) \\
&\leq n\psi\left(\frac{\varrho(\frac{\Psi(v)}{n})}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1+\sqrt{\frac{2\Psi(v)}{n}}}{1-\theta}\right) \leq \frac{n}{2}\left(\frac{\left(1+\sqrt{\frac{2\tau}{n}}\right)^2}{1-\theta}-1\right) \\
&= \frac{n}{2}\left(\frac{\frac{2\tau}{n}+2\sqrt{\frac{2\tau}{n}}+\theta}{1-\theta}\right) = \frac{2\tau+2\sqrt{2n\tau}+\theta n}{2(1-\theta)}.
\end{aligned}$$

(ii) Using Theorem 3.2 in [1], Lemma 3.4(i) and (ii), we have

$$\begin{aligned}
&\Psi\left(\frac{v}{\sqrt{1-\theta}}\right) \\
&\leq n\psi\left(\frac{\varrho(\frac{\Psi(v)}{n})}{\sqrt{1-\theta}}\right) \leq \frac{(1+q)n}{2}\left(\frac{\varrho(\frac{\Psi(v)}{n})}{\sqrt{1-\theta}}-1\right)^2 \\
&\leq \frac{(1+q)n}{2}\left(\frac{1+\sqrt{\frac{2\Psi(v)}{n}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \leq \frac{(1+q)n}{2}\left(\frac{\theta+\sqrt{\frac{2\Psi(v)}{n}}}{\sqrt{1-\theta}}\right)^2 \\
&= \frac{(1+q)(\theta\sqrt{n}+\sqrt{2\tau})^2}{2(1-\theta)}.
\end{aligned}$$

The last inequality holds because $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$. □

We define

$$\tilde{\Psi}_0 := \frac{2\tau+2\sqrt{2n\tau}+\theta n}{2(1-\theta)} \quad \text{and} \quad \hat{\Psi}_0 := \frac{(1+q)(\theta\sqrt{n}+\sqrt{2\tau})^2}{2(1-\theta)}. \tag{12}$$

We will use $\tilde{\Psi}_0$ for large-update methods and $\hat{\Psi}_0$ for small-update methods.

4. Complexity analysis

In this section we compute the total number of iterations of Algorithm. Since $P_*(\kappa)$ LCPs are generalization of LO problems, we loose the orthogonality of the search direction vectors d_x and d_s. After a damped step for fixed μ we have

$$x_+ = x + \alpha\Delta x, \quad s_+ = s + \alpha\Delta s.$$

Then by (3), we have

$$x_+ = x\left(e+\alpha\frac{\Delta x}{x}\right) = x\left(e+\alpha\frac{d_x}{v}\right) = \frac{x}{v}(v+\alpha d_x),$$

$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).$$

Then we have

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

Throughout the paper we assume that the step size α is such that the coordinates of the vectors $v + \alpha d_x$ and $v + \alpha d_s$ are positive. Hence by Corollary 3.3, we have

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\ ) \ \leq \ \frac{1}{2}\ (\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

For given $\mu > 0$ by letting $f(\alpha)$ be the difference of the new and old proximity measures, i.e.,

$$f(\alpha) = \Psi(v_+) - \Psi(v),$$

we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v).$$

Note that $f(0) = f_1(0) = 0$. By taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi^{'}(v_i + \alpha[d_x]_i)[d_x]_i + \psi^{'}(v_i + \alpha[d_s]_i)[d_s]_i),$$

where $[d_x]_i$ and $[d_s]_i$ denote the i-th components of the vectors d_x and d_s, respectively. From (5) and the definition of δ,

$$f_1^{'}(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{13}$$

By taking the derivative of $f_1^{'}(\alpha)$ with respect to α, we have

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi^{''}(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi^{''}(v_i + \alpha[d_s]_i)[d_s]_i^2). \tag{14}$$

Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (7), for $\Delta x \in R^n$ we have

$$(1 + 4\kappa)\sum_{i\in J_+}\Delta x_i\Delta s_i + \sum_{i\in J_-}\Delta x_i\Delta s_i \geq 0,$$

where $J_+ = \{\ i \in J\ :\ \Delta x_i\Delta s_i \geq 0\ \}$, $J_- = J - J_+$. Since $d_x d_s = \frac{v^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}$ and $\mu > 0$,

$$(1 + 4\kappa)\sum_{i\in J_+}[d_x]_i[d_s]_i + \sum_{i\in J_-}[d_x]_i[d_s]_i \geq 0. \tag{15}$$

For notational convenience we define

$$\delta := \delta(v), \quad \sigma_+ = \sum_{i\in J_+}[d_x]_i[d_s]_i\ , \quad \sigma_- = -\sum_{i\in J_-}[d_x]_i[d_s]_i.$$

In the following we cite some lemmas in [3] for the analysis of the algorithm.

Lemma 4.1 (Modification of Lemma 4.1 in [3]). $\sigma_+ \leq \delta^2 \quad and \quad \sigma_- \leq (1 + 4\kappa)\delta^2$.

Lemma 4.2 (Modification of Lemma 4.2 in [3]). $\sum_{i=1}^{n}([d_x]_i^2 + [d_s]_i^2) \leq 4(1 + 2\kappa)\delta^2, \|d_x\| \leq 2\sqrt{1+2\kappa}\ \delta$, *and* $\|d_s\| \leq 2\sqrt{1+2\kappa}\ \delta$.

Lemma 4.3 (Modification of Lemma 4.3 in [3]). $f_1^{''}(\alpha) \leq 2(1+2\kappa)\ \delta^2\psi^{''}(v_{\min} - 2\alpha\sqrt{1+2\kappa}\ \delta)$.

Lemma 4.4 (Modification of Lemma 4.4 in [3]). $f_1^{'}(\alpha) \leq 0$ *if* α *is satisfying*

$$-\psi'(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) + \psi^{'}(v_{\min}) \leq \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{16}$$

Lemma 4.5 (Modification of Lemma 4.5 in [3]). *Let* $\rho : [0, \infty) \to (0, 1]$ *denote the inverse function of the restriction of* $-\frac{1}{2}\psi^{'}(t)$ *to the interval* $(0, 1]$. *Then the largest step size* α *that satisfies* (16) *is given by*

$$\bar{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \tag{17}$$

Lemma 4.6 (Modification of Lemma 4.6 in [3]). *Let* ρ *and* $\bar{\alpha}$ *be as defined in Lemma 4.5. Then we have*

$$\bar{\alpha} \geq \frac{1}{1+2\kappa}\ \frac{1}{\psi''(\rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta))}.$$

Define

$$\tilde{\alpha} = \frac{1}{1+2\kappa}\ \frac{1}{\psi''(\rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta))} \tag{18}$$

and we will use $\tilde{\alpha}$ as the default step size in our Algorithm. By Lemma 4.6, we have $\bar{\alpha} \geq \tilde{\alpha}$. In the following, we want to evaluate the decrease of the proximity function value. We cite the following result in [12] without proof.

Lemma 4.7 (Lemma 12 in [12]). *Let* $h(t)$ *be a twice differentiable convex function with* $h(0) = 0$, $h^{'}(0) < 0$ *and let* $h(t)$ *attains its* (*global*) *minimum at* $t^* > 0$. *If* $h^{''}(t)$ *is increasing for* $t \in [0, t^*]$, *then*

$$h(t) \leq \frac{th^{'}(0)}{2}, \quad 0 \leq t \leq t^*.$$

Lemma 4.8 (Modification of Lemma 4.8 in [3]). *If the step size* α *is such that* $\alpha \leq \bar{\alpha}$, *then* $f(\alpha) \leq -\alpha\delta^2$.

Theorem 4.9. *Let* $\tilde{\alpha}$ *be a step size as defined in* (18). *Then we have*

$$f(\tilde{\alpha}) \leq -\frac{1}{1+2\kappa}\ \frac{\delta^2}{\psi''(\rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta))}. \tag{19}$$

Proof. By Lemma 4.6 and (18), $\tilde{\alpha} \leq \bar{\alpha}$. By Lemma 4.8, we get the desired result. □

Lemma 4.10. *The right hand side in* (19) *is monotonically decreasing in* δ.

Proof. Let $t = \rho(a\delta)$ where $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$. Then $0 < t \leq 1$ and $-\psi^{'}(\rho(a\delta)) = 2a\delta$, i.e., $\frac{1}{2}\psi^{'}(t) = -\frac{1}{2}\psi^{'}(\rho(a\delta)) = a\delta$. Then

$$\frac{1}{1+2\kappa}\ \frac{\delta^2}{\psi^{''}(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))} = \frac{1}{4a^2(1+2\kappa)}\frac{\psi^{'}(t)^2}{\psi^{''}(t)}.$$

Define

$$g(t) = \frac{1}{4a^2(1+2\kappa)}\frac{\psi^{'}(t)^2}{\psi^{''}(t)}.$$

Since ρ is monotonically decreasing, t is monotonically decreasing if δ increases. Hence the right hand in (19) is monotonically decreasing in δ if and only if the function $g(t)$ is monotonically decreasing for $0 < t \leq 1$. Note that $g(1) = 0$ and

$$g^{'}(t) = \frac{1}{4a^2(1+2\kappa)}\ \frac{\psi^{'}(t)\{2\psi^{''}(t)^2 - \psi^{'}(t)\psi^{'''}(t)\}}{\psi^{''}(t)^2}.$$

Since $\psi^{'}(1) = 0$ and $\psi^{''} > 0$, $\psi^{'}(t) \leq 0$ for $0 < t \leq 1$. By Lemma 3.2(iii), $g(t)$ is monotonically decreasing for $0 < t \leq 1$. Hence the lemma is proved. □

Lemma 4.11 (Lemma 14 in [12])**.** *Let* $t_0, t_1, \ldots, t_K$ *be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \kappa t_k^{1-\gamma},\ \ k = 0, 1, \ldots, K-1,$$

where $\kappa > 0$ *and* $0 < \gamma \leq 1$. *Then* $K \leq \lfloor \frac{t_0^{\gamma}}{\kappa\gamma} \rfloor$.

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration as Ψ_k, $k = 1,\ 2, \ldots$. Let K denote the total number of inner iterations in the outer iteration. Then by the definition of K, we have

$$\Psi_{K-1} > \tau, \quad 0 \leq \Psi_K \leq \tau.$$

Note that $\Psi_0 \leq \min\{\tilde{\Psi}_0, \hat{\Psi}_0\}$.

In the following lemma, we compute the upper bound for the total number of inner iterations which we needed to return to the τ-neighborhood again. For notational convenience we denote $\Psi(v)$ by Ψ and $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$.

Lemma 4.12. *Let* K *be the total number of inner iterations in an outer iteration. Then we have*

$$K \leq 8(1+\sqrt{2})(1+2\kappa)q\left(1 + \frac{1}{q}\log\left(1 + a\sqrt{2\Psi_0}\right)\right)^2 \Psi_0^{\frac{1}{2}},$$

where Ψ_0 *denotes the value of* $\Psi(v)$ *after the* μ*-update.*

Proof. Since $\psi^{'''}(t) < 0$, $\psi^{''}(t)$ is a decreasing function. Using (19) and (11), we have

$$f(\tilde{\alpha}) \le -\frac{1}{1+2\kappa}\ \frac{\delta^2}{\psi^{''}(\rho(\delta a))} \le -\frac{1}{1+2\kappa}\ \frac{\delta^2}{\psi^{''}\left(\frac{q}{q+\log(1+2a\delta)}\right)}.$$

By Lemma 4.10 and Lemma 3.1, we have

$$f(\tilde{\alpha}) \le -\frac{1}{1+2\kappa}\frac{\Psi}{2}\left(\psi^{''}\left(\frac{q}{q+\log\left(1+2a\sqrt{\frac{\Psi}{2}}\right)}\right)\right)^{-1}.$$

Since $\psi^{''}(t) = 1 + \frac{q}{t^2}e^{q\left(\frac{1}{t}-1\right)}$,

$$\psi^{''}\left(\frac{q}{q+\log\left(1+2a\sqrt{\frac{\Psi}{2}}\right)}\right) = 1 + q(1+a\sqrt{2\Psi})\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi})\right)^2.$$

Hence we have

$$f(\tilde{\alpha}) \le -\frac{\Psi}{2(1+2\kappa)}\frac{1}{1+q(1+a\sqrt{2\Psi})\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi})\right)^2}. \tag{20}$$

Assuming $\Psi_0 \ge \Psi \ge \tau \ge 1$ and using $a = 1 + \frac{1}{\sqrt{1+2\kappa}} \le 2$, we have

$$1 + a\sqrt{2\Psi} \le 1 + 2\sqrt{2\Psi} \le (1+2\sqrt{2})\sqrt{\Psi}.$$

From (20), we have

$$\begin{aligned} f(\tilde{\alpha}) &\le -\frac{\Psi}{2(1+2\kappa)}\frac{1}{\sqrt{\Psi}q(2+2\sqrt{2})\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi})\right)^2} \\ &\le -\frac{\sqrt{\Psi}}{4(1+2\kappa)}\frac{1}{q(1+\sqrt{2})\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi_0})\right)^2}. \end{aligned}$$

This implies that

$$\Psi_{k+1} \le \Psi_k - \beta {\Psi_k}^{1-\gamma}, \qquad k = 0, 1, 2, \ldots, K-1,$$

where

$$\beta = \frac{1}{4(1+\sqrt{2})(1+2\kappa)q\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi_0})\right)^2}, \quad \gamma = \frac{1}{2}.$$

Hence by Lemma 4.11, we have

$$K \le \frac{\Psi_0^{\gamma}}{\beta\gamma} = 8(1+\sqrt{2})(1+2\kappa)q\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi_0})\right)^2 \Psi_0^{\frac{1}{2}}. \tag{21}$$

This completes the proof. □

From (12) and (21), we have

$$K \leq 8(1+\sqrt{2})(1+2\kappa)q\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi_0})\right)^2\left(\frac{2\tau+2\sqrt{2n\tau}+\theta n}{2(1-\theta)}\right)^{\frac{1}{2}}.$$

The upper bound for the total number of iterations is obtained by multiplying the number K by the number of central path parameter updates. If the central path parameter μ has the initial value μ^0 and is updated by multiplying $1-\theta$, with $0<\theta<1$, then after at most

$$\lceil\ \frac{1}{\theta}\ \log\frac{n\mu^0}{\epsilon}\ \rceil \tag{22}$$

iterations we have $n\mu \leq \epsilon$. Thus the total number of iterations is bounded above by

$$8(1+\sqrt{2})(1+2\kappa)q\left(1+\frac{1}{q}\log(1+a\sqrt{2\Psi_0})\right)^2\left(\frac{2\tau+2\sqrt{2n\tau}+\theta n}{2(1-\theta)}\right)^{\frac{1}{2}}\frac{1}{\theta}\log\frac{n\mu^0}{\epsilon}.$$

In the following we give the main result.

Theorem 4.13. *Let a $P_*(\kappa)$ linear complementarity problem be given, where $\kappa \geq 0$. Assume that a strictly feasible starting point (x^0, s^0) is available with $\Psi(x^0, s^0, \mu^0) \leq \tau$ for some $\mu^0 > 0$. Then the total number of iterations for our Algorithm is bounded above by*

$$\lceil\ 8(1+\sqrt{2})(1+2\kappa)q\left(1+\tfrac{1}{q}\log(1+a\sqrt{2\Psi_0})\right)^2\left(\tfrac{2\tau+2\sqrt{2n\tau}+\theta n}{2(1-\theta)}\right)^{\frac{1}{2}}\ \rceil\lceil\ \tfrac{1}{\theta}\log\tfrac{n\mu^0}{\epsilon}\ \rceil.$$

Remark 4.14. For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we have $\tilde{\Psi}_0 = O(n)$. When the parameter $q = \log\left(1+a\sqrt{\frac{2\tau+2\sqrt{2n\tau}+\theta n}{1-\theta}}\right)$ where $a = 1+\frac{1}{\sqrt{1+2\kappa}}$, we have $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\varepsilon})$ iteration complexity which is the best known complexity result so far. For small-update methods with $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we have $\hat{\Psi}_0 = O(q)$. Hence we have $O((1+2\kappa)q\sqrt{qn}\log\frac{n}{\varepsilon})$ iteration complexity for small-update methods which is so far the best known result for such methods.

Example 4.15. *Consider the LCP in which*

$$M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},\ q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Then M is a $P_*(1/4)$ matrix and the solution of this problem is $(x^*; s^*) = (0, 0, 2, 3)^T$. Numerical results of the algorithm with $\tau = 1,\ \varepsilon = 10^{-2},\ \theta = 0.5$, $\mu^0 = 1$, and $(x^0; s^0) = (0.1, 0.05, 1.5, 2.7)^T$ are given in the following table.

outer iterations	v_1	v_2	$\Psi(v)$
1	0.5477	0.5196	0.5847
2	0.7746	0.7348	0.1402
3	1.0914	1.0392	0.0102
4	1.5492	1.4697	0.4633
5	2.1909	2.0785	2.9325

Then after 5 outer iterations we have $x = (0.0005, 0.0016)^T$ *and* $s = (1.4516, 2.8990)^T$.

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