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PROVING UNIFIED COMMON FIXED POINT THEOREMS VIA COMMON PROPERTY (E-A) IN SYMMETRIC SPACES

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Abstract. A metrical common fixed point theorem proved for a pair of self mappings due to Sastry and Murthy ([16]) is extended to symmetric spaces which in turn unifies certain fixed point theorems due to Pant ([13]) and Cho et al. ([4]) besides deriving some related results. Some illustrative examples to highlight the realized improvements are also furnished.

1. Introduction with preliminaries

A symmetric d in respect of a non-empty set X is a function $d : X \times X \to [0, \infty)$ which satisfies $d(x, y) = d(y, x)$ and $d(x, y) = 0 \Leftrightarrow x = y$ (for all $x, y \in X$). If d is a symmetric on a set X, then for $x \in X$ and $\epsilon > 0$, we write $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is given by the sets U (along with empty set) in which for each $x \in U$, one can find some $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A set $S \subset X$ is a neighbourhood of $x \in X$ if and only if there is U containing x such that $x \in U \subset S$. A symmetric d is said to be a semi-metric if for each $x \in X$ and for each $\epsilon > 0$, $B(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. Thus a symmetric (resp. a semi-metric) space X is a topological space whose topology $\tau(d)$ on X is induced by a symmetric (resp. a semi-metric) d. Notice that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$. The distinction between a symmetric and a semi-metric is apparent as one can easily construct a semi-metric d such that $B(x, \epsilon)$ need not be a neighbourhood of x in $\tau(d)$. As symmetric spaces are not essentially Hausdorff, therefore in order to prove fixed point theorms, some additional axioms are required. The following axioms are relevant to this note which are available in Galvin and Shore [5], Wilson [17], Hicks and Rhoades [6], Aliouche [2] and Cho et al. [4]. From now on symmetric as well as semi-metric spaces will be denoted by (X, d) whereas a nonempty arbitrary set will be denoted by Y.

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(W_3): [17] Given $\{x_n\}, x$ and y in X with $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply $x = y$.

(W_4): [17] Given $\{x_n\}, \{y_n\}$ and x in X with $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply $d(y_n, x) \to 0$.

(HE): [2] Given $\{x_n\}, \{y_n\}$ and x in X with $d(x_n, x) \to 0$ and $d(y_n, x) \to 0$ imply $d(x_n, y_n) \to 0$.

($1C$): [4] A symmetric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

(CC): [17] A symmetric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where x_n, y_n are sequences in X and $x, y \in X$.

Clearly, the continuity (i.e., (CC)) of a symmetric is a stronger property than 1-continuity, i.e., (CC) implies ($1C$) but not conversely. Also (W_4) implies (W_3) and ($1C$) implies (W_3) but converse implications are not true. All other possible implications amongst (W_3), ($1C$) and (HE) are not true in general whose nice illustration via demonstrative examples are available in Cho et al. [4]. But (CC) implies all the remaining four conditions namely:(W_3), (W_4), (HE) and ($1C$).

Recall that a sequence $\{x_n\}$ in a semi-metric space (X, d) is said to be a d-Cauchy sequence if it satisfies the usual metric condition. Here, one needs to notice that in a semi-metric space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if semi-metric is suitably restricted (see Wilson [17]). In [3], Burke furnished an illustrative example to show that a convergent sequence in semi-metric spaces need not admit Cauchy subsequence. But he was able to formulate an equivalent condition under which every convergent sequence in semi-metric space admits a Cauchy subsequence. There are several concept of completeness in semi-metric space, e.g. S-completeness, d-Cauchy completeness, strong and weak completeness whose details are available in Wilson [17] but we omit the details as such notions are not relevant to this note.

Lastly, we recall that a pair of self-mappings (f, g) defined on a symmetric (semi-metric) space (X, d) is said to be

(i) compatible (cf. [8]) if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X,

(ii) R-weakly commuting (cf. [12]) on X if $d(fgx, gfx) \leq Rd(fx, gx)$ for some $R > 0$ where x varies over X,

(iii) pointwise R-weakly commuting (cf. [12]) on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$,

(iv) non-compatible (cf. [15]) if there exists some sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but $\lim_{n\to\infty}(fgx_n, gfx_n)$ is either non-zero or non-existent,

(v) tangential (or satisfying the property (E.A) (cf. [1, 16]) if there exists a sequence $\{x_n\}$ in X and some $t \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$,

(vi) partially commuting (or weakly compatible or coincidentally commuting (cf. [9])) if pair commutes on the set of coincidence points,

(vii) occasionally weakly compatible (in short OWC) (cf. [10]) if there is at least one coincidence point x of (f,g) in X at which (f,g) commutes and

(viii) let f and g be two selfmaps defined on a symmetric space (X,d). Then f is said to be g-continuous (cf. [16]) if $gx_n \to gx \Rightarrow fx_n \to fx$ whenever $\{x_n\}$ is a sequence in X and $x \in X$.

Notice that pointwise R-weakly commutativity is equivalent to commutativity at coincidence points whereas compatible maps are pointwise R-commutating as they commute at their coincidence points. Interestingly, the class of tangential maps contains as proper subsets the classes of compatible as well as non-compatible maps and this is the motivation to use the tangential property (or the property (E.A)) in place of compatibility or non-compatibility.

For the sake of completeness, we state the following theorems from Pant [13], Sastry and Murthy [16], and Cho et al. [4] respectively.

Theorem 1.1 (cf. [13]). *Let (f,g) be a pair of non-compatible pointwise R-weakly commuting self-mappings of a metric space (X,d) satisfying:*

(i) $\overline{fX} \subset gX$,

(ii) $d(fx,fy) \leq kd(gx,gy)$ *for all* $x,y \in X, k \geq 0$, *and*

(iii) $d(fx,f^2x) \neq \max\{d(fx,gfx), d(f^2x,gfx)\})$

whenever the right hand side is non-zero. Then f and g have a common fixed point.

A similar theorem also appears in V. Pant [14].

The following theorem due to Sastry and Murthy [16] generalizes Theorem 1.1.

Theorem 1.2 (cf. [16]). *If (in the setting of Theorem 1.1) $d(fx,fy) \leq kd(gx,gy)$ for all $x,y \in X, k \geq 0$ holds and further*

(i) *the pair (f,g) is weakly commuting,*

(ii) *the pair (f,g) is tangential,*

(iii) *f is g-continuous,*

(iv) *either $\overline{f(X)} \subset g(X)$ or $g(X)$ is closed.*

Then f and g have a common fixed point.

In an attempt to offer consolidation to certain results proved for contractive type mappings due to Imdad et al. [7], recently Cho et al. [4] proved two interesting fixed point theorems for nonexpasive type of mappings in symmetric spaces which run as follows:

Theorem 1.3 (cf. [4]). *Let $f,g,S,$ and T be self-mappings of a symmetric (semi-metric) space (X,d) where d satisfies (W_3) and (H.E). Suppose that*

(i) $fX \subset TX$ *and* $gX \subset SX$,

(ii) *the pair* (g, T) *satisfies the property* (E.A) (*resp.,* (f, S) *satisfies the property* (E.A)),

(iii) SX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X (*resp.,* TX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X) *and*

(iv) *for any* $x, y \in X$, $d(fx, gy) \leq m(x, y)$, *where*
$m(x,y) = \max\Big\{d(Sx,Ty), \min[d(fx,Sx), d(gy,Ty)], \min[d(fx,Ty), d(gy,Sx)]\Big\}.$

Then, there exist $u, w \in X$ *such that* $fu = Su = gw = Tw$.

Theorem 1.4 (cf. [4])**.** *Let* f, g, S, *and* T *be self-mappings of a symmetric (semi-metric) space* (X, d) *whereas* d *enjoys (1C) and (H.E). Suppose that*

(i) $fX \subset TX$ *and* $gX \subset SX$,

(ii) *the pair* (g, T) *enjoys the property* (E.A) (*resp.,* (f, S) *enjoys the property (E-A)*),

(iii) SX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X (*resp.,* TX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X),

(iv) *for any* $x, y \in X$, $d(fx, gy) \leq m_1(x, y)$, *where*
$m_1(x,y) = \max\Big\{d(Sx,Ty), \alpha[d(fx,Sx) + d(gy,Ty)], \alpha[d(fx,Ty) + d(gy,Sx)]\Big\}$
$0 < \alpha < 1$.

Then, there exist $u, w \in X$ *such that* $fu = Su = gw = Tw$.

The purpose of this paper is to prove unified theorems in symmetric (semi-metric) spaces which generalize various results due to Pant [13], V. Pant [14], Sastry and Murty [16], Imdad et al. [7], Cho et al. [4] and some others.

2. Results

In what follows, we utilize the common property (E.A.) instead of the property (E.A) to prove similar results. Firstly, on the lines of Liu et al. [11], we adopt the following:

Definition 2.1. Let Y be an arbitrary set and X be a non-empty set equipped with symmetric (semi-metric) d. Two pairs (f, S) and (g, T) of mappings from Y into X are said to share the common property (E.A.) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

We prove our first result by making use of S-continuity of f and T-continuity of g instead of utilizing some Lipschitzian or contractive type condition which runs as follows:

Theorem 2.1. *Let* Y *be an arbitrary nonempty set whereas* X *be another nonempty set equipped with a symmetric (semi-metric)* d *which enjoys* (W_3) (*Hausdorffness of* $\tau(d)$) *and (H.E). Let* $f, g, S, T : Y \to X$ *be four mappings which satisfy the following conditions*:

(i) *f is S-continuous and g is T-continuous,*
(ii) *the pair (f,S) and (g,T) share the common property* (E-A),
(iii) *SX and TX are d-closed ($\tau(d)$-closed) subset of X (resp., $fX \subset TX$ and $gX \subset SX$).*
Then there exist $u, w \in X$ such that $fu = Su = Tw = gw$.
Moreover, if $Y = X$ along with
(iv) *the pairs (f,S) and (g,T) are weakly compatible and*
(v) $d(fx, gfx) \neq \max\{d(Sx,Tfx), d(gfx,Tfx), d(fx,Tfx), d(fx,Sx), d(gfx,Sx)\}$, *whenever the right hand side is non-zero.*

Then f, g, S, and T have a common fixed point in X.

Proof. Notice that Y is an arbitrary set but fY lies in X, therefore a sequence $\{fx_n\}$ in a semi-metric space (X,d) converges to a point fx with respect to $\tau(d)$ if and only if $d(fx_n, fx) \to 0$. To substantiate this, suppose $fx_n \to fx$ and let $\epsilon > 0$. Since $S(fx,\epsilon)$ is a neighborhood of fx, there exists $U \in \tau(d)$ such that $fx \in U \subset S(fx,\epsilon)$. As $fx_n \to fx$, one can find a $m \in \mathbb{N}$ (where $\mathbb{N}$ stands for the set of natural numbers) such that $fx_n \in U \subset S(fx,\epsilon)$ for $n \geq m$ implying thereby $d(fx_n, fx) < \epsilon$ for $n \geq m$, i.e., $d(fx_n, fx) \to 0$. The converse part is obvious in view of the definition of $\tau(d)$.

Since the pairs (f,S) and (g,T) share the common property (E.A), there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X and a $t \in X$ such that

$$\lim_{n\to\infty} d(fx_n,t) = \lim_{n\to\infty} d(Sx_n,t) = \lim_{n\to\infty} d(gy_n,t) = \lim_{n\to\infty} d(Ty_n,t) = 0.$$

Since $S(X)$ is a d-closed (or $\tau(d)$-closed) subspace of X, one can find $u \in X$ such that $Su = t$ which in turn yields that $\lim_{n\to\infty} d(Sx_n, Su) = 0$. Now using S-continuity of f along with the condition (W_3), one finds $d(fu, Su) = 0$ yielding thereby fu=Su. Since $fX \subset TX$, there exists a point $w \in X$ such that $fu = Tw$. Similarly using the d-closedness of $T(X)$ and T-continuity of f along with condition (W_3), we can show that $gw = Tw$ yielding thereby $fu = Su = gw = Tw = t$. Thus both the pairs have a point of coincidence.

Now using weak compatibility of the pairs (f,S) and (g,T), we have $fSu = Sfu$, $ffu = fSu = Sfu = SSu$ and $gTw = Tgw = ggw$. Now, we assert that $fu = w$. Otherwise employing (v), we have

$$\begin{aligned} & d(fu, ffu) \\ &= d(ffu, gw) \\ &\neq \max\Big\{d(Sfu,Tw), d(gw,Tw), d(ffu,Tw), d(ffu,Sfu), d(gw,Sfu)\Big\} \\ &= \max\Big\{d(ffu,fu), 0, d(ffu,fu), 0, d(ffu,fu)\Big\} = d(ffu,fu) \end{aligned}$$

which is a contradiction yielding thereby $fu = w$. Similarly, in case $u \neq gw$, we again arrive at a contradiction. Thus, $fu = w = Su = Tw = gw = u$, and w is a common fixed point of f, g, S, and T.

By restricting f,g,S and T suitably, one can derive corollaries involving two as well as three mappings. Here, it may be pointed out that any result for three mappings is itself a new result. For the sake of brevity, we opt to mention just one such corollary by restricting Theorem 2.1 to a triod of mappings f, S and T which is still new and presents yet another sharpened form of Theorem 1.2 to symmetric (semi-metric) spaces besides admitting a non-self setting upto coincidence points. □

Corollary 2.1. *Let Y be an arbitrary set whereas (X, d) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and $(H.E)$. Let $f, S, T : Y \to X$ be a triod of mappings which satisfy the following conditions:*

(i) *f is S-continuous and f is T-continuous,*
(ii) *the pair (f, S) as well as (f, T) is tangential,*
(iii) *SX and TX are d-closed ($\tau(d)$-closed) subset of X $(fX \subset TX \cap SX)$.*

Then there exist $u, w \in X$ such that $fu = Su = Tw$.

Moreover, if $Y = X$ alongwith

(iv) *the pairs (f, S) and (f, T) are weakly compatible and*
(v) $d(fx, f^2x) \neq \max\Big\{d(Sx, Tfx), d(f^2x, Tfx), d(fx, Tfx), d(fx, Sx),$ $d(f^2x, Sx)\Big\}$*, whenever the right hand side is non-zero,*

then f, S, and T have a common fixed point in X .

Our next theorem is essentially inspired by the condition (iv) of Theorem 1.3 wherein a nonexpansive type condition is utilized. Here, we employ a corresponding Lipschitzian type generalized condition.

Theorem 2.2. *Let Y be an arbitrary set whereas (X, d) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE). Let $f, g, S, T : Y \to X$ be four mappings which satisfy the following conditions:*

(i) *the pair (g, T) satisfies the property* (E.A) *(resp., (f, S) satisfies the property* (E.A)*),*
(ii) *TX is a d-closed ($\tau(d)$-closed) subset of X and $gX \subset SX$ (resp., SX is a d-closed ($\tau(d)$-closed) subset of X and $fX \subset TX$) and*
(iii) *$d(fx, gy) \leq km(x, y)$, for any $x, y \in X$, where $k \geq 0$ and $m(x, y)$ is the same as in Theorem 1.3.*

Then there exist $u, w \in Y$ such that $fu = Su = Tw = gw$.

Moreover, if $Y = X$ along with

(iv) *the pairs (f, S) and (g, T) are weakly compatible and*
(v) $d(fx, gfx) \neq \max\Big\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx),$ $d(gfx, Sx)\Big\}$*, whenever the right hand side is non-zero,*

then f, g, T, and S have a common fixed point.

Proof. Since the pair (g, T) satisfies the property (E.A), there exists a sequence $\{x_n\}$ in X and a point $t \in X$ such that $\lim_{n\to\infty} d(Tx_n, t) = \lim_{n\to\infty} d(gx_n, t) = 0$. As SX is a d-closed ($\tau(d)$-closed) subset of X and $g(X) \subset S(X)$, one can always find a sequence $\{y_n\}$ in X such that $gx_n = Sy_n$ so that $\lim_{n\to\infty} d(Sy_n, t) = 0$. By the property (HE), $\lim_{n\to\infty} d(gx_n, Tx_n) = \lim_{n\to\infty} d(Sy_n, Tx_n) = 0$. Since SX is a d-closed ($\tau(d)$-closed) subset of X and $g(X) \subset S(X)$, there exists a point $u \in X$ such that $Su = t$.
Now using condition (iii), we have

$$\begin{aligned} & d(fu, gx_n) \\ \leq\ & k \max\Big\{d(Su, Tx_n), \min[d(fu, Su), d(gx_n, Tx_n)], \min[d(fu, Tx_n), d(gx_n, Su)]\Big\}, \end{aligned}$$

which on letting $n \to \infty$, gives rise $\lim_{n\to\infty} d(fu, gx_n) = 0$. Now appealing to (W_3)), we get $fu = Su$. Since $fX \subset TX$, there exists a point $w \in X$ such that $fu = Tw$. Now, we show that $Tw = gw$. To accomplish this, using (iii), we have

$$\begin{aligned} & d(fu, gw) \\ \leq\ & k \max\Big\{d(Su, Tw), \min[d(fu, Su), d(gw, Tw)], \min[d(fu, Tw), d(gw, Su)]\Big\} \\ =\ & k \max\Big\{d(Tw, Tw), \min[d(fu, fu), d(gw, Tw)], \min[d(fu, fu), d(gw, gu)]\Big\} \\ =\ & 0 \end{aligned}$$

implying thereby $fu = gw$ and hence in all $fu = Su = gw = Tw$ which shows that both the pairs have a point of coincidence each.
Now employing weak compatibility of the pairs (f, S) and (g, T), we have $fSu = Sfu$, $ffu = fSu = Sfu = SSu$ and $gTw = Tgw = ggw$.

If $fu \neq w$, then from (v), we have either

$$\begin{aligned} & d(fu, ffu) \\ =\ & d(ffu, gw) \\ >\ & \max\Big\{d(Sfu, Tw), d(gw, Tw), d(ffu, Tw), d(ffu, Sfu), d(gw, Sfu)\Big\} \\ =\ & \max\Big\{d(ffu, fu), 0, d(ffu, fu), 0, d(ffu, fu)\Big\} = d(ffu, fu) \end{aligned}$$

or

$$\begin{aligned} & d(fu, ffu) \\ =\ & d(ffu, gw) \\ <\ & \max\Big\{d(Sfu, Tw), d(gw, Tw), d(ffu, Tw), d(ffu, Sgu), d(gw, Sfu)\Big\} \\ =\ & \max\Big\{d(ffu, fu), 0, d(ffu, fu), 0, d(ffu, fu)\Big\} = d(ffu, fu) \end{aligned}$$

which gives a contradiction (in both the cases). Similarly, if $u \neq gw$, we again arrive at a contradiction. Thus, $fu = w = Su = Tw = gw = u$, and w is a common fixed point of f, g, S, and T. □

Remark 2.1. Choosing k=1 in Theorem 2.2, we drive a slightly sharpened form of Theorem 1.3 as conditions on the ranges of involved mappings are relatively lightened.

By restricting f, g, S and T suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.2 to a triod of mappings which is yet another sharpened and unified form of Theorem 1.1 due to Pant [13] (also relevant result in V. Pant [14]) in symmetric spaces.

Corollary 2.2. *Suppose that* (*in the setting of Theorem 2.2*) d *satisfies* (W_3) *and* (HE). *If* $f, S, T : Y \to X$ *are three mappings which satisfy the following conditions*:

(i) *the pair* (f, S) *satisfies the property* (E.A) (*resp.,* (f, T) *satisfies the property* (E.A)),
(ii) SX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X *and* $fX \subset TX$ (*resp.,* TX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X *and* $fX \subset SX$) *and*
(iii) $d(fx, fy) \leq km_2(x, y)$ *for any* $x, y \in X$, *where* $k \geq 0$ *and*
$$m_2(x,y) = \max\Big\{d(Sx,Ty), \min[d(fx,Sx), d(fy,Ty)], \min[d(fx,Ty), d(fy,Sx)]\Big\},$$

then, there exist $u, w \in Y$ *such that* $fu = Su = Tw$.

Moreover, if $Y = X$ *along with*

(iv) *the pairs* (f, S) *and* (f, T) *are weakly compatible and*
(v) $d(fx, f^2x) \neq \max\Big\{d(Sx,Tfx), d(f^2x,Tfx), d(fx,Tfx), d(fx,Sx), d(f^2x,Sx)\Big\}$, *whenever the right hand side is non-zero,*

then f, S *and* T *have a common fixed point.*

Corollary 2.3. *Let* (X, d) *be symmetric* (*semi metric*) *space wherein* d *satisfies* (W_3) (*Hausdoffness of* $\tau(d)$) *and* (*H.E*). *If* $f, g, S, T : X \to X$ *are four self mappings of* X *which satisfy the following conditions*:

(i) *the pair* (f, S) *satisfies the property* (E.A) (*resp.,* (g, T) *satisfies the property* (E.A)),
(ii) SX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X *and* $fX \subset TX$ (*resp.,* TX *is a* d*-closed* ($\tau(d)$*-closed*) *subset of* X *and* $gX \subset SX$).
(iii) $d(fx, gy) < m(x, y)$ *where* $m(x, y)$ *is nonzero and carries the same meaning as in Theorem 1.3.*

Then there exist $u, w \in X$ *such that* $fu = Su = Tw = gw$.

Moreover, if

(iv) *the pairs* (f, S) *and* (g, T) *are weakly compatible,*

then f, g, S *and* T *have a unique common fixed point.*

Proof. Notice that all the conditions of Theorem 2.2 are satisfied except (v) besides being $Y = X$. Therefore there exist $u, w \in X$ such that $fu = Su = gw = Tw$ which on using weak comptability of the pairs yields that $ffu = fSu = Sfu = SSu$ and $gTw = Tgw = TTw = ggw$. If $fu \neq w$, then employing (iii), we have

$$\begin{aligned} d(fu, ffu) &= d(ffu, gw) \\ &< \max\Big\{d(Sfu, Tw), \min[(d(gw, Tw), d(ffu, Tw))], \\ &\qquad \min[(d(ffu, Sgu), d(gw, Sfu))]\Big\} \\ &= \max\{d(ffu, fu), 0, 0\} = d(ffu, fu), \end{aligned}$$

which is a contradiction. Thus $fu = ffu = Sfu$ which shows that fu is a common fixed point of f and S. Similarly using $u \neq gw$, one can show that gw is a common fixed point of g and T. This concludes the proof. □

Our next theorem is essentially inspired by the condition (iv) of Theorem 1.4.

Theorem 2.3. *Theorem 2.2 remains true if* (W_3) *is replaced by* $(1C)$ *whereas condition* (iii) *of Corollary 2.3 is replaced by the following condition besides retaining rest of the hypotheses*:

$$d(fx, gy) \leq km_1(x, y)$$

for any $x, y \in X$, *where* $k \geq 0$ *with* $k\alpha < 1$ *and* $m_1(x, y)$ *is the same as Theorem 1.4.*

Proof. The proof can be completed on the lines of proof of Theorem 2.2, hence details are not included. □

By restricting f, g, S and T suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.3 to a triod of mappings which is yet another sharpened form of Theorem 1.1 due to Pant [13] (also relevant result in V. Pant [14]) in symmetric spaces.

Corollary 2.4. *Let* (*in the setting of Theorem 2.3*) d *satisfy* (IC) *and* (HE). *Suppose that the triod of mappings* $f, S, T : Y \to X$ *satisfy the following conditions*:

(i) *the pair* (f, S) *satisfies the property* (E.A) (*resp.,* (f, T) *satisfies the property* (E.A)),

(ii) SX *is a d-closed* ($\tau(d)$*-closed*) *subset of* X *and* $fX \subset TX$ (*resp.,* TX *is a d-closed* ($\tau(d)$*-closed*) *subset of* X *and* $fX \subset SX$) *and*

(iii) $d(fx, fy) \leq km_3(x, y)$,
$m_3(x, y) = \max\Big\{d(Sx, Ty), \alpha[d(fx, Sx) + d(fy, Ty)], \alpha[d(fx, Ty) + d(fy, Sx)]\Big\}$
for any $x, y \in X$, *where* $k \geq 0$, $0 < \alpha < 1$ *together with* $k\alpha < 1$.

Then there exist $u, w \in Y$ *such that* $fu = Su = Tw$.

Moreover, if $Y = X$ *together with*

(iv) *the pairs* (f, S) *and* (f, T) *are weakly compatible and*

(v) $d(fx, f^2x) \neq \max\Big\{d(Sx, Tfx), d(f^2x, Tfx), d(fx, Tfx), d(fx, Sx),$ $d(f^2x, Sx)\Big\}$, *whenever the right hand side is non-zero.*

Then f, T, and S have a common fixed point.

Corollary 2.5. *Let (X, d) be symmetric (semi-metric) space wherein d satisfies $(1C)$ (Hausdoffness of $\tau(d)$) and (HE). If f, g, S and T are four self mappings of X which satisfy the following conditions:*

(i) *the pair (f, S) satisfies the property* (E.A) (*resp., (g, T) satisfies the property* (E.A)),

(ii) *SX is a d-closed ($\tau(d)$-closed) subset of X and $fX \subset TX$ (resp., TX is a d-closed ($\tau(d)$-closed) subset of X and $gX \subset SX$).*

(iii) *$d(fx, gy) < m_1(x, y)$ where $m_1(x, y)$ carries the same meaning as in Theorem 1.4. Then there exist $u, w \in X$ such that $fu = Su = Tw = gw$.*

Moreover, if

(iv) *the pairs (f, S) and (g, T) are weakly compatible,*

then f, g, S and T have a unique common fixed point.

Proof. The proof can be completed on the lines of Corollary 2.3, hence details are not included. □

The following lemma enunciates a set of conditions which interrelates property (E.A) with common property (E.A).

Lemma 2.4. *Let Y be an arbitrary set whereas (X, d) be a symmetric (semi-metric) space wherein d satisfies (W_3) (Hausdorffness of $\tau(d)$)) and (H.E). If $f, g, S, T : Y \to X$ are four mappings which satisfy the following conditions:*

(i) *the pair (f, S) (or (g, T)) satisfies the property* (E.A.),

(ii) *$fX \subset TX$ or $(gX \subset SX)$,*

(iii) *$d(fx, gy) \leq km(x, y)$,*
for any $x, y \in X$, where $k \geq 0$ and $m(x, y)$ is the same as earlier,

then the pairs (f, S) and (g, T) share the common property (E.A.).

Proof. Since the pair (f, S) enjoys the property (E.A.), one can find a sequence $\{x_n\} \subset Y$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Since $fX \subset TX$, therefore for each $\{x_n\}$ one can find a $\{y_n\} \in X$ such that $fx_n = Ty_n$ which in turn yeilds that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = t$. Now we assert that $\lim_{n\to\infty} gy_n = t$. If not, then using (iii), we have

$$d(fx_n, gy_n) \leq k \max\Big\{ d(Sx_n, Ty_n), \min[d(fx_n, Sx_n), d(gy_n, Ty_n)], \min[d(fx_n, Ty_n), d(gy_n, Sx_n)]\Big\}$$

which on letting $n \to \infty$ and making use of (W_3) and (H.E.), one gets

$$\lim_{n\to\infty} d(t, gy_n) \leq k \max\{0, 0, 0\} \leq 0$$

yielding thereby $\lim_{n\to\infty} gy_n = t$ which shows that the pairs (f, S) and (g, T) share the common property (E.A.). □

Lemma 2.5. *Lemma 2.1 remains true if* (W_3) *is replaced by* $(1C)$ *whereas condition* (iii) *of Lemma 2.1 is replaced by*

$$d(fx, gy) \leq km_1(x, y)$$

for any $x, y \in X$, *where* $k \geq 0$ *with* $k\alpha < 1$ *and* $m_1(x, y)$ *is the same as earlier. besides retaining rest of the hypotheses.*

Proof. The proof can be completed on the lines of Lemma 2.1, hence details are not included. □

Theorem 2.6. *Let* $f, g, S, T : Y \to X$ *be four mappings where* Y *is an arbitrary non-empty set and* X *is a non-empty set equipped with a symmetric (semi-metric)* d *wherein* d *satisfies* (W_3) *(Hausdorffness of* $\tau(d)$*) and (H.E). Suppose that*

(i) *the pair* (f, S) *and* (g, T) *share the common property* (E.A.),
(ii) TX *and* SX *are* d*-closed* ($\tau(d)$ *closed) subsets of* X *and*
(iii) $d(fx, gy) \leq km(x, y)$,
for any $x, y \in X$, *where* $k \geq 0$ *and* $m(x, y)$ *is the same as in Theorem 1.3.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence each.*
Moreover if $Y = X$ *along with,*

(iv) *the pairs* (f, S) *and* (g, T) *are weakly compatible and*
(v) $d(fx, gfx) \neq \max\Big\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\Big\}$, *whenever the right hand side is non-zero,*

then f, g, S *and* T *have a common fixed point.*

Proof. Since the pairs (f, S) and (g, T) share the common property (E.A.), therefore there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} d(fx_n, t) = \lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(gy_n, t) = \lim_{n\to\infty} d(Ty_n, t) = 0$$

for some $t \in X$, which due to property (H.E.), gives rise $\lim_{n\to\infty} d(gy_n, Ty_n) = 0$ and $\lim_{n\to\infty} d(fx_n, Sx_n) = 0$. Since SX is a closed subset of X, hence $\lim_{n\to\infty} Sx_n = t \in SX$ and hence there exists a point $u \in X$ such that $Su = t$ which in turn yeilds that $\lim_{n\to\infty} d(gy_n, Ty_n) = 0$, $\lim_{n\to\infty} d(Su, Ty_n) = 0$ and $\lim_{n\to\infty} d(Su, gy_n) = 0$. Now we assert that $fu = Su$. If not, then using (iii), we have

$$d(fu, gy_n) \leq k \max\Big\{ d(Su, Ty_n), \min[d(fu, Su), d(gy_n, Ty_n)], \min[d(fu, Ty_n), d(gy_n, Su)]\Big\}$$

which on letting $n \to \infty$, and making use (W_3) and (H.E.) reduces to

$$d(gu, t) \leq 0$$

a contradiction, implying thereby $fu = t$. Hence $fu = Su$. Therefore, u is a coincidence point of the pair (f, S).

Since TX is also a closed subset of X, hence $\lim_{n\to\infty} Ty_n = t \in TX$ and also there exists a point $w \in X$ such that $Tw = t$ which in turn yeild $\lim_{n\to\infty} d(Sx_n, Tw) = 0, \lim_{n\to\infty} d(fx_n, Tw) = 0$ and $\lim_{n\to\infty} d(fx_n, Sx_n) = 0$. Now we assest that $gw = Tw$. If not, then again using (iii), we have

$$d(fx_n, gw) \leq k \ \max\Big\{ d(Sx_n, Tw), min[d(fx_n, Sx_n), d(gw, Tw)], \min[d(fx_n, Tw), d(gw, Sx_n)]\Big\}$$

which on letting $n \to \infty$, and making use of (W_3) and (H.E.) reduces to

$$\lim_{n\to\infty} d(fx_n, gw) = 0, \text{ i.e., } (\lim_{n\to\infty} fx_n = gw)$$

a contradiction, implying thereby $gw = t$. Hence $gw = Tw$, which shows that w is a coincidence point of the pair (g, T) and in all $fu = Su = gw = Tw = t$. The rest of the proof is similar to that of Theorem 2.1. hence it is omitted. □

Theorem 2.7. *Theorem 2.4 remains true if condition* (iii) (*of Theorem 2.4*) *is replaced by*

$$d(fx, gy) \leq km_1(x, y)$$

for all $x, y \in Y$, $k\alpha < 1$ *with* $m_1(x, y)$ *is the same as earlier whereas* (W_3) *is replaced by* $(1C)$ *besides retaining rest of the hypotheses.*

Proof. The proof can be completed on the lines of proof of Theorem 2.4, hence details are not included. □

Theorem 2.8. *Let* Y *be an arbitrary set whereas* (X, d) *be a symmetric (semi-metric) space equipped with a symmetric (semi-metric)* d *which enjoys* (W_3) *and* $(H.E)$. *Let* $f, g, S, T : Y \to X$ *be four mappings which satisfy the following conditions*:

(i) *the pair* (f, S) (*or* (g, T)) *satisfies property* (E.A.),
(ii) $fX \subset TX$ *or* $(gX \subset SX)$*and*
(iii) $d(fx, gy) \leq km(x, y)$ *for any* $x, y \in X$, *where* $k \geq 0$ *and* $m(x, y)$ *is the same as earlier.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence.*

Moreover, if $Y = X$ *along with*

(iv) *the pairs* (f, S) *and* (f, T) *are weakly compatible and*
(v) $d(fx, gfx) \neq \max\Big\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\Big\}$, *whenever the right hand side is non-zero,*

then f, g, S *and* T *have a common fixed point.*

Proof. Notice that all the conditions of Lemma 2.1 are satisfied, therefore the pairs (f, S) and (g, T) share the common property (E.A), i.e., there exist two sequences $\{x_n\}$, $\{y_n\} \subset Y$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} Ty_n = t \in X.$$

If $S(X)$ is closed subset of X, then in view of Theorem 2.2, the pair (f, S) has a point of coincidence say u, i.e., $fu = Su$. Since $fX \subset TX$ and $fu \in fX$, there exists $w \in Y$ such that $fu = Tw$. Now we assert that $gw = Tw$. If not, then using (iii) we have

$$d(fx_n, gw) \le k \max\Big\{ d(Sx_n, Tw), \min[d(fx_n, Sx_n), d(gw, Tw)], \min[d(Sx_n, gw), d(Tw, fx_n)]\Big\}$$

which on making $n \to \infty$, reduces to

$$d(t, gw) \le 0$$

which is a contradiction yielding thereby $gw = Tw$. The rest of the proof is similar to that of Theorem 2.4. hence it is omitted. □

Theorem 2.9. *Theorem 2.6. remains true if* (W_3) *is replaced by* $(1C)$ *whereas condition* (iii) (*of Theorem 2.6*) *is replaced by*

$$d(fx, gy) \le km_1(x, y)$$

for all $x, y \in Y$ *where* $m_1(x, y)$ *is the same as earlier with* $k\alpha < 1$ *besides retaining rest of the hypothesis.*

Proof. Proceeding on the lines of the proof of Theorem 2.4, one can complete the proof of this theorem, hence details are not included. □

3. Illustrative examples

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results over some recently established results due to Cho et al. [4] and others. Our first example demonstrates Theorem 2.1.

Example 3.1. Consider $X = [2, 20]$ equipped with the symmetric $d(x, y) = (x - y)^2$.

In order to illustrate Theorem 2.1, we set $f = g$ and $S = T$. Define $f, S : X \to X$ as

$$f(2) = 2,\ f(x) = 7 \text{ if } 2 < x \le 5,\ f(x) = 2 \text{ if } x > 5,$$

$$S(2) = 2,\ S(x) = 7 \text{ if } 2 < x \le 5,\ S(x) = \frac{x+1}{3} \text{ if } x > 5.$$

Then the pair (f, S) satisfies all the conditions of Theorem 2.1 and has a coincidence at $x = 2$ which also remains a common fixed point of the pair. Notice that f is not Lipschetzian whenever $x \in (2, 5]$ and $y = 20$ as (with $k \geq 0$)

$$d(f5, f20) \leq k\ d(S5, S20) \Rightarrow 25 \leq 0$$

which is a contradiction. Here it is interesting to note that the condition

$$\begin{aligned} &d(fx, ffx) \\ &\neq \max\Big\{d(Sx, Sfx), d(fx, Sx), d(ffx, Sfx), d(fx, Sfx), d(Sx, ffx)\Big\} \end{aligned}$$

is not satisfed:
$\Big($e.g. $x = 5,\ d(f5, ff5) \neq \max\Big\{d(S5, Sf5), d(f5, S5), d(ff5, Sf5), d(f5, Sf5),$ $d(S5, ff5)\Big\} \Rightarrow d(7, 2) \neq \max\{d(7, \frac{8}{3}), d(7, 7), d(2, \frac{8}{3}), d(7, \frac{8}{3}), d(7, 2)\} \Rightarrow 25 \neq 25\Big)$ by this example whenever right hand side of above inequality is nonzero. This confirms that condition (v) of Theorem 2.1 is only a necessary condition but not sufficient.

The following example exhibits that the axioms (H.E) and (1C) are necessary in Theorem 2.3. The idea of this example essentially appears in Cho et al. [4].

Example 3.2. Consider $X = [0, \infty)$ and define a symmetric d on X as

$$d(x, y) = \begin{cases} \mid x - y \mid, \text{ if } x \neq 0,\ y \neq 0 \\ \\ \dfrac{1}{x}\ , \text{ if } x \neq 0,\ y = 0 \\ \\ \dfrac{1}{y}\ , \text{ if } y \neq 0,\ x = 0. \end{cases}$$

This d does not satisfy (H.E) and (1C). Set $Y = X,\ f = g, S = T$ and define f and S as follows:

$$fx = x,\ x \geq 0 \text{ and}$$

$$Sx = \begin{cases} \frac{x}{3},\ x > 0 \\ \\ 0,\ x = 0 \end{cases}$$

In order to verify $d(fx, fy) \leq\ kn_1(x, y)$ where $n_1(x, y)$ denotes the restriction of $m_1(x, y)$ to mappings f and S, we distinguish two cases:
Case (i). If $x > 0,\ y > 0$, then

$$d(fx, fy) = \mid x - y \mid =\ 3 \mid \frac{x - y}{3} \mid = 3d(Sx, Sy) \leq 3m_1(n, y)$$

where $m_1(n, y)$ is the same as earlier.
Case (ii). If $x = 0$ and $y > 0$, then

$$\begin{aligned} d(fx, fy) = d(0, y) &= \frac{1}{y} = \frac{1}{4}\,\frac{4}{y} = \frac{1}{4}\left[\frac{3}{y} + \frac{1}{y}\right] \\ &= \left[\frac{1}{4}\left(\frac{3}{y} + \frac{1}{y}\right)\right] = \left[\frac{1}{4}(d(fx, Sy) + d(Sx, fy))\right] \\ &< 3m_1(n, y) \end{aligned}$$

which shows that the condition (iii) of Theorem 2.3 is satisfied (for all $x, y \in X$) with $k = 3$, $\alpha = \frac{1}{4}$ and $m_1(n, y)$ is same as earlier. Also the pair (f, S) enjoys the property (E-A) (e.g. $x_n = n$) whereas $f(X)$ is d-closed (or $\tau(d)$) closed subset of X. Thus all the conditions of Theorem 2.3 are satisfied.

Notice that the pair (f, S) has no coincidence or common fixed point.
The following example demostrates Theorem 2.4.

Example 3.3. Consider $X = [-1, 1]$ equipped with the symmetric $d(x, y) = (x - y)^2$ which satisfies (W_3) and (HE). Define self mappings f, g, S and T on X as

$$f(-1) = f1 = 3/5,\ fx = x/4,\ -1 < x < 1,$$
$$g(-1) = g1 = 3/5,\ gx = -x/4,\ -1 < x < 1,$$
$$S(-1) = -1/8,\ Sx = x/8,\ -1 < x < 1, \text{ and } S1 = -1/8, \text{ and}$$
$$T(-1) = -1/8,\ Tx = -x/8,\ -1 < x < 1, \text{ and } T1 = 1/8.$$

Consider sequences $\{x_n = \frac{1}{n}\}$ and $\{y_n = \frac{-1}{n}\}$ in X. Clearly,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = 0$$

which show that pairs (f, S) and (g, T) share the common property $(E.A)$. Notice that $f(X) = g(X) = \{\frac{3}{5}\} \cup (\frac{-1}{4}, \frac{1}{4}) \not\subset S(X) = T(X) = [\frac{-1}{8}, \frac{1}{8}]$. In order to verify conditon (iii) of Theorem 1.3 notice that

$$\begin{aligned} d(fx, gy) &= (x/4 + y/4)^2 = ((x + y)/4)^2 \\ &= 4((x + y)/8)^2 \leq\ 4\ d(Sx, Ty) \leq 4\ m(x, y) \end{aligned}$$

where $m(x, y)$ is the same as earlier.

Therefore, all the conditions of Theorem 2.4 are satisfied and 0 is a common fixed point of the pairs (f, S) and (g, T) which is also their coincidence point as well.

Here it is worth noting that Theorems 1.3. and 1.4. due to Cho et al. [4] can be used only when k is at the most 1 whereas our results are valid for any $k \geq 0$. Notice that in the present example k is 4 and hence Theorems 1.3 and 1.4 due to Cho et al. [4] can not work in the context of this example which substantiate the utility of our results over earlier ones.
Our last example highlights the non-uniqueness of common fixed points in the present context.

Example 3.4. In order to highlight the non-uniqueness of common fixed point in Theorems 2.1, consider $X = \{0, 1, 1/2, 1/3, \ldots 1/n, \ldots\}$ under the symmetric $d(x, y) = e^{|x-y|} - 1$. Set $f = S$, $g = T$ and define f and g on X by $f(1/n) =$

$1/n^2$, $g(1/n) = 1/n^3$, $f(0) = 0 = g(0)$. Clearly $\overline{f(X)} \not\subset g(X)$ but $g(X)$ is a closed subset of X. Also, rest of the conditions of Theorem 2.1 are trivially satisfied. Notice that f and g have two common fixed points namely: 0 and 1.

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