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CONVERGENCE THEOREMS FOR TWO FAMILIES OF WEAK RELATIVELY NONEXPANSIVE MAPPINGS AND A FAMILY OF EQUILIBRIUM PROBLEMS

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Abstract. The purpose of this paper is to prove strong convergence theorems for common fixed points of two families of weak relatively nonexpansive mappings and a family of equilibrium problems by a new monotone hybrid method in Banach spaces. Because the hybrid method presented in this paper is monotone, so that the method of the proof is different from the original one. We shall give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping in Banach space l^2. Our results improve and extend the corresponding results announced in [W. Takahashi and K. Zembayashi, *Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings*, Fixed Point Theory Appl. (2008), Article ID 528476, 11 pages; doi:10.1155/2008/528476] and [Y. Su, Z. Wang, and H. Xu, *Strong convergence theorems for a common fixed point of two hemi-relatively nonexpansive mappings*, Nonlinear Anal. **71** (2009), no. 11, 5616–5628] and some other papers.

1. Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Iterative methods for approximation of fixed points of a nonexpansive mapping have been studied by many researchers; see [10, 14, 20, 21, 22, 25, 31, 34] and others.

On the other hand, a closed hemi-relatively nonexpansive mapping, which is another generalization of a nonexpansive mapping and a relatively nonexpansive mapping, has been considered recently. Its properties and iterative schemes for such a mapping have been studied in [16, 22, 25] and others.

Equilibrium problems which were introduced by Blum and Oettli [3] in 1994 have had a great impact and influence in the development of several branches

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of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization.

Let E be a real Banach space and let E^* be the dual of E. Let C be a nonempty closed convex subset of E, and f a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. The equilibrium problem (for short, EP) is to find $p \in C$ such that

$$f(p, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $p \in EP(f)$ if and only if $\langle Tp, y - p\rangle \geq 0$ for all $y \in C$; i.e., p is a solution of the variational inequality, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP; see, for example [3] and references therein. Matsushita and Takahashi [18] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in [0, 1], T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Many authors studied the problem of finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces, see, for instance, [31] and the references therein.

Recently, Kimura and Takahashi [15] established strong convergence theorems by the hybrid method for a family of relatively nonexpensive mappings as follows:

Theorem 1.1. *Let E be a strictly convex reflexive Banach space having the Kadec-Klee property and a Fréchet differentiable norm, let C be a non-empty and closed convex subset of E and $\{S_\lambda : \lambda \in \Lambda\}$ a family of relatively nonexpansive mappings of C into itself having a common fixed point. Let $\{\alpha_n\}$ be a sequence in* $[0, 1]$ *such that* $\liminf_{n\to\infty} \alpha_n < 1$. *For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme*: $x_1 \in C$, *$C_1 = C$, and*

$$\begin{cases} y_n(\lambda) = J^*(\alpha_n Jx_n + (1 - \alpha_n)JS_\lambda x_n) \quad \text{for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup_{\lambda\in\Lambda} \phi(z, y_n(\lambda)) \leq \phi(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}}(x), \end{cases}$$

for every $n \in \mathbb{N}$*, then* $\{x_n\}$ *converges strongly to* $P_F x \in C$*, where* $F = \cap_{\lambda \in \Lambda} F(S_\lambda)$ *is the set of common fixed points of* $\{S_\lambda\}$ *and* P_K *is the metric projection of* E *onto a nonempty closed convex subset* K *of* E*.*

Very recently, Y. Su, Z. Wang, and H. Xu [28] proposed the new strong convergence theorems, one of which as follows:

Theorem 1.2. *Let* E *be a uniformly convex and uniformly smooth real Banach space, let* C *be a non-empty and closed convex subset of* E*, let* $T, S : C \to C$ *be two closed hemi-relatively non-expansive mappings such that* $F := F(T) \cap F(S) \neq \emptyset$*. Define a sequence* $\{x_n\}$ *in* C *by the following algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{cases}$$

with the conditions:
(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
(iii) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0, 1)$*.*
Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection of* C *onto* F*.*

Motivated by these results above, we prove strong convergence theorems for common fixed points of two families of weak relatively nonexpansive mappings and a family of equilibrium problems by a new hybrid method in Banach spaces. The main results are more general than the theorems of Y. Su, Z. Wang, and H. Xu [28], and at the same time, our hybrid algorithm and the method of the proof are all different from that of [28]. In addition, we succeed in applying our algorithm to a family of equilibrium problems, which is different from others' method.

In recent years, the definition of weak relatively nonexpansive mapping has been presented and studied by many authors [26, 28, 36], but they have not given the example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. In this paper, we give an example which is a weak relatively nonexpansive mapping but not a relatively nonexpansive mapping in a Banach space l^2.

2. Preliminaries

In what follows, E denotes a real Banach space with norm $\|\cdot\|$ and let E^* the dual of E. The norm of E^* is also denoted by $\|\cdot\|$. For $y^* \in E^*$, its value at $x \in E$ is denoted by $\langle x, y^* \rangle$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. E is said to have the Kadec-Klee property if a weakly convergent sequence $\{x_n\}$ in E with limit $x_0 \in E$ satisfies that $\lim_{n\to\infty} \|x_n\| = \|x_0\|$, then $\{x_n\}$ converges strongly to x_0.

Let $S_E = \{x \in E : \|x\| = 1\}$ and define $f : S_E \times S_E \times \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by

$$f(x, y, t) = \frac{\|x + ty\| - \|x\|}{t}$$

for $x, y \in S_E$ and $t \in \mathbb{R}\backslash\{0\}$. A norm of E is said to be *Gâteaux* differentiable if $\lim_{t\to\infty} f(x, y, t)$ has a limit for each $x, y \in S_E$. In this case, E is said to be smooth. A norm of E is said to be *Fréchet* differentiable if $\lim_{t\to\infty} f(x, y, t)$ is attained uniformly for $y \in S_E$ for each $x \in E$. It is known that E^* has a *Fréchet* differentiable norm if and only if E is strictly convex and reflexive, and has the Kadec-Klee property.

Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalize dduality mapping J from E to E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E is a smooth, strictly convex, and reflexive Banach space, then J is a single-valued one-to-one mapping onto E^*. In this case, the inverse mapping J^{-1} coincides with the duality mapping on E^*. For more details, see [9, 18, 29].

Suppose that a Banach space E is smooth. Then J is a single-valued mapping. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in E$. We know several fundamental properties of ϕ as follows: $\phi(x, y) \geq 0$ for all $x, y \in E$. For a sequence $\{y_n\}$ in E and $x \in E$, $\{y_n\}$ is bounded if and only if $\{\phi(x, y_n)\}$ is bounded. For more details, see [5].

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T.

A point p in C is said to be an asymptotic fixed point of T [24] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\widehat{F}(T)$.

A mapping T of C into itself is said to be relatively nonexpansive [6, 7, 17] if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T), x \in C$;
(3) $\widehat{F}(T) = F(T)$.

The hybrid algorithms for fixed point of relatively nonexpensive mappings and applications have been studied by many authors, for example [6, 7, 17, 27, 33, 35].

A point p in C is said to be a strong asymptotic fixed point of T [26, 36] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called weak relatively nonexpansive if

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u, x)$, $\forall u \in F(T), x \in C$;
(3) $\widetilde{F}(T) = F(T)$.

Remark 1. In [36], the weak relatively nonexpansive mapping is also said to be relatively weak nonexpansive.

Remark 2. In [27], the authors have given the definition of hemi-relatively nonexpansive mapping as follows. A mapping T from C into itself is called hemi-relatively nonexpansive if

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u, x)$, $\forall u \in F(T), x \in C$.

It is obvious that a relatively nonexpansive mapping is a weak relatively nonexpansive mapping and a weak relatively nonexpansive mapping is a hemi-relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

If E is a strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}0 \neq \emptyset$, then it is proved in [31] that $J_r := (J + rA)^{-1}J$ for $r > 0$ is relatively nonexpansive. Moreover, if $T : E \to E$ is relatively nonexpansive, then using the definition of ϕ one can show that $F(T)$ is closed and convex.

The following conclusion is obvious.

Conclusion. *A mapping is closed hemi-relatively nonexpansive if and only if it is weak relatively nonexpansive.*

Let C_n be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We denote two subsets $s - Li_n C_n$ and $w - Ls_n C_n$ as follows: $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in w - Ls_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. We define the Mosco convergence [19] of $\{C_n\}$ as follows: If C_0 satisfies that $C_0 = s - Li_n C_n = w - Ls_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco and we write $C_0 = M - \lim_{n\to\infty} C_n$. For more details, see [2].

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator Π_C follow from the properties of the C functional $\phi(x, y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$.

The following theorem plays an important role in our results.

Theorem 2.1 (See Ibaraki, Kimura, and Takahashi [11]). *Let E be a smooth, reflexive, and strictly convex Banach space having the Kadec-Klee property. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of E. If $K_0 = M - \lim_{n\to\infty} K_n$ exists and is nonempty, then $\{\Pi_{K_n} x\}$ converges strongly to $\{\Pi_{K_0} x\}$ for each $x \in C$.*

We also need the following lemmas for the proof of our main results.

Lemma 2.2 (Kamimura and Takahashi [13]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $\|y_n - z_n\| \to 0$.*

Lemma 2.3 (Alber [1]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Lemma 2.4 (Cho et al. [8]). *Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.5 (Kamimure and Takahashi [13]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.*

For solving the equilibrium problem let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0, \ \forall\ x \in C$,
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0, \ \forall\ x, y \in E$,
(A3) for all $x, y, z \in C, \ \limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$,
(A4) for all $x \in C$, $y \to f(x, y)$ is convex and lower semi-continuous.

Lemma 2.6 (Blum and Oettli [3]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4)*, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall \ y \in C.$$

Lemma 2.7 (Takahashi and Zembayashi [32, Lemma 2.8]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4)*. For $r > 0$, define a mapping $T_r : E \to C$ as follows*:

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall \ y \in C\}$$

for all $x \in E$. Then, the following hold:

(1) *T_r is single-valued*;

(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = EP(f)$*;

(4) *$EP(f)$ is closed and convex*;

Lemma 2.8 (Takahashi and Zembayashi [31]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4)*, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

3. Main results

Theorem 3.1. *Let E be a strictly convex, reflexive and uniformly smooth Banach space having the Kadec-Klee property, let C be a non-empty and closed convex subset of E and Let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$, satisfying* (A1)-(A4)*. Let $\{S_\lambda : \lambda \in \Lambda\}$ and $\{T_\lambda : \lambda \in \Lambda\}$ be two families of weak relatively nonexpansive mappings of C into itself such that $F := \cap_{\lambda\in\Lambda} F(S_\lambda) \cap \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset$. For an arbitrarily chosen point $x_0 \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme*: *$x_1 \in C$, $C_1 = C$, and*

(3.1)

$$\begin{cases} z_n(\lambda) = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n), \\ y_n(\lambda) = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n Jz_n(\lambda)), \\ u_n(\lambda) \in C \ \ \textit{such that } f_\lambda(u_n(\lambda), y) + \frac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \ \textit{for all } y \in C \textit{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:

(i) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0$;

(iii) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0,1)$;

(iv) $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0 \in C$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. Firstly, we show that C_n is closed and convex for each $n \in \mathbb{N}$.

From the definition of ϕ, we may show that

$$\begin{aligned}
C_{n+1} &= \{z \in C_n : \sup_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq \phi(z, x_n)\} \\
&= \cap_{\lambda\in\Lambda}\{z \in C_n : \phi(z, u_n(\lambda)) \leq \phi(z, x_n)\} \\
&= \cap_{\lambda\in\Lambda}\{z \in C : 2\langle z, Jx_n - Ju_n(\lambda)\rangle + \|u_n(\lambda)\|^2 - \|x_n\|^2 \leq 0\} \cap C_n,
\end{aligned}$$

and thus C_n is closed and convex for every $n \in \mathbb{N}$.

Secondly, we prove that $F \subset C_n$ for all $n \in \mathbb{N}$.

Let $p \in F$. Putting $u_n(\lambda) = T_{r_n} y_n(\lambda)$ for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$. On the other hand, from Lemma 2.7, one has T_{r_n} is a hemi-relatively nonexpansive mapping, then, for all $p \in F$ we obtain

$$\begin{aligned}
\phi(p, z_n(\lambda)) &= \phi(p, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n)) \\
&= \|p\|^2 - 2\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n\rangle \\
&\quad + \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n\|^2 \\
&\leq \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, T_\lambda x_n) + \beta_n^{(3)}\phi(p, S_\lambda x_n) \\
&\leq \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, x_n) + \beta_n^{(3)}\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned}$$

By the similar reason, we have, for all $p \in F$ that

$$\begin{aligned}
\phi(p, u_n(\lambda)) &= \phi(p, T_{r_n} y_n(\lambda)) \\
&\leq \phi(p, y_n(\lambda)) \\
&= \phi(p, J^{-1}(\alpha_n Jz_n(\lambda) + (1-\alpha_n)Jx_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jz_n(\lambda) + (1-\alpha_n)Jx_n\rangle \\
&\quad + \|\alpha_n Jz_n(\lambda) + (1-\alpha_n)Jx_n)\|^2 \\
&\leq \alpha_n\phi(p, z_n(\lambda)) + (1-\alpha_n)\phi(p, x_n) \\
&\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned} \tag{3.2}$$

That is, $p \in C_n$ for all $n \in \mathbb{N}$, and hence $F \subset C_n$ for all $n \in \mathbb{N}$. Since F is nonempty, C_n is a nonempty closed convex subset of E and thus Π_{C_n} exists for every $n \in \mathbb{N}$. Hence $\{x_n\}$ is well defined.

Thirdly, we shall show that $\lim_{n\to\infty} x_n = \overline{x} = \Pi_{C_0} x_0$.

Since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $C_0 = \cap_{n=1}^{\infty} C_n$ is nonempty, it follows that

$$M - \lim_{n\to\infty} C_n = C_0 = \cap_{n=1}^{\infty} C_n \neq \emptyset.$$

By Theorem 2.1, $\{x_n\} = \{\Pi_{C_n} x_0\}$ converges strongly to $\{\overline{x}\} = \{\Pi_{C_0} x_0\}$.

Fourthly, we prove that $\overline{x} \in F$.

Since $\overline{x} \in C_n$ for every $n \in \mathbb{N}$, it follows that $\sup_{\lambda\in\Lambda} \phi(\overline{x}, u_n(\lambda)) \leq \phi(\overline{x}, x_n)$ for every $n \in \mathbb{N}$. So we get

$$0 \leq \lim_{n\to\infty} \sup_{\lambda\in\Lambda} \phi(\overline{x}, u_n(\lambda)) \leq \lim_{n\to\infty} \phi(\overline{x}, x_n) = 0.$$

Hence, from Lemma 2.2, we can get

$$\lim_{n\to\infty} \|u_n(\lambda) - \overline{x}\| = 0 \quad \text{for any} \quad \lambda \in \Lambda.$$

So we obtain

$$\lim_{n\to\infty} \|u_n(\lambda) - x_n\| = 0 \quad \text{for any} \quad \lambda \in \Lambda. \tag{3.3}$$

It follows from (3.2) that

$$0 \leq \phi(p, u_n(\lambda)) \leq \phi(p, y_n(\lambda)) \leq \phi(p, x_n)$$

so we obtain

$$0 \leq \phi(p, x_n) - \phi(p, y_n(\lambda)) \leq \phi(p, x_n) - \phi(p, u_n(\lambda))$$

therefore,

$$0 \leq \lim_{n\to\infty} (\phi(p, x_n) - \phi(p, y_n(\lambda))) \leq \lim_{n\to\infty} (\phi(p, x_n) - \phi(p, u_n(\lambda))) = 0,$$

so one easily obtains

$$\lim_{n\to\infty} (y_n(\lambda) - x_n) = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} (1 - \alpha_n)\|Jz_n(\lambda) - Jx_n\| = \|Jy_n(\lambda) - Jx_n\| = 0.$$

Since $0 \leq \alpha_n \leq \alpha < 1$, then

$$\lim_{n\to\infty} \|Jz_n(\lambda) - Jx_n\| = 0.$$

Further, since E has the Kadec-Klee property, the norm of E^* is *Fréchet* differentiable and therefore J^{-1} is norm-to-norm continuous, hence we have that

$$\lim_{n\to\infty} \|z_n(\lambda) - x_n\| = 0,$$

so that $z_n(\lambda) \to \overline{x}$ as $n \to \infty$.

Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. For any $\lambda \in \Lambda$, let $r = \sup_{n\in\mathbb{N}\cup\{0\}}\{\|x_n\|, \|S_\lambda x_n\|, \|T_\lambda x_n\|\}$. From Lemma 2.4, we have

$$\begin{aligned}
\phi(p, z_n(\lambda)) &= \phi(p, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n)) \\
&= \|p\|^2 - 2\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n\rangle \\
&\quad + \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n\|^2 \\
&\le \|p\|^2 - 2\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n\rangle \\
&\quad + \beta_n^{(1)}\|Jx_n\|^2 + \beta_n^{(2)}\|JT_\lambda x_n\|^2 \\
&\quad + \beta_n^{(3)}\|JS_\lambda x_n\|^2 - \beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_\lambda x_n\|) \\
&\le \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, T_\lambda x_n) \\
&\quad + \beta_n^{(3)}\phi(p, S_\lambda x_n) - \beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_\lambda x_n\|) \\
&\le \phi(p, x_n) - \beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_\lambda x_n\|)
\end{aligned}$$

and hence

$$\beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_\lambda x_n\|) \le \phi(p, x_n) - \phi(p, z_n(\lambda)) \to 0$$

as $n \to \infty$. By using the same way, we can prove that

$$\beta_n^{(1)}\beta_n^{(3)} g(\|Jx_n - JS_\lambda x_n\|) \le \phi(p, x_n) - \phi(p, z_n(\lambda)) \to 0$$

as $n \to \infty$. From the properties of the mapping g and the conditions (i), (ii) we have

$$\|Jx_n - JT_\lambda x_n\| \to 0$$

as $n \to \infty$, and

$$\|Jx_n - JS_\lambda x_n\| \to 0$$

as $n \to \infty$. Further, since E has the Kadec-Klee property, the norm of E^* is *Fréchet* differentiable and therefore J^{-1} is norm-to-norm continuous, hence we have that

$$\lim_{n\to\infty} \|x_n - T_\lambda x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - S_\lambda x_n\| = 0.$$

Since T_λ and S_λ are two weak relatively nonexpansive mappings for any $\lambda \in \Lambda$, we have that $\overline{x} \in F(T_\lambda)$ and $\overline{x} \in F(S_\lambda)$ for any $\lambda \in \Lambda$ and thus $\overline{x} \in \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} F(S_\lambda)$.

Then, we show $\overline{x} \in \cap_{\lambda\in\Lambda} EP(f_\lambda)$. From $u_n(\lambda) = T_{r_n} y_n(\lambda)$ and Lemma 2.8, we obtain

$$\begin{aligned}
\phi(u_n(\lambda), y_n(\lambda)) &= \phi(T_{r_n} y_n(\lambda), y_n(\lambda)) \\
&\le \phi(\overline{x}, y_n(\lambda)) - \phi(\overline{x}, T_{r_n} y_n(\lambda)) \\
&\le \phi(\overline{x}, x_n) - \phi(\overline{x}, T_{r_n} y_n(\lambda)) \\
&= \phi(\overline{x}, x_n) - \phi(\overline{x}, u_n(\lambda)).
\end{aligned}$$

It follows from (3.3) that

$$\phi(u_n(\lambda), y_n(\lambda)) \to 0 \quad \text{as} \quad n \to \infty.$$

Noticing that Lemma 2.2, we get

$$\|u_n(\lambda) - y_n(\lambda)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\|Ju_n(\lambda) - Jy_n(\lambda)\| \to 0.$$

From the (A2), we note that

$$\begin{aligned}\|y - u_n(\lambda)\|\frac{\|Ju_n(\lambda) - Jy_n(\lambda)\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \\ &\geq -f_\lambda(u_n(\lambda), y) \\ &\geq f_\lambda(y, u_n(\lambda)), \quad \forall y \in C, \quad \forall \lambda \in \Lambda.\end{aligned}$$

By taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_n(\lambda) \to \overline{x}$, we have $f_\lambda(y, \overline{x}) \leq 0$ for all $y \in C$, for all $\lambda \in \Lambda$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\overline{x}$. Noticing that $y, \overline{x} \in C$, we obtain $y_t \in C$, which yields that $f_\lambda(y_t, \overline{x}) \leq 0$. It follows from (A1) that

$$0 = f_\lambda(y_t, y_t) \leq tf_\lambda(y_t, y) + (1-t)f_\lambda(y_t, \overline{x}) \leq tf_\lambda(y_t, y).$$

That is, $f_\lambda(y_t, y) \geq 0$.

Let $t \downarrow 0$, from (A3), we obtain $f_\lambda(\overline{x}, y) \geq 0$, $\forall y \in C$. This implies that $\overline{x} \in EP(f_\lambda)$. This shows that $\overline{x} \in F$.

Finally, since $\overline{x} = \Pi_{C_0}x_0 \in F$ and F is a nonempty closed convex subset of $C_0 = \cap_{n=1}^{\infty} C_n$, we conclude that $\overline{x} = \Pi_F x_0$ This completes the proof. □

Taking $\alpha_n \equiv 0$, Theorem 3.1 reduces to the following result.

Theorem 3.2. *Let E be a strictly convex, reflexive and uniformly smooth Banach space having the Kadec-Klee property, let C be a non-empty and closed convex subset of E and Let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$, satisfying* (A1)-(A4). *Let $\{S_\lambda : \lambda \in \Lambda\}$ and $\{T_\lambda : \lambda \in \Lambda\}$ be two families of weak relatively nonexpansive mappings of C into itself such that $F := \cap_{\lambda\in\Lambda}F(S_\lambda) \cap \cap_{\lambda\in\Lambda}F(T_\lambda) \cap \cap_{\lambda\in\Lambda}EP(f_\lambda) \neq \emptyset$. For an arbitrarily chosen point $x_0 \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

(3.4)

$$\begin{cases} y_n(\lambda) = J^{-1}(\beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_\lambda x_n + \beta_n^{(3)}JS_\lambda x_n), \\ u_n(\lambda) \in C \text{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \text{ for all } y \in C, \text{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda}\phi(z, u_n(\lambda)) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:
(i) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.
Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0 \in C$, *where* Π_F *is the generalized projection of* E *onto* F.

Taking $T_\lambda \equiv S_\lambda$, Theorem 3.1 reduces to the following result.

Theorem 3.3. *Let* E *be a strictly convex, reflexive and uniformly smooth Banach space having the Kadec-Klee property, let* C *be a non-empty and closed convex subset of* E *and Let* $\{f_\lambda : \lambda \in \Lambda\}$ *be a family of bifunctions from* $C \times C$ *to* $\mathbb{R}$, *satisfying* (A1)-(A4). *Let* $\{T_\lambda : \lambda \in \Lambda\}$ *be a family of weak relatively nonexpansive mappings of* C *into itself such that*

$$F := \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset.$$

For an arbitrarily chosen point $x_0 \in E$, *generate a sequence* $\{x_n\}$ *by the following iterative scheme*: $x_1 \in C$, $C_1 = C$, *and*
(3.5)

$$\begin{cases} z_n(\lambda) = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_\lambda x_n) \quad \text{for all } \lambda \in \Lambda, \\ y_n(\lambda) = J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jz_n(\lambda)) \quad \text{for all } \lambda \in \Lambda, \\ u_n(\lambda) \in C \text{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \ \text{for all } y \in C, \ \text{for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:
(i) $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$;
(ii) $\limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.
Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0 \in C$, *where* Π_F *is the generalized projection of* E *onto* F.

Next, we prove a convergence theorem for Halpern-type iterative algorithm.

Theorem 3.4. *Let* E *be a strictly convex, reflexive and uniformly smooth Banach space having the Kadec-Klee property, let* C *be a non-empty and closed convex subset of* E *and Let* $\{f_\lambda : \lambda \in \Lambda\}$ *be a family of bifunctions from* $C \times C$ *to* $\mathbb{R}$, *satisfying* (A1)-(A4). *Let* $\{S_\lambda : \lambda \in \Lambda\}$ *and* $\{T_\lambda : \lambda \in \Lambda\}$ *be two families of weak relatively nonexpansive mappings of* C *into itself such that*

$$F := \cap_{\lambda\in\Lambda} F(S_\lambda) \cap \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset.$$

For an arbitrarily chosen point $x_0 \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, *and*
(3.6)

$$\begin{cases} z_n(\lambda) = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n), \\ y_n(\lambda) = J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jz_n(\lambda)), \\ u_n(\lambda) \in C \text{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \text{ for all } y \in C, \text{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq (1-\alpha_n\beta_n^{(1)})\phi(z, x_n) + \alpha_n\beta_n^{(1)}\phi(z, x_0)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:
(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0$;
(iii) $\liminf_{n\to\infty} \alpha_n > 0$.
(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.
Then $\{x_n\}$ converges strongly to $\Pi_F x_0 \in C$, where Π_F is the generalized projection of E onto F.

Proof. Firstly, we show that C_n is closed and convex for each $n \in \mathbb{N}$.
From the definition of ϕ, we may show that

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \sup_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq (1-\alpha_n\beta_n^{(1)})\phi(z, x_n) + \alpha_n\beta_n^{(1)}\phi(z, x_0)\} \\ &= \cap_{\lambda\in\Lambda}\{z \in C_n : \phi(z, u_n(\lambda)) \leq (1-\alpha_n\beta_n^{(1)})\phi(z, x_n) + \alpha_n\beta_n^{(1)}\phi(z, x_0)\} \\ &= \cap_{\lambda\in\Lambda}\{z \in C : \|u_n(\lambda)\|^2 + 2\langle z, (1-\beta_n^{(1)})Jx_n + \beta_n^{(1)}Jx_0 - Ju_n(\lambda)\rangle \\ &\leq (1-\beta_n^{(1)})\|x_n\|^2 + \beta_n^{(1)}\|x_0\|^2\} \cap C_n, \end{aligned}$$

and thus C_n is closed and convex for every $n \in \mathbb{N}$.
Secondly, we prove that $F \subset C_n$ for all $n \in \mathbb{N}$.
Let $p \in F$. Putting $u_n(\lambda) = T_{r_n} y_n(\lambda)$ for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$. On the other hand, from Lemma 2.7, one has T_{r_n} is a hemi-relatively nonexpansive mapping, then, for all $p \in F$ we obtain

$$\begin{aligned} \phi(p, z_n(\lambda)) &= \phi(p, J^{-1}(\beta_n^{(1)}Jx_0 + \beta_n^{(2)}JT_\lambda x_n + \beta_n^{(3)}JS_\lambda x_n)) \\ &= \|p\|^2 - 2\langle p, \beta_n^{(1)}Jx_0 + \beta_n^{(2)}JT_\lambda x_n + \beta_n^{(3)}JS_\lambda x_n\rangle \\ &\quad + \|\beta_n^{(1)}Jx_0 + \beta_n^{(2)}JT_\lambda x_n + \beta_n^{(3)}JS_\lambda x_n\|^2 \\ &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, T_\lambda x_n) + \beta_n^{(3)}\phi(p, S_\lambda x_n) \\ &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, x_n) + \beta_n^{(3)}\phi(p, x_n) \\ &= \beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})\phi(p, x_n). \end{aligned}$$

By the similar reason and the results above, we have, for all $p \in F$ that

$$\begin{aligned} \phi(p, u_n(\lambda)) &= \phi(p, T_{r_n} y_n(\lambda)) \\ &\leq \phi(p, y_n(\lambda)) \\ &= \phi(p, J^{-1}(\alpha_n J z_n(\lambda) + (1-\alpha_n) J x_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n J z_n(\lambda) + (1-\alpha_n) J x_n \rangle \\ &\quad + \|\alpha_n J z_n(\lambda) + (1-\alpha_n) J x_n\|^2 \\ &\leq \alpha_n \phi(p, z_n(\lambda)) + (1-\alpha_n)\phi(p, x_n) \\ &\leq \alpha_n \beta_n^{(1)} \phi(p, x_0) + (1-\alpha_n \beta_n^{(1)})\phi(p, x_n). \end{aligned} \tag{3.7}$$

That is, $p \in C_n$ for all $n \in \mathbb{N}$, and hence $F \subset C_n$ for all $n \in \mathbb{N}$. Since F is nonempty, C_n is a nonempty closed convex subset of E and thus Π_{C_n} exists for every $n \in \mathbb{N}$. Hence $\{x_n\}$ is well defined.

Thirdly, we shall show that $\lim_{n\to\infty} x_n = \overline{x} = \Pi_{C_0} x_0$.

Since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $C_0 = \cap_{n=1}^{\infty} C_n$ is nonempty, it follows that

$$M - \lim_{n\to\infty} C_n = C_0 = \cap_{n=1}^{\infty} C_n \neq \emptyset.$$

By Theorem 2.1, $\{x_n\} = \{\Pi_{C_n} x_0\}$ converges strongly to $\{\overline{x}\} = \{\Pi_{C_0} x_0\}$.

Fourthly, we prove that $\overline{x} \in F$.

Since $\overline{x} \in C_n$ for every $n \in \mathbb{N}$, it follows that

$$\sup_{\lambda\in\Lambda} \phi(\overline{x}, u_n(\lambda)) \leq (1-\alpha_n\beta_n^{(1)})\phi(\overline{x}, x_n) + \alpha_n \beta_n^{(1)} \phi(\overline{x}, x_0)$$

for every $n \in \mathbb{N}$. So we get

$$0 \leq \lim_{n\to\infty} \sup_{\lambda\in\Lambda} \phi(\overline{x}, u_n(\lambda)) \leq \lim_{n\to\infty} (1-\alpha_n\beta_n^{(1)})\phi(\overline{x}, x_n) + \alpha_n \beta_n^{(1)} \phi(\overline{x}, x_0) = 0.$$

Hence, from Lemma 2.2, we can get

$$\lim_{n\to\infty} \|u_n(\lambda) - \overline{x}\| = 0 \quad \text{for any} \quad \lambda \in \Lambda.$$

So we obtain

$$\lim_{n\to\infty} \|u_n(\lambda) - x_n\| = 0 \quad \text{for any} \quad \lambda \in \Lambda. \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned} 0 &\leq \phi(p, u_n(\lambda)) \\ &\leq \phi(p, y_n(\lambda)) \\ &\leq \alpha_n \beta_n^{(1)} \phi(p, x_0) + (1-\alpha_n\beta_n^{(1)})\phi(p, x_n), \end{aligned}$$

therefore,

$$\begin{aligned}0 &\le \lim_{n\to\infty}(\alpha_n\beta_n^{(1)}\phi(p,x_0)+(1-\alpha_n\beta_n^{(1)})\phi(p,x_n)-\phi(p,y_n(\lambda)))\\ &\le \lim_{n\to\infty}(\alpha_n\beta_n^{(1)}\phi(p,x_0)+(1-\alpha_n\beta_n^{(1)})\phi(p,x_n)-\phi(p,u_n(\lambda)))\\ &=0,\end{aligned}$$

so one easily obtains

$$\lim_{n\to\infty}(y_n(\lambda)-x_n)=0,$$

so that $y_n(\lambda)\to\overline{x}$ as $n\to\infty$.

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\alpha_n\|Jz_n(\lambda)-Jx_n\|=\|Jy_n(\lambda)-Jx_n\|=0.$$

Since $\liminf_{n\to\infty}\alpha_n>0$, then

$$\lim_{n\to\infty}\|Jz_n(\lambda)-Jx_n\|=0.$$

Further, since E has the Kadec-Klee property, the norm of E^* is *Fréchet* differentiable and therefore J^{-1} is norm-to-norm continuous, hence we have that

$$\lim_{n\to\infty}\|z_n(\lambda)-x_n\|=0,$$

so that $z_n(\lambda)\to\overline{x}$ as $n\to\infty$.

Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. For any $\lambda\in\Lambda$, let $r=\sup_{n\in\mathbb{N}\cup\{0\}}\{\|x_0\|,\|S_\lambda x_n\|,\|T_\lambda x_n\|\}$. From Lemma 2.4, we have for all $p\in F$ that

$$\begin{aligned}\phi(p,z_n(\lambda)) &= \phi(p,J^{-1}(\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_\lambda x_n+\beta_n^{(3)}JS_\lambda x_n))\\ &= \|p\|^2-2\langle p,\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_\lambda x_n+\beta_n^{(3)}JS_\lambda x_n\rangle\\ &\quad+\|\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_\lambda x_n+\beta_n^{(3)}JS_\lambda x_n\|^2\\ &\le \|p\|^2-2\langle p,\beta_n^{(1)}Jx_0+\beta_n^{(2)}JT_\lambda x_n+\beta_n^{(3)}JS_\lambda x_n\rangle\\ &\quad+\beta_n^{(1)}\|Jx_0\|^2+\beta_n^{(2)}\|JT_\lambda x_n\|^2\\ &\quad+\beta_n^{(3)}\|JS_\lambda x_n\|^2-\beta_n^{(3)}\beta_n^{(2)}g(\|JT_\lambda x_n-JS_\lambda x_n\|)\\ &\le \beta_n^{(1)}\phi(p,x_0)+\beta_n^{(2)}\phi(p,T_\lambda x_n)\\ &\quad+\beta_n^{(3)}\phi(p,S_\lambda x_n)-\beta_n^{(3)}\beta_n^{(2)}g(\|JT_\lambda x_n-JS_\lambda x_n\|)\\ &\le \beta_n^{(1)}\phi(p,x_0)+\beta_n^{(2)}\phi(p,x_n)\\ &\quad+\beta_n^{(3)}\phi(p,x_n)-\beta_n^{(3)}\beta_n^{(2)}g(\|JT_\lambda x_n-JS_\lambda x_n\|)\\ &\le \beta_n^{(1)}\phi(p,x_0)+(1-\beta_n^{(1)})\phi(p,x_n)-\beta_n^{(3)}\beta_n^{(2)}g(\|JT_\lambda x_n-JS_\lambda x_n\|)\end{aligned}$$

and from condition (i) and $x_n\to\overline{x}, z_n(\lambda)\to\overline{x}$ as $n\to\infty$ for all $\lambda\in\Lambda$, we obtain

$$\beta_n^{(3)}\beta_n^{(2)}g(\|JT_\lambda x_n-JS_\lambda x_n\|)\le\beta_n^{(1)}\phi(p,x_0)+(1-\beta_n^{(1)})\phi(p,x_n)-\phi(p,z_n(\lambda))\to 0$$

as $n \to \infty$. From the properties of the mapping g and the condition (ii) we have

$$\|JT_\lambda x_n - JS_\lambda x_n\| \to 0 \tag{3.9}$$

as $n \to \infty$.

We can also get

$$\begin{aligned}
&\|Jx_n - Jz_n(\lambda)\| \\
&= \|Jx_n - (\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n)\| \\
&= \|\beta_n^{(1)}(Jx_n - Jx_0) + \beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n)\| \\
&\geq \|\beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n)\| - \|\beta_n^{(1)}(Jx_n - Jx_0)\|,
\end{aligned}$$

which leads to

$$\|\beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n)\| \leq \|Jx_n - Jz_n(\lambda)\| + \|\beta_n^{(1)}(Jx_n - Jx_0)\|.$$

Since $x_n \to \overline{x}, z_n(\lambda) \to \overline{x}$ as $n \to \infty$ and $\lim_{n\to\infty} \beta_n^{(1)} = 0$, then from above inequality we obtain

$$\|\beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n)\| \to 0 \tag{3.10}$$

as $n \to \infty$.

On the other hand, by using the property of norm $\|\cdot\|$, we have

$$\begin{aligned}
&\|\beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n)\| \\
&= \|\beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n) + \beta_n^{(3)}(Jx_n - JT_\lambda x_n) \\
&\quad - \beta_n^{(3)}(Jx_n - JT_\lambda x_n)\| \\
&= \|(\beta_n^{(2)} + \beta_n^{(3)})(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(JT_\lambda x_n - JS_\lambda x_n)\| \\
&\geq \|(\beta_n^{(2)} + \beta_n^{(3)})(Jx_n - JT_\lambda x_n)\| - \|\beta_n^{(3)}(JT_\lambda x_n - JS_\lambda x_n)\|,
\end{aligned}$$

which leads to the following inequality

$$\begin{aligned}
&\|(\beta_n^{(2)} + \beta_n^{(3)})(Jx_n - JT_\lambda x_n)\| \\
&\leq \|\beta_n^{(2)}(Jx_n - JT_\lambda x_n) + \beta_n^{(3)}(Jx_n - JS_\lambda x_n)\| + \|\beta_n^{(3)}(JT_\lambda x_n - JS_\lambda x_n)\|.
\end{aligned}$$

Therefore, by using (3.9) and (3.10) we have

$$\|(\beta_n^{(2)} + \beta_n^{(3)})(Jx_n - JT_\lambda x_n)\| \to 0,$$

this together with condition (ii) one has

$$\lim_{n\to\infty} \|(Jx_n - JT_\lambda x_n)\| = 0.$$

Further, since E has the Kadec-Klee property, the norm of E^* is *Fréchet* differentiable and therefore J^{-1} is norm-to-norm continuous, hence we have that

$$\lim_{n\to\infty} \|x_n - T_\lambda x_n\| = 0.$$

By the same way, we can prove that

$$\lim_{n\to\infty} \|x_n - S_\lambda x_n\| = 0.$$

Since T_λ and S_λ are two weak relatively nonexpansive mappings for any $\lambda \in \Lambda$, we have that $\overline{x} \in F(T_\lambda)$ and $\overline{x} \in F(S_\lambda)$ for any $\lambda \in \Lambda$ and thus $\overline{x} \in \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} F(S_\lambda)$.

Then, we show $\overline{x} \in \cap_{\lambda\in\Lambda} EP(f_\lambda)$. From $u_n(\lambda) = T_{r_n} y_n(\lambda)$ and Lemma 2.8, we obtain

$$\begin{aligned}
\phi(u_n(\lambda), y_n(\lambda)) &= \phi(T_{r_n} y_n(\lambda), y_n(\lambda)) \\
&\le \phi(\overline{x}, y_n(\lambda)) - \phi(\overline{x}, T_{r_n} y_n(\lambda)) \\
&\le \alpha_n \beta_n^{(1)} \phi(\overline{x}, x_0) + (1 - \alpha_n \beta_n^{(1)}) \phi(\overline{x}, x_n) - \phi(\overline{x}, T_{r_n} y_n(\lambda)) \\
&= \alpha_n \beta_n^{(1)} \phi(\overline{x}, x_0) + (1 - \alpha_n \beta_n^{(1)}) \phi(\overline{x}, x_n) - \phi(\overline{x}, u_n(\lambda)).
\end{aligned}$$

It follows from (3.8) that

$$\phi(u_n(\lambda), y_n(\lambda)) \to 0 \quad \text{as} \quad n \to \infty.$$

Noticing that Lemma 2.2, we get

$$\|u_n(\lambda) - y_n(\lambda)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\|Ju_n(\lambda) - Jy_n(\lambda)\| \to 0.$$

From the (A2), we note that

$$\begin{aligned}
\|y - u_n(\lambda)\| \frac{\|Ju_n(\lambda) - Jy_n(\lambda)\|}{r_n} &\ge \frac{1}{r_n} \langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda) \rangle \\
&\ge -f_\lambda(u_n(\lambda), y) \\
&\ge f_\lambda(y, u_n(\lambda)), \quad \forall y \in C, \quad \forall \lambda \in \Lambda.
\end{aligned}$$

By taking the limit as $n \to \infty$ in above inequality and from (A4) and $u_n(\lambda) \to \overline{x}$, we have $f_\lambda(y, \overline{x}) \le 0$ for all $y \in C$, for all $\lambda \in \Lambda$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\overline{x}$. Noticing that $y, \overline{x} \in C$, we obtain $y_t \in C$, which yields that $f_\lambda(y_t, \overline{x}) \le 0$. It follows from (A1) that

$$0 = f_\lambda(y_t, y_t) \le t f_\lambda(y_t, y) + (1-t) f_\lambda(y_t, \overline{x}) \le t f_\lambda(y_t, y).$$

That is, $f_\lambda(y_t, y) \ge 0$.

Let $t \downarrow 0$, from (A3), we obtain $f_\lambda(\overline{x}, y) \ge 0$, $\forall y \in C$. This implies that $\overline{x} \in EP(f_\lambda)$. This shows that $\overline{x} \in F$.

Finally, since $\overline{x} = \Pi_{C_0} x_0 \in F$ and F is a nonempty closed convex subset of $C_0 = \cap_{n=1}^{\infty} C_n$, we conclude that $\overline{x} = \Pi_F x_0$ This completes the proof. □

Taking $\alpha_n \equiv 1$, Theorem 3.3 reduces to the following result.

Theorem 3.5. *Let E be a strictly convex, reflexive and uniformly smooth Banach space having the Kadec-Klee property, let C be a non-empty and closed convex subset of E and Let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$, satisfying* (A1)-(A4). *Let $\{S_\lambda : \lambda \in \Lambda\}$ and $\{T_\lambda : \lambda \in \Lambda\}$ be two families of weak relatively nonexpansive mappings of C into itself such that*

$$F := \cap_{\lambda\in\Lambda} F(S_\lambda) \cap \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset.$$

For an arbitrarily chosen point $x_0 \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, *and*

(3.11)

$$\begin{cases} y_n(\lambda) = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_\lambda x_n + \beta_n^{(3)} JS_\lambda x_n), \\ u_n(\lambda) \in C \text{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \text{ for all } y \in C, \text{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq (1-\beta_n^{(1)})\phi(z, x_n) + \beta_n^{(1)}\phi(z, x_0)\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:

(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0$;

(iii) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0 \in C$, where Π_F is the generalized projection of E onto F.

Taking $T_\lambda \equiv S_\lambda$, Theorem 3.4 reduces to the following result.

Theorem 3.6. *Let E be a strictly convex, reflexive and uniformly smooth Banach space having the Kadec-Klee property, let C be a non-empty and closed convex subset of E and Let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$, satisfying* (A1)-(A4). *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of weak relatively nonexpansive mappings of C into itself such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \cap \cap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset$. For an arbitrarily chosen point $x_0 \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme*: $x_1 \in C$, $C_1 = C$, *and*

(3.12)

$$\begin{cases} z_n(\lambda) = J^{-1}(\beta_n Jx_0 + (1-\beta_n) JT_\lambda x_n) \quad \text{for all } \lambda \in \Lambda, \\ y_n(\lambda) = J^{-1}((1-\alpha_n) Jx_n + \alpha_n Jz_n(\lambda)) \quad \text{for all } \lambda \in \Lambda, \\ u_n(\lambda) \in C \text{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \text{ for all } y \in C, \text{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq (1-\alpha_n\beta_n)\phi(z, x_n) + \alpha_n\beta_n\phi(z, x_0)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$;
(ii) $\liminf_{n\to\infty} \alpha_n > 0$;
(iii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0 \in C$, *where* Π_F *is the generalized projection of* E *onto* F.

Remark 3. We may use more general projections for this result than the generalized projections; see [14]. Namely, the Bregman projections, which include various important examples in the convex analysis, are applicable to this result. See also [4, 5].

4. An example of weak relatively nonexpansive mapping

Next, we give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping in a Banach space l^2.

Example 1. Let $E = l^2$, where

$$l^2 = \{\xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |\xi_n|^2 < \infty\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{\frac{1}{2}}, \ \forall\, \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \ \forall\, \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots),\ \eta = (\eta_1, \eta_2, \eta_3, \dots, \eta_n \dots) \in l^2.$$

It is well known that, l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 =& (1, 0, 0, 0, \dots) \\ x_1 =& (1, 1, 0, 0, \dots) \\ x_2 =& (1, 0, 1, 0, 0, \dots) \\ x_3 =& (1, 0, 0, 1, 0, 0, \dots) \\ & \dots\dots\dots\dots\dots\dots\dots \\ x_n =& (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \dots, \xi_{n,k}, \dots) \\ & \dots\dots\dots\dots\dots\dots\dots, \end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if} \quad k = 1,\ n+1, \\ 0 & \text{if} \quad k \neq 1, k \neq n+1 \end{cases}$$

for all $n \geq 1$. Define a mapping $T : E \to E$ as follows

$$T(x) = \begin{cases} \frac{n}{n+1} x_n & \text{if} \quad x = x_n \ (\exists\, n \geq 1), \\ -x & \text{if} \quad x \neq x_n \ (\forall\, n \geq 1). \end{cases}$$

Conclusion 4.1. *$\{x_n\}$ converges weakly to x_0.*

Proof. For any $f = (\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_k, \dots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \to 0$$

as $n \to \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

Conclusion 4.2. *$\{x_n\}$ is not a Cauchy sequence, so that, it does not converges strongly to any element of l^2.*

Proof. In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

Conclusion 4.3. *T has a unique fixed point 0, that is $F(T) = \{0\}$.*

Proof. The conclusion is obvious. □

Conclusion 4.4. *x_0 is an asymptotic fixed point of T.*

Proof. Since $\{x_n\}$ converges weakly to x_0 and

$$\|Tx_n - x_n\| = \|\frac{n}{n+1}x_n - x_n\| = \frac{1}{n+1}\|x_n\| \to 0$$

as $n \to \infty$, so that, x_0 is an asymptotic fixed point of T. □

Conclusion 4.5. *T has a unique strong asymptotic fixed point 0, so that, $F(T) = \widetilde{F}(T)$.*

Proof. In fact that, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, from conclusion 4.2, there exist sufficiently large nature number N such that $z_n \neq x_m$ for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. □

Conclusion 4.6. *T is a weak relatively nonexpansive mapping.*

Proof. Since $E = L^2$ is a Hilbert space, we have

$$\phi(0, Tx) = \|0 - Tx\|^2 = \|Tx\|^2 \le \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall \ x \in E.$$

From conclusion 4.5, we have $F(T) = \widetilde{F}(T)$, then T is a weak relatively nonexpansive mapping. □

Conclusion 4.7. *T is not a relatively nonexpansive mapping.*

Proof. From Conclusions 4.3 and 4.4, we have $F(T) \neq \widehat{F}(T)$, so that, T is not a relatively nonexpansive mapping. □

5. Approximation of a zero of a maximal monotone operator

In this section, we consider the problem of finding a zero of a maximal monotone operator A, which can be applied to various kinds of problems such as equilibrium problems, variational inequalities, convex minimization problems, and others.

Now, we apply the theorem 3.1 to prove a strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let A be a multi-valued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator A is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. A monotone operator A is said to be maximal if it's graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well-known.

Theorem 5.1 (Rockafellar [24])**.** *Let E be a reflexive, strictly convex and smooth Banach space and let A be a monotone operator from E to E^*. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Remark 4.1 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique x_r such that

$$Jx \in Jx_r + rAx_r.$$

Then we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such J_r is called the resolvent of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$, and J_r is a relatively nonexpansive mapping (see [12] for more details). Using Theorem 3.1, we can consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [17, 27, 31] and some other references therein.

Theorem 5.2. *Let E be a strictly convex reflexive and uniformly smooth Banach space having the Kadec-Klee property. Let A be a maximal monotone operator of E into E^*, and let $\{f_\lambda : \lambda \in \Lambda\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$, satisfying* (A1)-(A4). *Let J_r be the resolvent of A, where $r > 0$, such that $A^{-1}0 \cap \cap_{\lambda \in \Lambda} EP(f_\lambda) \neq \emptyset$. For an arbitrarily chosen point $x_0 \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$,* *and*

$$\begin{cases} z_n(\lambda) = J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_r x_n) \ \ \textit{for all } \lambda \in \Lambda, \\ y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n(\lambda)) \ \ \textit{for all } \lambda \in \Lambda, \\ u_n(\lambda) \in C \textit{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \ \textit{for all } y \in C, \textit{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:
(i) $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$;
(ii) $\limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.
Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0 \in C$*, where* Π_F *is the generalized projection of* E *onto* F*.*

Proof. Since J_r is a closed hemi-relatively non-expansive mapping, and $A^{-1}0 = F(J_r)$, from Theorem 3.3, we obtain Theorem 5.2. □

Theorem 5.3. *Let* E *be a strictly convex reflexive and uniformly smooth Banach space having the Kadec-Klee property. Let* A *be a maximal monotone operator of* E *into* E^**, and let* $\{f_\lambda : \lambda \in \Lambda\}$ *be a family of bifunctions from* $C \times C$ *to* R*, satisfying (A1)-(A4). Let* J_r *be the resolvent of* A*, where* $r > 0$*, such that* $A^{-1}0 \cap \cap_{\lambda\in\Lambda} EP(f_\lambda) \neq \emptyset$*. For an arbitrarily chosen point* $x_0 \in E$*, generate a sequence* $\{x_n\}$ *by the following iterative scheme*: $x_1 \in C$*,* $C_1 = C$*, and*

$$\begin{cases} z_n(\lambda) = J^{-1}(\beta_n Jx_0 + (1-\beta_n)JJ_r x_n) \ \ \textit{for all } \lambda \in \Lambda, \\ y_n(\lambda) = J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jz_n(\lambda)) \ \ \textit{for all } \lambda \in \Lambda, \\ u_n(\lambda) \in C \textit{ such that } f_\lambda(u_n(\lambda), y) + \dfrac{1}{r_n}\langle y - u_n(\lambda), Ju_n(\lambda) - Jy_n(\lambda)\rangle \geq 0 \\ \ \textit{for all } y \in C, \textit{ for all } \lambda \in \Lambda, \\ C_{n+1} = \{z \in C_n : \sup\limits_{\lambda\in\Lambda} \phi(z, u_n(\lambda)) \leq (1-\alpha_n\beta_n)\phi(z, x_n) + \alpha_n\beta_n\phi(z, x_0)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases}$$

with the conditions:
(i) $\lim_{n\to\infty} \beta_n = 0$;
(ii) $\liminf_{n\to\infty} \alpha_n > 0$;
(iii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iv) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.
Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0 \in C$*, where* Π_F *is the generalized projection of* E *onto* F*.*

Proof. Since J_r is a closed hemi-relatively non-expansive mapping, and $A^{-1}0 = F(J_r)$, from Theorem 3.6, we obtain Theorem 5.3. □

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