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IMPLICIT ITERATION PROCESS FOR COMMON FIXED POINTS OF AN INFINITE FAMILY OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. Some convergence theorems for approximating to a common fixed point of an infinite family of strictly pseudocontractive mappings of Browder-Petryshyn type are proved in the setting of Banach spaces by using a new composite implicit iterative process with errors. The results presented in the paper generalize and improve the main results of Bai and Kim [1], Gu [4], Osilike [5], Su and Li [7], and Xu and Ori [8].

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E and $J : E \to E^*$ is the *normalized duality mapping* defined by

$$J(x) = \{f \in E^* : \langle x,\ f\rangle = ||x||^2 = ||f||^2\},\ \ \forall x \in E.$$

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *semi-compact* if, for any bounded sequence $\{x_n\} \subset D(T)$ such that $||x_n - Tx_n|| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x^* \in D(T)$. A mapping $T : D(T) \to R(T)$ is said to be λ-*strictly pseudocontractive* in the terminology of Browder-Petryshyn [2] if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty,\ j(x - y)\rangle \le ||x - y||^2 - \lambda||x - y - (Tx - Ty)||^2 \tag{1.1}$$

for all $x, y \in D(T)$ and $j(x - y), \in J(x - y)$. (1.1) can be written as

$$\langle (I - T)x - (I - Ty),\ j(x - y)\rangle \ge \lambda||(I - T)x - (I - T)y)||^2. \tag{1.2}$$

It is easy to see that each λ-strictly pseudocontractive mapping is L-Lipschitzian continuous. Indeed, it follows from (1.2) that

$$\lambda||(I - T)x - (I - T)y)||^2 \le ||x - y - (Tx - Ty)||||x - y||.$$

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Therefore, we have

$$\lambda||(I-T)x-(I-T)y)|| \le ||x-y||.$$

Again, since

$$\lambda(||Tx-Ty||-||x-y||) \le \lambda||(I-T)x-(I-T)y)||, \tag{1.3}$$

this implies that

$$||Tx-Ty|| \le L||x-y||, \tag{1.4}$$

where $L = 1 + \frac{1}{\lambda}$.

Concerning the convergence problem of implicit iterative processes to approximating a common fixed point for a finite family of strictly pseudocontractive mappings of Browder-Petryshyn type in the setting of Hilbert spaces or Banach spaces have been considered by several authors (see, for example, [1, 2, 4-8]).

In this paper, we introduce the following new implicit iteration process with errors for an infinite family of strictly pseudocontractive mappings of Browder-Petryshyn type:

$$\begin{cases} x_0 \in K, \\ x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n y_n + \gamma_n u_n, \\ y_n = (1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x_n + \delta_n v_n, \quad \forall n \ge 1, \end{cases} \tag{1.5}$$

where K is a nonempty closed convex subset of a real Banach space E, $\{T_n\}$ is an infinite family of strictly pseudocontractive mappings $T_n : K \to K$ of Browder-Petryshyn type, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are sequences in [0, 1] with $\alpha_n + \gamma_n \le 1$, $\beta_n + \delta_n \le 1$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in K.

Observe that, if K is a nonempty closed convex subset of E and $T_n : K \to K$ is a λ_n-strictly pseudocontractive mapping, then it is a L_n-Lipschitzian mapping with $L_n = 1 + \frac{1}{\lambda_n}$. Hence, for any x_{n-1}, $\gamma_n u_n$, $\delta_n v_n \in K$, the mapping $S_n : K \to K$ defined by

$$\begin{aligned} S_n(x) &= (1-\alpha_n-\gamma_n)x_{n-1} \\ &\quad + \alpha_n T_n\{(1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x + \delta_n v_n\} + \gamma_n u_n, \ \forall n \ge 1, \end{aligned}$$

satisfies the following:

$$\begin{aligned} ||S_n x - S_n y|| &= \alpha_n ||T_n\{(1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x_n + \delta_n v_n\} \\ &\qquad - T_n x\{(1-\beta_n-\delta_n)x_{n-1} + \alpha_n T_n y + \gamma_n v_n\}|| \\ &\le \alpha_n L_n ||\beta_n(T_n x - T_n y)|| \\ &\le \alpha_n \beta_n L_n^2 ||x-y||, \ \forall x, y \in K. \end{aligned}$$

This shows that, if $\alpha_n \beta_n L_n^2 < 1$, then $S_n : K \to K$ is a contractive mapping. By Banach contraction principle, there exists a unique fixed point $x_n \in K$ such that

$$\begin{cases} x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n y_n + \gamma_n u_n, \\ y_n = (1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x_n + \delta_n v_n, \quad \forall n \geq 1. \end{cases}$$

Therefore, if $\alpha_n \beta_n L_n^2 < 1$ for all $n \geq 1$, then the iterative sequence (1.5) can be employed for the approximation of common fixed points of an infinite family of strictly pseudocontractive mappings.

The purpose of this paper is to investigate the convergence problem of approximating to a common fixed point of an infinite family of strictly pseudocontractive mappings of Browder-Petryshyn type in an arbitrary real Banach space by this new iteration sequence (1.5). The results presented in the paper generalize and improve the results in Gu [4], Osilike [5], Su and Li [7], Xu and Ori [8].

In order to prove the main results of this paper, we need the following lemmas.

Lemma 1.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_n : K \to K$, $n = 1, 2, \ldots$, be a λ_n-strictly pseudocontractive mappings of Browder-Petryshyn type. If $\xi = \inf_{n\geq 1} \lambda_n > 0$, then we have the following*:

(1) *$\{T_n\}$ is an infinite family of uniformly ξ-strictly pseudocontractive and uniformly L-Lipschitzian with $L = 1 + \frac{1}{\xi}$, i.e., for all $x, y \in K$ and $j(x-y) \in J(x-y)$,*

$$(1.6)\quad \langle T_n x - T_n y,\ j(x-y)\rangle \leq ||x-y||^2 - \xi||x-y-(T_n x - T_n y)||^2, \ \ \forall n \geq 1,$$

and

$$(1.7)\qquad ||T_n x - T_n y|| \leq L||x-y||,$$

where $L = 1 + \frac{1}{\xi}$ for all $n \geq 1$.

(2) *If $F = \bigcap_{n=1}^{\infty} F(T_n)$ (the set of common fixed points of $\{T_n\}$) is nonempty, then F is a closed set in K.*

Proof. (1) For each $n = 1, 2, \ldots$, since T_n is a λ_n-strictly pseudocontractive mapping, we have

$$\begin{aligned} \langle T_n x - T_n y,\ j(x-y)\rangle &\leq ||x-y||^2 - \lambda_n||x-y-(T_n x - T_n y)||^2 \\ &\leq ||x-y||^2 - \xi||x-y-(T_n x - T_n y)||^2 \end{aligned}$$

for all $x, y \in K$ and $j(x-y) \in J(x-y)$. The conclusion (1.6) is proved.

Again, by (1.4), for each $n = 1, 2, \ldots$, we have

$$||T_n x - T_n y|| \leq L_n||x-y|| \leq L||x-y||, \ \ \forall n \geq 1,$$

where $L_n = 1 + \frac{1}{\lambda_n}$ and $L = 1 + \frac{1}{\xi}$.

(2) Let $\{x_n\} \subset F$ be any sequence such that $x_n \to p$. Now, we prove that $p \in F$. In fact, for any $i = 1, 2, \ldots$, from (1.7), it follows that

$$\begin{aligned}||p - T_i p|| &\le ||p - x_n|| + ||x_n - T_i p|| \\ &= ||p - x_n|| + ||T_i x_n - T_i p|| \\ &\le (1 + L)||p - x_n|| \to 0 \ \ (n \to \infty),\end{aligned}$$

which implies that $p \in F(T_i)$ for all $i \ge 1$, i.e., $p \in F$. The conclusion of Lemma 1.1 is proved. □

Lemma 1.2 ([6]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition*:

$$a_{n+1} \le (1 + b_n)a_n + c_n, \ \ \forall n \ge n_0,$$

where n_0 is some nonnegative integer such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

In addition, if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0 \ \ (n \to \infty)$.

Lemma 1.3 ([3]). *Let E be a real Banach space and J be the normalized duality mapping. Then, for any $x, y \in E$,*

$$||x + y||^2 \le ||x||^2 + 2\langle y,\ j(x + y)\rangle, \ \ \forall j(x + y) \in J(x + y).$$

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_n : K \to K$, $n = 1, 2, \ldots$, be an infinite family of λ_n-strictly pseudocontractive mappings with $F = \bigcap_{i=1}^{\infty} F(T_n) \neq \emptyset$ (the set of common fixed points of $\{T_n\}$) and $0 < \xi = \inf_{n\ge1} \lambda_n < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be sequences in $[0, 1]$ satisfying the conditions $\alpha_n + \beta_n \le 1$ and $\beta_n + \delta_n \le 1$ for all $n \ge 1$, $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and $\{x_n\}$ be the iterative sequence with errors defined by* (1.5). *If the following conditions are satisfied*:

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{i=1}^{\infty} \{\alpha_n^2 + \alpha_n\beta_n + \alpha_n\delta_n + \gamma_n\} < \infty$;
(iii) $\alpha_n\beta_n L^2 < 1$, *where* $L = 1 + \frac{1}{\xi}$,

then we have the following:

(1) *The limit* $\lim_{n\to\infty} ||x_n - p||$ *exists for any* $p \in F$.

(2) $\liminf_{n\to\infty} ||x_n - T_n x_n|| = 0$.

(3) *The sequence $\{x_n\}$ converges strongly to a common fixed point $x^* \in F$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0, \tag{2.1}$$

where $d(x,F)$ denotes the distance from x to the set F, i.e.,

$$d(x,F) = \inf_{y\in F} \|x-y\|.$$

Proof. For any $p \in F$, it follows from Lemma 1.1, Lemma 1.3 and (1.5) that
(2.2)

$$\begin{aligned}
&||x_n - p||^2 \\
&= ||(1-\alpha_n-\gamma_n)(x_{n-1}-p) + \alpha_n(T_n y_n - p) + \gamma_n(u_n - p)|| \\
&\le (1-\alpha_n-\gamma_n)^2||x_{n-1}-p||^2 + 2\alpha_n\langle T_n y_n - p,\ j(x_n-p)\rangle \\
&\quad + 2\gamma_n\langle u_n - p,\ j(x_n-p)\rangle \\
&\le (1-\alpha_n)^2||x_{n-1}-p||^2 + 2\alpha_n\langle T_n y_n - T_n x_n,\ j(x_n-p)\rangle \\
&\quad + 2\alpha_n\langle T_n x_n - p,\ j(x_n-p)\rangle + 2\gamma_n\langle u_n - p,\ j(x_n-p)\rangle \\
&\le (1-\alpha_n)^2||x_{n-1}-p||^2 + 2\alpha_n||T_n y_n - T_n x_n||||x_n-p|| \\
&\quad + 2\alpha_n||x_n-p||^2 - 2\alpha_n\xi||T_n x_n - x_n||^2 + 2\gamma_n||u_n-p||||x_n-p|| \\
&\le (1-\alpha_n)^2||x_{n-1}-p||^2 + 2\alpha_n L||y_n-x_n||||x_n-p|| \\
&\quad + 2\alpha_n||x_n-p||^2 - 2\alpha_n\xi||T_n x_n - x_n||^2 + \gamma_n(||u_n-p||^2 + ||x_n-p||^2) \\
&\le (1-\alpha_n)^2||x_{n-1}-p||^2 + 2\alpha_n L||y_n-x_n||||x_n-p|| \\
&\quad + (2\alpha_n+\gamma_n)||x_n-p||^2 - 2\alpha_n\xi||T_n x_n - x_n||^2 + \gamma_n M,
\end{aligned}$$

where $M = \sup_{n\ge1}\{||u_n-p||^2, ||u_n-p||, ||v_n-p||, ||v_n-p||^2\}$.

Now, we make an estimation for $||y_n - x_n||$. It follows from (1.5) and Lemma 1.1 that

$$\begin{aligned}
&||y_n - x_n|| \\
&= ||(1-\beta_n-\delta_n)(x_{n-1}-x_n) + \beta_n(T_n x_n - x_n) + \delta_n(v_n - x_n)|| \\
&\le (1-\beta_n-\delta_n)||x_{n-1}-x_n|| + \beta_n||T_n x_n - x_n|| + \delta_n||v_n - x_n|| \\
&\le ||(\alpha_n+\gamma_n)x_{n-1} - \alpha_n T_n y_n - \gamma_n u_n|| \\
&\quad + \beta_n(||T_n x_n - p|| + ||x_n-p||) + \delta_n(||v_n-p|| + ||x_n-p||) \\
&\le \alpha_n||x_{n-1} - T_n y_n|| + \gamma_n(x_{n-1} - u_n)|| \\
&\quad + \beta_n(L+1)||x_n-p|| + \delta_n(M + ||x_n-p||) \\
&\le \alpha_n(||x_{n-1}-p|| + ||T_n y_n - p||) + \gamma_n(||x_{n-1}-p|| + ||u_n-p||) \\
&\quad + (\beta_n(L+1)+\delta_n)||x_n-p|| + \delta_n M \\
&\le \alpha_n L||y_n-p|| + (\alpha_n+\gamma_n)||x_{n-1}-p|| \\
&\quad + (\beta_n(L+1)+\delta_n)||x_n-p|| + (\delta_n+\gamma_n)M \\
&\le \alpha_n L\{(1-\beta_n-\delta_n)||x_{n-1}-p|| + \beta_n||T_n x_n - p|| + \delta_n||v_n-p||\} \\
&\quad + (\alpha_n+\gamma_n)||x_{n-1}-p|| + (\beta_n(L+1)+\delta_n)||x_n-p|| \\
&\quad + (\delta_n+\gamma_n)M
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\leq \alpha_n L\{||x_{n-1}-p|| + \beta_n L||x_n - p|| + \delta_n M\} \\
&\quad + (\alpha_n+\gamma_n)||x_{n-1}-p|| + (\beta_n(L+1)+\delta_n)||x_n-p|| \\
&\quad + (\delta_n+\gamma_n)M \\
&\leq \{\alpha_n L + \alpha_n + \gamma_n\}||x_{n-1}-p|| \\
&\quad + (\alpha_n\beta_n L^2 + \beta_n(L+1)+\delta_n)||x_n-p|| + (\alpha_n\delta_n L + \delta_n + \gamma_n)M.
\end{aligned}$$

Substituting (2.3) into (2.2) and noting that

$$2||x_{n-1}-p||||x_n-p|| \leq \{||x_{n-1}-p||^2 + ||x_n-p||^2\}$$

and

$$2||x_n-p|| \leq \{1+||x_n-p||^2\},$$

we have

$$\begin{aligned}
&||x_n-p||^2 \\
&\leq (1-\alpha_n)^2||x_{n-1}-p||^2 \\
&\quad + \alpha_n L(\alpha_n L+\alpha_n+\gamma_n)(||x_{n-1}-p||^2+||x_n-p||^2) \\
&\quad + \{2\alpha_n L(\alpha_n\beta_n L^2+\beta_n(L+1)+\delta_n) + (2\alpha_n+\gamma_n)\}||x_n-p||^2 \\
&\quad + L\alpha_n M(\alpha_n\delta_n L+\gamma_n+\delta_n)(1+||x_n-p||^2) \\
&\quad - 2\alpha_n\xi||T_n x_n - x_n||^2 + \gamma_n M.
\end{aligned} \tag{2.4}$$

Let

$$\begin{aligned}
\eta_n = {} & \alpha_n L(\alpha_n L+\alpha_n+\gamma_n) + 2\alpha_n L(\alpha_n\beta_n L^2+\beta_n(L+1)+\delta_n) \\
& + \gamma_n + LM\alpha_n(\alpha_n\delta_n L+\gamma_n+\delta_n).
\end{aligned} \tag{2.5}$$

By the condition (ii), $\eta_n \to 0$ as $n \to \infty$. From (2.4), we have

$$\begin{aligned}
&(1-\eta_n-2\alpha_n)||x_n-p||^2 \\
&\leq \{(1-\alpha_n)^2 + L\alpha_n(L\alpha_n+\alpha_n+\gamma_n)\}||x_{n-1}-p||^2 \\
&\quad + M\{L\alpha_n(\alpha_n\delta_n L+\gamma_n+\delta_n)+\gamma_n\} - 2\alpha_n\xi||T_n x_n - x_n||^2.
\end{aligned} \tag{2.6}$$

Simplifying (2.6), we have

$$\begin{aligned}
&||x_n-p||^2 \\
&\leq \left\{1+\frac{\alpha_n^2+\eta_n+L\alpha_n(L\alpha_n+\alpha_n+\gamma_n)}{(1-\eta_n-2\alpha_n)}\right\}||x_{n-1}-p||^2 \\
&\quad + \frac{M\{L\alpha_n(\alpha_n\delta_n L+\gamma_n+\delta_n)+\gamma_n\} - 2\alpha_n\xi||T_n x_n - x_n||^2}{(1-\eta_n-2\alpha_n)}.
\end{aligned} \tag{2.7}$$

Since $\eta_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$, there exists a positive integer n_0 such that $\frac{1}{2} < 1-\eta_n-2\alpha_n < 1$ for all $n \geq n_0$. Therefore, from (2.7), it follows that

(2.8)
$$
\begin{aligned}
&||x_n - p||^2 \\
&\le \{1 + 2(\alpha_n^2 + \eta_n + L\alpha_n(L\alpha_n + \alpha_n + \gamma_n)\}||x_{n-1} - p||^2 \\
&\quad + 2M\{L\alpha_n(\alpha_n\delta_n L + \gamma_n + \delta_n) + \gamma_n)\} - 2\alpha_n\xi||T_n x_n - x_n||^2, \ \forall n \ge n_0.
\end{aligned}
$$

Letting

$$
\begin{aligned}
b_n &= 2(\alpha_n^2 + \eta_n + L\alpha_n(L\alpha_n + \alpha_n + \gamma_n), \\
c_n &= 2M\{L\alpha_n(\alpha_n\delta_n L + \gamma_n + \delta_n) + \gamma_n)\},
\end{aligned}
$$

by the condition (ii), we have $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$.

On the other hand, it follows from (2.8) that

$$
||x_n - p||^2 \le (1 + b_n)||x_{n-1} - p||^2 + c_n, \ \forall n \ge n_0. \tag{2.9}
$$

By Lemma 1.3, the limit $\lim_{n\to\infty} ||x_n - p||$ exists. Hence $\{||x_n - p||\}$ is bounded. Without loss of generality, we can assume that

$$
||x_n - p|| \le B, \ \forall n \ge 1.
$$

Furthermore, it follows from (2.8) and (2.9) that

$$
2\alpha_n\xi||T_n x_n - x_n||^2 \le ||x_{n-1} - p||^2 - ||x_n - p||^2 + Bb_n + c_n, \ \forall n \ge n_0. \tag{2.10}
$$

Therefore, we have

$$
2\xi \sum_{n=n_0}^{\infty} \alpha_n||T_n x_n - x_n||^2 \le ||x_{n_0-1} - p||^2 + B\sum_{n=n_0}^{\infty} b_n + \sum_{n=n_0}^{\infty} c_n < \infty.
$$

This implies that

$$
\sum_{n=1}^{\infty} \alpha_n||T_n x_n - x_n||^2 < \infty.
$$

Next, we prove that

$$
\liminf_{n\to\infty} ||T_n x_n - x_n|| = 0. \tag{2.11}
$$

In fact, suppose that the conclusion is not true. That is, if $\liminf_{n\to\infty} ||T_n x_n - x_n|| = C > 0$, then, for any $\varepsilon \in (0, C)$, there exists an positive n_1 such that

$$
||T_n x_n - x_n||^2 > (C - \varepsilon)^2, \ \forall n \ge n_1.
$$

Hence we have

$$
\infty = \sum_{n=n_1}^{\infty} (C - \varepsilon)^2 \alpha_n \le \sum_{n=n_1}^{\infty} \alpha_n||T_n x_n - x_n||^2 < \infty.
$$

This is a contradiction. Therefore, the conclusion (2.11) is proved.

Finally, we prove the conclusion (3). The necessity of condition (2.1) is obvious.

Now, we prove the sufficiency of the condition (2.1). Indeed, it follows from (2.9) that

$$
||x_n - p||^2 \le (1 + b_n)||x_{n-1} - p||^2 + c_n, \ \forall n \ge n_0, \tag{2.12}
$$

where the sequences $\{b_n\}$ and $\{c_n\}$ satisfy $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$. Hence we have

$$(d(x_n, F))^2 \leq (1+b_n)((d(x_{n-1}, F))^2) + c_n, \ \ \forall n \geq n_0. \tag{2.13}$$

By Lemma 1.2, the condition (2.1) and (2.13), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum_{n=1}^{\infty} b_n < \infty$ and $1+t \leq \exp\{t\}$ for all $t > 0$, by (2.12), we have

$$||x_n - p||^2 \leq \exp\{b_n\}||x_{n-1} - p||^2 + c_n, \ \ \forall n \geq n_0. \tag{2.14}$$

Hence, for any positive integers $n, m \geq n_0$, from (2.14), we have

$$\begin{aligned}
&||x_{n+m} - p||^2 \\
\leq\ & \exp\{b_{n+m}\}||x_{n+m-1} - p||^2 + c_{n+m} \\
\leq\ & \exp\{b_{n+m}\}[\exp\{b_{n+m-1}\}||x_{n+m-2} - p||^2 + c_{n+m-1}] + c_{n+m} \\
\leq\ & \exp\{b_{n+m} + b_{n+m-1}\}||x_{n+m-2} - p||^2 + \exp\{b_{n+m}\}c_{n+m-1} + c_{n+m} \\
\leq\ & \cdots \\
\leq\ & \exp\{\sum_{i=n+1}^{n+m} b_i\}||x_n - p||^2 + \exp\{\sum_{i=n+2}^{n+m} b_i\}\sum_{i=n+1}^{n+m} c_i \\
\leq\ & Q(||x_n - p||^2 + \sum_{i=n+1}^{n+m} c_i),
\end{aligned} \tag{2.15}$$

where $Q = \exp\{\sum_{n=1}^{\infty} b_n\} < \infty$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=1}^{\infty} c_i < \infty$, for any given $\epsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that

$$[d(x_n, F)]^2 < \frac{\epsilon^2}{8(Q+1)}, \quad \sum_{i=n+1}^{n+m} c_i < \frac{\epsilon^2}{4Q}, \ \ \forall n \geq n_1.$$

Hence there exists $p_1 \in F$ such that

$$||x_n - p||^2 < \frac{\epsilon^2}{4(Q+1)}, \ \ \forall n \geq n_1. \tag{2.16}$$

Consequently, it follows from (2.15) and (2.16) that, for any $n \geq n_1$ and $m \geq 1$,

$$\begin{aligned}
||x_{n+m} - x_n||^2 &\leq 2(||x_{n+m} - p_1||^2 + ||x_n - p_1||^2) \\
&\leq 2\{Q(||x_n - p_1||^2 + \sum_{i=n+1}^{\infty} c_i) + ||x_n - p_1||^2\} \\
&= 2\{(1+Q)||x_n - p_1||^2 + Q\sum_{i=n+1}^{\infty} c_i\} \\
&\leq 2(\frac{\epsilon^2}{4} + \frac{\epsilon^2}{4}) = \epsilon^2.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. Since K is closed, let $x_n \to p \in K$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and F is a closed subset by Lemma 1.1, we have $p \in F$. This completes the proof. □

Taking $\gamma_n = 0$ and $\delta_n = 0$ for all $n \geq 1$ in Theorem 2.1, we can obtain the following theorem, which is a generalization of Su and Li [7, Theorem 2.1] and the corresponding results of Osilike [5], Xu and Ori [8].

Theorem 2.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_n : K \to K$, $n = 1, 2, \ldots$, be an infinite family of λ_n-strictly pseudocontractive mappings with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $0 < \xi = \inf_{n\geq 1} \lambda_n < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ and $\{x_n\}$ be the iterative sequence defined by*

$$\begin{cases} x_0 \in K, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n y_n, \\ y_n = (1-\beta_n)x_{n-1} + \beta_n T_n x_n, \quad \forall n \geq 1. \end{cases} \tag{2.17}$$

If the following conditions are satisfied:

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{i=1}^{\infty} \{\alpha_n^2 + \alpha_n\beta_n\} < \infty$;
(iii) $\alpha_n\beta_n L^2 < 1$, *where* $L = 1 + \frac{1}{\xi}$,

then all the conclusions in Theorem 2.1 are still hold.

In Theorem 2.2, if $\beta_n = 0$ for all $n \geq 1$, then we have the following result.

Theorem 2.3. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_n : K \to K$, $n = 1, 2, \ldots$, be an infinite family of λ_n-strictly pseudocontractive mappings with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $0 < \xi = \inf_{n\geq 1} \lambda_n < 1$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{x_n\}$ be the iterative sequence defined by*

$$\begin{cases} x_0 \in K, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_{n-1}, \quad \forall n \geq 1. \end{cases} \tag{2.18}$$

If the following conditions are satisfied:

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{i=1}^{\infty} \alpha_n^2 < \infty$,

then all the conclusions in Theorem 2.1 still hold.

Theorem 2.4. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a semi-compact λ-strictly pseudocontractive mappings with $F(T) \neq \emptyset$ (the set of common fixed points of T). Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be sequences in $[0, 1]$ satisfying the conditions $\alpha_n + \beta_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$, $\{v_n\}$ be two bounded sequences in K and*

$\{x_n\}$ be the iterative sequence with errors defined by

$$\begin{cases} x_0 \in K, \\ x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T y_n + \gamma_n u_n, \\ y_n = (1-\beta_n-\delta_n)x_{n-1} + \beta_n T x_n + \delta_n v_n, \quad \forall n \geq 1. \end{cases} \tag{2.19}$$

If the following conditions are satisfied:

(i) $\sum_{i=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{i=1}^{\infty} \{\alpha_n^2 + \alpha_n\beta_n + \alpha_n\delta_n + \gamma_n\} < \infty$;
(iii) $\alpha_n\beta_n L^2 < 1$, *where* $L = 1 + \frac{1}{\lambda}$,

then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Theorem 2.1, we know that

$$\liminf_{n\to\infty} ||x_n - Tx_n|| = 0$$

and so there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\lim_{n\to\infty} ||x_{n_k} - Tx_{n_k}|| = 0. \tag{2.20}$$

By the semi-compactness of T, there exists a subsequence $\{x_{n_i}\} \subset \{x_{n_k}\}$ such that $\lim_{i\to\infty} x_{n_i} = x^*$. Hence it follows from (2.20) that $x^* = Tx^*$. Since $\lim_{n\to\infty} ||x_n - x^*||$ exists, we know that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

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