REGIONS OF VARIABILITY FOR GENERALIZED
α-CONVEX AND β-STARLIKE FUNCTIONS,
AND THEIR EXTREME POINTS

Shaolin Chen and Aiwu Huang

Abstract. Suppose that n is a positive integer. For any real number α (β resp.) with $\alpha < 1$ ($\beta > 1$ resp.), let $K^{\langle n \rangle}(\alpha)$ ($K^{\langle n \rangle}(\beta)$ resp.) be the class of analytic functions in the unit disk $\mathbb{D}$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = f^{(n)}(0) - 1 = 0$, $\mathrm{Re}(\frac{zf^{(n+1)}(z)}{f^{(n)}(z)} + 1) > \alpha$ ($\mathrm{Re}(\frac{zf^{(n+1)}(z)}{f^{(n)}(z)} + 1) < \beta$ resp.) in $\mathbb{D}$, and for any $\lambda \in \overline{\mathbb{D}}$, let $K^{\langle n \rangle}(\alpha, \lambda)$ ($K^{\langle n \rangle}(\beta, \lambda)$ resp.) denote a subclass of $K^{\langle n \rangle}(\alpha)$ ($K^{\langle n \rangle}(\beta)$ resp.) whose elements satisfy some condition about derivatives. For any fixed $z_0 \in \mathbb{D}$, we shall determine the two regions of variability $V^{\langle n \rangle}(z_0, \alpha)$ ($V^{\langle n \rangle}(z_0, \beta)$ resp.) and $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ ($V^{\langle n \rangle}(z_0, \beta, \lambda)$ resp.). Also we shall determine the extreme points of the families of analytic functions which satisfy $f(\mathbb{D}) \subset V^{\langle n \rangle}(z_0, \alpha)$ ($f(\mathbb{D}) \subset V^{\langle n \rangle}(z_0, \beta)$ resp.) when f ranges over the classes $K^{\langle n \rangle}(\alpha)$ ($K^{\langle n \rangle}(\beta)$ resp.) and $K^{\langle n \rangle}(\alpha, \lambda)$ ($K^{\langle n \rangle}(\beta, \lambda)$ resp.), respectively.

1. Introduction and preliminaries

We denote the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by $\mathcal{H}(\mathbb{D})$, and we think of $\mathcal{H}(\mathbb{D})$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. We let $K^{\langle n \rangle}(\alpha)$ ($K^{\langle n \rangle}(\beta)$ resp.), where $\alpha < 1$ ($\beta > 1$ resp.), denote the set of analytic functions $f \in \mathcal{H}(\mathbb{D})$ which are generalized convex of order α in $\mathbb{D}$ (generalized β-starlike functions in $\mathbb{D}$ resp.). We recall that $f \in K^{\langle n \rangle}(\alpha)$ ($f \in K^{\langle n \rangle}(\beta)$ resp.) if and only if $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = f^{(n)}(0) - 1 = 0$ and $\mathrm{Re}(P_f^{\langle n \rangle}(z)) > \alpha$ ($\mathrm{Re}(P_f^{\langle n \rangle}(z)) < \beta$ resp.), where

$$P_f^{\langle n \rangle}(z) = \frac{zf^{(n+1)}(z)}{f^{(n)}(z)} + 1 \ \ (z \in \mathbb{D})$$

and "Re" denotes the real part.

Let S_n^* denote the class of univalent starlike functions in $\mathbb{D}$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = f^{(n)}(0) - 1 = 0$, where $f^{(n-1)}$ is univalent.

We recall that $K^{\langle n \rangle}(\beta) \subset S_n^*$ for $\beta = \frac{3}{2}$ (See [9]). It is well known that if $-\frac{1}{2} \leq \alpha < 1$ and $f \in K^{\langle n \rangle}(\alpha)$, then $f^{(n-1)}$ is univalent (cf. [4]).

For $f \in K^{\langle n \rangle}(\alpha)$ ($f \in K^{\langle n \rangle}(\beta)$ resp.), we denote by $\log f^{(n)}$ the single-valued branch of the logarithm of $f^{(n)}$ with $\log f^{(n)}(0) = 0$. Using the well known Herglotz representation for analytic functions with positive real part in $\mathbb{D}$, we know that if $f \in K^{\langle n \rangle}(\alpha)$, then there exists a unique positive unit measure μ on $(-\pi, \pi]$ such that

$$\frac{z f^{(n+1)}(z)}{f^{(n)}} + 1 = (1 - \alpha) \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) + \alpha.$$

Hence

$$(1.1) \qquad \log f^{(n)}(z) = 2(1 - \alpha) \int_{-\pi}^{\pi} \log \frac{1}{1 - ze^{-it}} d\mu(t).$$

It follows that for each fixed $z_0 \in \mathbb{D}$ the region of variability

$$V^{\langle n \rangle}(z_0, \alpha) = \{\log f^{(n)}(z_0) : f \in K^{\langle n \rangle}(\alpha)\}$$

coincides with the set

$$(1.2) \qquad\qquad \{-2(1 - \alpha) \log(1 - z) : |z| \leq |z_0|\}.$$

Similarly, if $f \in K^{\langle n \rangle}(\beta)$, then

$$\frac{z f^{(n+1)}(z)}{f^{(n)}} + 1 = \beta - (\beta - 1) \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t).$$

Hence

$$\log f^{(n)}(z) = 2(\beta - 1) \int_{-\pi}^{\pi} \log(1 - ze^{-it}) d\mu(t).$$

And so for each fixed $z_0 \in \mathbb{D}$, the region of variability

$$V^{\langle n \rangle}(z_0, \beta) = \{\log f^{(n)}(z_0) : f \in K^{\langle n \rangle}(\beta)\}$$

coincides with the set

$$\{2(\beta - 1) \log(1 - z) : |z| \leq |z_0|\}.$$

Let B_0 be the class of analytic functions ω in $\mathbb{D}$ such that $|\omega(z)| \leq 1$ in $\mathbb{D}$ and $\omega(0) = 0$. We see that if $f \in K^{\langle n \rangle}(\alpha)$, then

$$(1.3) \qquad\qquad \omega_f(z) = \frac{P_f^{\langle n \rangle}(z) - 1}{P_f^{\langle n \rangle}(z) + 1 - 2\alpha} \in B_0.$$

If $f \in K^{\langle n \rangle}(\beta)$, then

(1.4)
$$\omega_f(z) = \frac{P_f^{\langle n \rangle}(z) - 1}{P_f^{\langle n \rangle}(z) - 2\beta + 1} \in B_0;$$

and conversely.

Schwarz Lemma implies that if $f \in K^{\langle n \rangle}(\alpha)$, then

$$|f^{(n+1)}(0)| = |2(1 - \alpha)\omega_f'(0)| \leq 2(1 - \alpha).$$

If $f \in K^{\langle n \rangle}(\beta)$, then

$$|f^{(n+1)}(0)| = |2(1 - \beta)\omega_f'(0)| \leq 2(\beta - 1).$$

For $\lambda \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $z_0 \in \mathbb{D}$, we introduce the following six notations.

$$K^{\langle n \rangle}(\alpha, \lambda) = \{f \in K^{\langle n \rangle}(\alpha) : f^{(n+1)}(0) = 2(1 - \alpha)\lambda\},$$

$$K^{\langle n \rangle}(\beta, \lambda) = \{f \in K^{\langle n \rangle}(\beta) : f^{(n+1)}(0) = 2(1 - \beta)\lambda\},$$

$$V^{\langle n \rangle}(z_0, \alpha, \lambda) = \{\log f^{(n)}(z_0) : f \in K^{\langle n \rangle}(\alpha, \lambda)\},$$

$$V^{\langle n \rangle}(z_0, \beta, \lambda) = \{\log f^{(n)}(z_0) : f \in K^{\langle n \rangle}(\beta, \lambda)\}$$

and

$$\mathcal{F}_\alpha = \{P_f^{\langle n \rangle}(z) : \mathrm{Re}(P_f^{\langle n \rangle}(z)) > \alpha, \text{ where } \alpha < 1\},$$

$$\mathcal{F}_\beta = \{P_f^{\langle n \rangle}(z) : \mathrm{Re}(P_f^{\langle n \rangle}(z)) < \beta, \text{ where } \beta > 1\}.$$

Herglotz formula shows that the extreme points of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are precisely the functions

$$f(z) = (1 - \alpha)\frac{e^{i\theta} + z}{e^{i\theta} - z} + \alpha$$

and

$$f(z) = \beta - (\beta - 1)\frac{e^{i\theta} + z}{e^{i\theta} - z},$$

respectively.

Recently, the regions of variability for convex functions, functions of bounded derivative, and spirallike functions etc have been discussed by Ponnusamy, Vasudevarao, and Yanagihara. See [10, 11, 14] for the details.

The main aim of this paper is to determine the sets $V^{\langle n \rangle}(z_0, \alpha)$, $V^{\langle n \rangle}(z_0, \alpha, \lambda)$, $V^{\langle n \rangle}(z_0, \beta)$, $V^{\langle n \rangle}(z_0, \beta, \lambda)$ and the extreme points of the families of analytic functions which satisfy $f(\mathbb{D}) \subset V^{\langle n \rangle}(z_0, \alpha)$ or $f(\mathbb{D}) \subset V^{\langle n \rangle}(z_0, \beta)$, respectively. Our main results are Theorems 2.2, 2.4, 2.6, 2.7, 2.8 and 2.9, where Theorems 2.2 and 2.4 are generalizations of the corresponding results in [10, 14]. They will be proved in Section 3.

2. Basic properties and main results

For our presentation, we need some more preparation. Let $S^* = S_1^*$, i.e., the class of analytic functions f in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ which map $\mathbb{D}$ conformally onto a starlike domain (with respect to the origin). Each function $f \in S^*$ is called *starlike* (univalent) in $\mathbb{D}$ and the function $f \in S^*$ is characterized by the analytic condition $\mathrm{Re}(zf'(z)/f(z)) > 0$ in $\mathbb{D}$ (cf. [2, 3]). For a positive integer p, let $(S^*)^p = \{f = f_0^p : f_0 \in S^*\}$. Now we recall the following result which is from [3].

Lemma 2.1. *Let f be an analytic function in $\mathbb{D}$ with $f(z) = z^p + \cdots$. If for any $z \in \mathbb{D}$, $\mathrm{Re}(1 + \frac{zf''(z)}{f'(z)}) > 0$, then $f \in (S^*)^p$.*

We shall introduce some basic properties of $V^{\langle n \rangle}(z_0, \alpha)$, $V^{\langle n \rangle}(z_0, \alpha, \lambda)$, $V^{\langle n \rangle}(z_0, \beta)$ and $V^{\langle n \rangle}(z_0, \beta, \lambda)$. Since the two sets $V^{\langle n \rangle}(z_0, \alpha)$, $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ and the two sets $V^{\langle n \rangle}(z_0, \beta)$, $V^{\langle n \rangle}(z_0, \beta, \lambda)$ have the similar corresponding properties, in the following, we only list down some basic properties of $V^{\langle n \rangle}(z_0, \alpha)$ and $V^{\langle n \rangle}(z_0, \alpha, \lambda)$.

(I) $V^{\langle n \rangle}(z_0, \alpha, \lambda) \subset V^{\langle n \rangle}(z_0, \alpha)$.

(II) The sets $V^{\langle n \rangle}(z_0, \alpha)$ and $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ are compact.
This statement follows from (1.2) and the fact that both $K^{\langle n \rangle}(\alpha)$ and $K^{\langle n \rangle}(\alpha, \lambda)$ are closed in $\mathcal{H}(\mathbb{D})$.

(III) The sets $V^{\langle n \rangle}(z_0, \alpha)$ and $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ are convex.
Indeed, if $f_0, f_1 \in K^{\langle n \rangle}(\alpha)$ and $0 \le t \le 1$, then the function

$$f_t(z) = \int_0^z \int_0^{\zeta_n} \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_2} \exp\{ (1-t)\log f_0^{(n)}(\zeta_1) \\ + t\log f_1^{(n)}(\zeta_1)\}d\zeta_1 d\zeta_2 \cdots d\zeta_{n-1}d\zeta_n$$

belongs to $K^{\langle n \rangle}(\alpha)$. Since $\log f_t^{(n)}(z_0) = (1-t)\log f_0^{(n)}(z_0) + t\log f_1^{(n)}(z_0)$, the convexity of $V^{\langle n \rangle}(z_0, \alpha)$ follows.

Similar reasoning shows that $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is also convex.

(IV) If $|\lambda| = 1$ or $z_0 = 0$, then $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ consists of only one point which is

$$-2(1-\alpha)\log(1 - \lambda z_0).$$

If $|\lambda| < 1$ and $z_0 \ne 0$, then

$$(2.1) \qquad\qquad -2(1-\alpha)\log(1 - \lambda z_0)$$

is an interior point of the set $V^{\langle n \rangle}(z_0, \alpha, \lambda)$.

Indeed, if $|\lambda| = |\omega'_f(0)| = 1$, then it follows from Schwarz Lemma that $\omega_f(z) = \lambda z$, which implies that $P_f^{\langle n \rangle}(z) = \frac{1+(1-2\alpha)\lambda z}{1-\lambda z}$ and $\log f^{(n)}(z) = -2(1-\alpha)\log(1-\lambda z)$. This also trivially holds for the case $z_0 = 0$.

For $\lambda \in \mathbb{D}$ and $a \in \overline{\mathbb{D}}$, let $\delta(z, \lambda) = \frac{z + \lambda}{1 + \overline{\lambda}z}$ and

(2.2)
$$F_{\langle n \rangle, a, \lambda}(z)$$
$$= \int_0^z \left\{ \int_0^{\zeta_{n+1}} \int_0^{\zeta_n} \cdots \int_0^{\zeta_3} \exp\left\{ \int_0^{\zeta_2} \frac{2(1 - \alpha)\delta(a\zeta_1, \lambda)}{1 - \delta(a\zeta_1, \lambda)\zeta_1} d\zeta_1 \right\} d\zeta_2 \cdots d\zeta_n \right\} d\zeta_{n+1}$$

for $z \in \mathbb{D}$.

Then, obviously,

$$F_{\langle n \rangle, a, \lambda} \in K^{\langle n \rangle}(\alpha, \lambda)$$

and

(2.3)
$$\omega_{F_{\langle n \rangle, a, \lambda}}(z) = z\delta(az, \lambda).$$

(V) The mapping $\mathbb{D} \ni a \mapsto \log F_{\langle n \rangle, a, \lambda}^{(n)}(z_0)$ is a non-constant analytic function of a for any fixed $z_0 \in \mathbb{D} \backslash \{0\}$ and $\lambda \in \mathbb{D}$.

Let

$$h(z) = \frac{1}{(1 - \alpha)(1 - |\lambda|^2)} \frac{\partial}{\partial a} \{\log F_{\langle n \rangle, a, \lambda}^{(n)}(z)\}_{a=0}$$
$$= 2 \int_0^z \frac{\zeta}{(1 - \lambda\zeta)^2} d\zeta = z^2 + \cdots .$$

Then it is easy to see that $\frac{zh''(z)}{h'(z)} + 1 = \frac{2}{1 - z\lambda}$ and $\mathrm{Re}\{\frac{zh''(z)}{h'(z)} + 1\} > 0$ for $z \in \mathbb{D}$.

By Lemma 2.1, there exists a function $h_0 \in S^*$ with $h = h_0^2$. The univalence of h_0 and $h_0(0) = 0$ imply that $h(z_0) \neq 0$ for $z_0 \in \mathbb{D} \backslash \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto \log F_{\langle n \rangle, a, \lambda}^{(n)}(z_0)$ is a non-constant analytic function of a. Property (V) is proved.

It follows from Property (V) that the mapping in (V) is an open mapping. Hence $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ contains the open set $\{\log F_{\langle n \rangle, a, \lambda}^{(n)}(z_0) : |a| < 1\}$. In particular,

$$\log F_{\langle n \rangle, 0, \lambda}^{(n)}(z_0) = -2(1 - \alpha)\log(1 - z_0\lambda)$$

is an interior point of the set $\{\log F_{\langle n \rangle, a, \lambda}^{(n)}(z_0) : a \in \mathbb{D}\} \subset V^{\langle n \rangle}(z_0, \alpha, \lambda)$. Up to now, we finish the proof of Property (IV).

Finally, since $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, we see that the boundary $\partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$ of $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is a Jordan curve and $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is the union of $\partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$ and its inner domain.

(VI) $V^{\langle n \rangle}(z_0 e^{i\theta}, \alpha, \lambda) = V^{\langle n \rangle}(z_0, \alpha, e^{i\theta}\lambda)$ for $\theta \in \mathbb{R}$.

This is a consequence of the fact that $e^{-in\theta}f(e^{i\theta}z) \in K^{\langle n \rangle}(\alpha, \lambda e^{i\theta})$ if and only if $f \in K^{\langle n \rangle}(\alpha, \lambda)$.

The following are our main results.

Theorem 2.2. *For $0 \leq \lambda < 1$ and $z_0 \in \mathbb{D}\backslash\{0\}$, the boundary $\partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \log F^{(n)}_{\langle n \rangle, e^{i\theta}, \lambda}(z_0) = \int_0^{z_0} \frac{2(1-\alpha)\delta(e^{i\theta}\zeta, \lambda)}{1 - \delta(e^{i\theta}\zeta, \lambda)\zeta} d\zeta.$$

If $\log f^{(n)}(z_0) = \log F^{(n)}_{\langle n \rangle, e^{i\theta}, \lambda}(z_0)$ for some $f \in K^{\langle n \rangle}(\alpha, \lambda)$ and $\theta \in (-\pi, \pi]$, then $f = F_{\langle n \rangle, e^{i\theta}, \lambda}$.

Remark 2.3. When $\alpha = 0$, Theorem 2.2 coincides with Theorem 1.1 in [14], and when $\alpha = -\frac{1}{2}$, Theorem 2.2 coincides with Theorem 2.8 in [10].

Theorem 2.4. *For $0 \leq \lambda < 1$ and $z_0 \in \mathbb{D}\backslash\{0\}$, the boundary $\partial V^{\langle n \rangle}(z_0, \beta, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \log G^{(n)}_{\langle n \rangle, e^{i\theta}, \lambda}(z_0) = \int_0^{z_0} \frac{2(\beta - 1)\delta(e^{i\theta}\zeta, \lambda)}{\delta(e^{i\theta}\zeta, \lambda)\zeta - 1} d\zeta,$$

where

$$G_{\langle n \rangle, a, \lambda}(z)$$
$$= \int_0^z \left\{ \int_0^{\zeta_{n+1}} \int_0^{\zeta_n} \cdots \int_0^{\zeta_3} \exp\left\{ \int_0^{\zeta_2} \frac{2(\beta-1)\delta(a\zeta_1, \lambda)}{\delta(a\zeta_1, \lambda)\zeta_1 - 1} d\zeta_1 \right\} d\zeta_2 \cdots d\zeta_n \right\} d\zeta_{n+1}$$

for $z \in \mathbb{D}$.

If $\log f^{(n)}(z_0) = \log G^{(n)}_{\langle n \rangle, e^{i\theta}, \lambda}(z_0)$ for some $f \in K^{\langle n \rangle}(\beta, \lambda)$ and $\theta \in (-\pi, \pi]$, then $f = G_{\langle n \rangle, e^{i\theta}, \lambda}$.

Remark 2.5. When $\beta = \frac{3}{2}$, Theorem 2.4 coincides with Theorem 2.6 in [10].

As in [12], a proper domain G of $\mathbb{C}$ is called *a uniform domain* provided there exists a constant $c \ (> 0)$ such that each pair of points $z_1, z_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$l(\gamma) \leq c|z_1 - z_2|$$

and

$$\min_{j=1,2} l(\gamma[z_j, z]) \leq c \operatorname{dist}(z, \partial D)$$

for all $z \in \gamma$. Here $l(\gamma)$ denotes the Euclidean length of γ, $\gamma(z_j, z)$ the part of γ between z_j and z, and $\operatorname{dist}(z, \partial D)$ the Euclidean distance from z to ∂D which is the boundary of D.

The following two results easily follow from Properties (II), (III) as above and the well known fact that any bounded and convex proper domain of $\mathbb{C}$ is uniform (cf. [13]).

Theorem 2.6. *For $z_0 \in \mathbb{D}\backslash\{0\}$, the domains*

$$V^{\langle n \rangle}(z_0, \alpha)\backslash\partial V^{\langle n \rangle}(z_0, \alpha)$$

and

$$V^{\langle n \rangle}(z_0, \alpha, \lambda) \backslash \partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$$

are uniform.

Theorem 2.7. *For $z_0 \in \mathbb{D} \backslash \{0\}$, the domains*

$$V^{\langle n \rangle}(z_0, \beta) \backslash \partial V^{\langle n \rangle}(z_0, \beta)$$

and

$$V^{\langle n \rangle}(z_0, \beta, \lambda) \backslash \partial V^{\langle n \rangle}(z_0, \beta, \lambda)$$

are uniform.

Theorem 2.8. *Let F_α^* denote the set of analytic functions in $\mathbb{D}$ so that $f(\mathbb{D}) \subset V^{\langle n \rangle}(z_0, \alpha)$ and $f(0) = 0$. Then a function f in $\mathbb{D}$ is an extreme point of F_α^* if and only if $f \in F_\alpha^*$ and $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$, where $\lambda(\theta)$ denotes the distance between $f(e^{i\theta})(= \lim_{r \to 1} f(re^{i\theta}))$ and $\partial V^{\langle n \rangle}(z_0, \alpha)$.*

Theorem 2.9. *Let F_β^* denote the set of analytic functions in $\mathbb{D}$ so that $f(\mathbb{D}) \subset V^{\langle n \rangle}(z_0, \beta)$ and $f(0) = 0$. Then a function f in $\mathbb{D}$ is an extreme point of F_β^* if and only if $f \in F_\beta^*$ and $\int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty$, where $\lambda(\theta)$ denotes the distance between $f(e^{i\theta})(= \lim_{r \to 1} f(re^{i\theta}))$ and $\partial V^{\langle n \rangle}(z_0, \beta)$.*

As a simple application of Theorems 2.8 and 2.9, we see that if

$$f_\alpha(z) = 2(1 - \alpha) \log \frac{1}{1 - z|z_0|e^{-i\theta}}$$

and

$$f_\beta(z) = 2(\beta - 1) \log(1 - z|z_0|e^{-i\theta}),$$

then $f_\alpha(z)$ and $f_\beta(z)$ are extreme points of F_α^* and F_β^*, respectively.

3. Proofs of the main results

It is enough to prove Theorems 2.2 and 2.8 since the proofs of Theorems 2.3 and 2.9 are similar.

We start with the following proposition which plays a key role in the proof of Theorem 2.2.

Proposition 3.1. *For any $f \in K^{\langle n \rangle}(\alpha, \lambda)$, we know that for any $z \in \mathbb{D}$,*

$$(3.1) \qquad \left| \frac{f^{(n+1)}(z)}{f^{(n)}(z)} - c(z, \lambda) \right| \leq r(z, \lambda) \ (z \in \mathbb{D}),$$

where

$$c(z, \lambda) = \frac{2(1 - \alpha)\{\lambda(1 - |z|^2) + \overline{z}(|z|^2 - \lambda^2)\}}{(1 - |z|^2)(1 - \lambda(z + \overline{z}) + |z|^2)}$$

and

$$r(z, \lambda) = \frac{2(1 - \alpha)(1 - \lambda^2)|z|}{(1 - |z|^2)(1 - \lambda(z + \overline{z}) + |z|^2)}.$$

For each $z \in \mathbb{D}\setminus\{0\}$, the equality sign in (3.1) holds if and only if $f = F_{\langle n \rangle, e^{i\theta}, \lambda}$ for some $\theta \in \mathbb{R}$.

Proof. For any $f \in K^{\langle n \rangle}(\alpha, \lambda)$, let $\omega_f \in B_0$ be as in (1.3). Then $\omega_f'(0) = \lambda$. It follows from Schwarz Lemma (see for example [2] or [6, 7, 8]) that

$$(3.2) \qquad \left| \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \frac{\lambda \omega_f(z)}{z}} \right| \leq |z|.$$

From (1.3), the inequality (3.2) is equivalent to

$$(3.3) \qquad \left| \frac{\frac{f^{(n+1)}(z)}{f^{(n)}(z)} - A(z, \lambda)}{\frac{f^{(n+1)}(z)}{f^{(n)}(z)} + B(z, \lambda)} \right| \leq |z||T(z, \lambda)|,$$

where

$$(3.4) \qquad A(z, \lambda) = \frac{2(1-\alpha)\lambda}{1 - \lambda z}, \ \ B(z, \lambda) = \frac{2(1-\alpha)}{z - \lambda} \ \ \text{and} \ \ T(z, \lambda) = \frac{z - \lambda}{1 - z\lambda}.$$

We can see that the inequality (3.3) is equivalent to

$$(3.5) \qquad \begin{aligned} & \left| \frac{f^{(n+1)}(z)}{f^{(n)}(z)} - \frac{[A(z, \lambda) + |z|^2|T(z, \lambda)|^2 B(z, \lambda)]}{1 - |z|^2|T(z, \lambda)|^2} \right| \\ & \leq \frac{|z||T(z, \lambda)||A(z, \lambda) + B(z, \lambda)|}{1 - |z|^2|T(z, \lambda)|^2}. \end{aligned}$$

It follows from (3.4) that

$$1 - |z|^2|T(z, \lambda)|^2 = \frac{(1 - |z|^2)(1 + |z|^2 - 2\lambda \mathrm{Re}(z))}{|1 - \lambda z|^2},$$

$$A(z, \lambda) + B(z, \lambda) = \frac{2(1 - \lambda^2)(1 - \alpha)}{(1 - \lambda z)(z - \lambda)}$$

and

$$A(z, \lambda) + |z|^2|T(z, \lambda)|^2 B(z, \lambda) = \frac{2(1 - \alpha)\{\lambda(1 - |z|^2) + \overline{z}(|z|^2 - \lambda^2)\}}{|1 - \lambda z|^2}.$$

Then we see that

$$\frac{A(z, \lambda) + |z|^2|T(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2|T(z, \lambda)|^2} = c(z, \lambda)$$

and

$$\frac{|z||T(z, \lambda)||A(z, \lambda) + B(z, \lambda)|}{1 - |z|^2|T(z, \lambda)|^2} = r(z, \lambda).$$

By (3.5), the inequality (3.1) follows.

It is easy to see that the equality sign occurs for any $z \in \mathbb{D}\setminus\{0\}$ in (3.1) if $f = F_{\langle n \rangle, e^{i\theta}, \lambda}$ for some $\theta \in \mathbb{R}$. Conversely, if the equality sign in (3.1) occurs for some $z \in \mathbb{D}\setminus\{0\}$, then the equality sign must hold in (3.2). Then Schwarz

Lemma shows that there exists $\theta \in \mathbb{R}$ such that $\omega_f(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \mathbb{D}$. This implies that $f = F_{\langle n \rangle, e^{i\theta}, \lambda}$. $\qquad\square$

Corollary 3.2. *Let* $\gamma: t \longmapsto z(t)$ $(0 \leq t \leq 1)$ *be a* C^1*-curve in* $\mathbb{D}$ *with* $z(0) = 0$ *and* $z(1) = z_0$. *Then we have*

$$V^{\langle n \rangle}(z_0, \alpha, \lambda) \subset \overline{\mathbb{D}}(C(\lambda, \gamma), R(\lambda, \gamma)) \triangleq \{w \in \mathbb{C} : |w - C(\lambda, \gamma)| \leq R(\lambda, \gamma)\},$$

where $C(\lambda, \gamma) = \int_0^1 c(z(t), \lambda)z'(t)dt$ *and* $R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda)|z'(t)|dt$.

Proof. For any $f \in K^{\langle n \rangle}(\alpha, \lambda)$, it follows from Proposition 3.1 that

$$
\begin{aligned}
|\log f^{(n)}(z_0) - C(\lambda, \gamma)| &= \left| \int_0^1 \left\{ \frac{f^{(n+1)}(z(t))}{f^{(n)}(z(t))} - c(z(t), \lambda) \right\} z'(t)dt \right| \\
&\leq \int_0^1 \left| \frac{f^{(n+1)}(z(t))}{f^{(n)}(z(t))} - c(z(t), \lambda) \right| |z'(t)|dt \\
&\leq \int_0^1 r(z(t), \lambda)|z'(t)|dt = R(\lambda, \gamma).
\end{aligned}
$$

The proof is complete. $\qquad\square$

We recall the following result from [10], which is useful for the proof of Proposition 3.4.

Lemma 3.3. *For* $\theta \in \mathbb{R}$ *and* $\lambda \in [0, 1]$, *the function*

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta}{\{1 + \lambda(e^{i\theta} - 1)\zeta - e^{i\theta}\zeta^2\}^2} d\zeta \quad (z \in \mathbb{D})$$

has a double zero at the origin and no zeros elsewhere in $\mathbb{D}$.

Furthermore, there exists a starlike univalent function G_0 *in* $\mathbb{D}$ *such that* $G = 2^{-1}e^{i\theta}G_0^2$ *and* $G_0(0) = G_0'(0) - 1 = 0$.

Proposition 3.4. *Let* $z_0 \in \mathbb{D}\backslash\{0\}$. *Then for any* $\theta \in (-\pi, \pi]$, *we have* $\log F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z_0) \in \partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$.

Furthermore, if $\log f^{(n)}(z_0) = \log F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z_0)$ *for some* $f \in K^{\langle n \rangle}(\alpha, \lambda)$ *and* $\theta \in (-\pi, \pi]$, *then* $f = F_{\langle n \rangle, e^{i\theta}, \lambda}$.

Proof. From (2.2), we have

$$
\begin{aligned}
\frac{F_{\langle n \rangle, a, \lambda}^{(n+1)}(z)}{F_{\langle n \rangle, a, \lambda}^{(n)}(z)} &= \frac{2(1-\alpha)\delta(az, \lambda)}{1 - \delta(az, \lambda)} \\
&= \frac{2(1-\alpha)(\lambda + az)}{1 + \lambda(a-1)z - az^2}.
\end{aligned}
$$

It follows from (3.4) that

$$\frac{F_{\langle n \rangle, a, \lambda}^{(n+1)}(z)}{F_{\langle n \rangle, a, \lambda}^{(n)}(z)} - A(z, \lambda) = \frac{(1 - \lambda^2)az2(1-\alpha)}{(1 - z\lambda)(1 + \lambda(a-1)z - az^2)}$$

and

$$\frac{F^{(n+1)}_{\langle n \rangle, a, \lambda}(z)}{F^{(n)}_{\langle n \rangle, a, \lambda}(z)} + B(z, \lambda) = \frac{2(1 - \lambda^2)(1 - \alpha)}{(z - \lambda)(1 + \lambda(a - 1)z - az^2)}.$$

Hence we obtain that

$$\frac{F^{(n+1)}_{\langle n \rangle, a, \lambda}(z)}{F^{(n)}_{\langle n \rangle, a, \lambda}(z)} - c(z, \lambda) = \frac{F^{(n+1)}_{\langle n \rangle, a, \lambda}(z)}{F^{(n)}_{\langle n \rangle, a, \lambda}(z)} - \frac{A(z, \lambda) + |z|^2 |T(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2 |T(z, \lambda)|^2}$$

$$= \frac{1}{1 - |z|^2 |T(z, \lambda)|^2} \left\{ \left(\frac{F^{(n+1)}_{\langle n \rangle, a, \lambda}(z)}{F^{(n)}_{\langle n \rangle, a, \lambda}(z)} - A(z, \lambda) \right) \right.$$

$$\left. - |z|^2 |T(z, \lambda)|^2 \left(\frac{F^{(n+1)}_{\langle n \rangle, a, \lambda}(z)}{F^{(n)}_{\langle n \rangle, a, \lambda}(z)} + B(z, \lambda) \right) \right\}$$

$$= \frac{2(1 - \alpha)(1 - \lambda^2)\{az(1 - \lambda \bar{z}) - |z|^2(\bar{z} - \lambda)\}}{(1 - |z|^2)\{1 - 2\lambda \mathrm{Re}(z) + |z|^2\}\{1 + \lambda(a - 1)z - az^2\}}.$$

Substituting a by $e^{i\theta}$ in the above equalities we see that

$$\frac{F^{(n+1)}_{\langle n \rangle, e^{i\theta}, \lambda}(z)}{F^{(n)}_{\langle n \rangle, e^{i\theta}, \lambda}(z)} - c(z, \lambda)$$

$$= r(z, \lambda) \frac{|1 + \lambda(e^{i\theta} - 1)z - e^{i\theta} z^2|^2}{|z|} \frac{e^{i\theta} z}{[1 + \lambda(e^{i\theta} - 1)z - e^{i\theta} z^2]^2}.$$

It follows from Lemma 3.3 that

$$(3.6) \qquad \frac{F^{(n+1)}_{\langle n \rangle, e^{i\theta}, \lambda}(z)}{F^{(n)}_{\langle n \rangle, e^{i\theta}, \lambda}(z)} - c(z, \lambda) = r(z, \lambda) \frac{G'(z)}{|G'(z)|}.$$

Since the function G_0 is starlike, we see that for any $z_0 \in \mathbb{D} \backslash \{0\}$, the linear segment joining 0 and $G_0(z_0)$ entirely lies in $G_0(\mathbb{D})$. Now we define γ_0 as follows.

$$(3.7) \qquad \gamma_0 : \ t \longmapsto z(t) = G_0^{-1}(tG_0(z_0)) \ (0 \leq t \leq 1).$$

Since

$$G(z(t)) = 2^{-1} e^{i\theta} G_0(z(t))^2 = 2^{-1} e^{i\theta} (tG_0(z_0))^2 = t^2 G(z_0),$$

we have

$$(3.8) \qquad G'(z(t))z'(t) = 2tG(z_0) \ (t \in [0, 1]).$$

By (3.6) and (3.8), we have that

(3.9)

$$
\log F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z_0) - C(\lambda, \gamma_0) = \int_0^1 \left\{ \frac{F^{(n+1)}_{\langle n\rangle, e^{i\theta}, \lambda}(z(t))}{F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z(t))} - c(z(t), \lambda) \right\} z'(t) dt
$$

$$
= \int_0^1 r(z(t), \lambda) \frac{G'(z(t)) z'(t)}{|G'(z(t)) z'(t)|} |z'(t)| dt
$$

$$
= \frac{G(z_0)}{|G(z_0)|} \int_0^1 r(z(t), \lambda) |z'(t)| dt
$$

$$
= \frac{G(z_0)}{|G(z_0)|} R(\lambda, \gamma_0).
$$

It yields that

$$
\log F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z_0) \in \partial \overline{\mathbb{D}}(C(\lambda, \gamma_0), R(\lambda, \gamma_0)).
$$

Also from Corollary 3.2, we know that

$$
\log F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z_0) \in V^{\langle n\rangle}(z_0, \alpha, \lambda) \subset \overline{\mathbb{D}}(C(\lambda, \gamma_0), R(\lambda, \gamma_0)).
$$

Hence we can conclude that

$$
\log F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z_0) \in \partial V^{\langle n\rangle}(z_0, \alpha, \lambda).
$$

Assume that there is some $f \in K^{\langle n\rangle}(\alpha, \lambda)$ and $\theta \in (-\pi, \pi]$ such that

(3.10)
$$
\log f^{(n)}(z_0) = \log F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z_0).
$$

Let

$$
h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|} \left\{ \frac{f^{(n+1)}(z(t))}{f^{(n)}(z(t))} - c(z(t), \lambda) \right\} z'(t),
$$

where $z(t) \in \gamma_0$ which is given by (3.7). Then $h(t)$ is a continuous function in $[0, 1]$ and (3.6) yields that

$$
|h(t)| \leq r(z(t), \lambda) |z'(t)|.
$$

Furthermore, (3.9) and (3.10) yield that

$$
\int_0^1 \mathrm{Re}(h(t)) dt = \int_0^1 \mathrm{Re} \left\{ \frac{\overline{G(z_0)}}{|G(z_0)|} \left[\frac{f^{(n+1)}(z(t))}{f^{(n)}(z(t))} - c(z(t), \lambda) \right] z'(t) \right\} dt
$$

$$
= \mathrm{Re} \left\{ \frac{\overline{G(z_0)}}{|G(z_0)|} \{ \log f^{(n)}(z_0) - C(\lambda, \gamma_0) \} \right\}
$$

$$
= \mathrm{Re} \left\{ \frac{\overline{G(z_0)}}{|G(z_0)|} \{ \log F^{(n)}_{\langle n\rangle, e^{i\theta}, \lambda}(z_0) - C(\lambda, \gamma_0) \} \right\}
$$

$$
= \int_0^1 r(z(t), \lambda) |z'(t)| dt.
$$

Thus we have $h(t) = r(z(t), \lambda)|z'(t)|$ for all $t \in [0, 1]$. It follows from (3.6) and (3.8) that for any $z \in \gamma_0$,

$$\frac{f^{(n+1)}(z)}{f^{(n)}(z)} = \frac{F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n+1)}(z)}{F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z)}.$$

Hence for any $z \in \mathbb{D}$,

$$\frac{f^{(n+1)}(z)}{f^{(n)}(z)} = \frac{F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n+1)}(z)}{F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z)}.$$

This implies that $f = F_{\langle n \rangle, e^{i\theta}, \lambda}$ in $\mathbb{D}$. $\square$

Proof of Theorem 2.2. At first, we prove that the closed curve

$$(-\pi, \pi] \ni \theta \mapsto \log F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z_0)$$

is simple.

Suppose not. Then there are $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$ such that

$$\log F_{\langle n \rangle, e^{i\theta_1}, \lambda}^{(n)}(z_0) = \log F_{\langle n \rangle, e^{i\theta_2}, \lambda}^{(n)}(z_0).$$

By Proposition 3.4, we have $\log F_{\langle n \rangle, e^{i\theta_1}, \lambda}^{(n)} = \log F_{\langle n \rangle, e^{i\theta_2}, \lambda}^{(n)}$. It follows from (2.3) that $\theta_1 = \theta_2$. This contradiction shows that the curve must be simple.

Since $V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, we see that the boundary $\partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is a simple closed curve. It follows from Proposition 3.4 that the curve $(-\pi, \pi] \ni \theta \mapsto \log F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z_0)$ is a subcurve of $\partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$.

The fact that a simple closed curve cannot contain any simple closed curve other than itself yields that $\partial V^{\langle n \rangle}(z_0, \alpha, \lambda)$ is given by

$$(-\pi, \pi] \ni \theta \mapsto \log F_{\langle n \rangle, e^{i\theta}, \lambda}^{(n)}(z_0).   \square$$

The proof of Theorem 2.8. From (1.2), we know that $V^{\langle n \rangle}(z_0, \alpha)$ is a bounded convex domain and has smooth boundary with positive curvature. Hence, Theorem 2.8 follows from Theorem 3 in [1]. $\square$

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SHAOLIN CHEN
DEPARTMENT OF MATHEMATICS
HUNAN NORMAL UNIVERSITY
CHANGSHA, HUNAN 410081, P. R. CHINA
E-mail address: `shlchen1982@yahoo.com.cn`

AIWU HUANG
DEPARTMENT OF MATHEMATICS
HUNAN UNIVERSITY OF CHINESE MEDICINE
CHANGSHA, HUNAN 410081, P. R. CHINA
E-mail address: `hezuoyun@sina.com`