Commun. Korean Math. Soc. **25** (2010), No. 4, pp. 547–556
DOI 10.4134/CKMS.2010.25.4.547

REPRESENTATION OF BOUNDED LINEAR OPERATORS WITH EQUAL SPECTRAL PROJECTIONS AT ZERO

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Abstract. In this paper, we present the reprentation of all operators B which are Drazin invertible and sharing the spectral projections at 0 with a given Drazin invertible operator A. Meanwhile, some related results for EP operators with closed range are obtained.

1. Introduction

As we see in [2], the Drazin inverse has proved helpful in analyzing Markov chains, difference equations and iterative procedures. Applications could then be made to denumerable Markov chains abstract Cauchy problems, infinite systems of linear differential equations, and possibly differential equations.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces. Denote by $B(\mathcal{H},\mathcal{K})$ the Banach space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. For an operator $A \in B(\mathcal{H},\mathcal{K})$, the symbols $N(A)$ and $R(A)$ will denote the null space and the range space of A, respectively. Let $A \in B(\mathcal{H})(= B(\mathcal{H},\mathcal{H}))$. The spectrum of A is denoted by $\sigma(A)$. If there exists an operator $Y \in B(\mathcal{H})$ satisfied the following relations

$$AY = YA, YAY = Y, A^{k+1}Y = A^k, \tag{1.1}$$

then Y is called the Drazin inverse of A (see [1]) and denoted by A^D. Recall that $asc(A)(des(A))$, the ascent (descent) of $A \in B(\mathcal{H})$, is the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$ $(R(A^n) = R(A^{n+1}))$. If no such n exists, then $asc(A) = \infty$ $(des(A) = \infty)$. It is well known that $asc(A) = des(A)$, if $asc(A)$ and $des(A)$ are finite. An operator $A \in B(\mathcal{H})$ has its Drazin inverse A^D if and only if it has finite ascent and descent. In such case $i(A) = asc(A) = des(A) = n$. The spectral projection A^π of A corresponding to 0 is the uniquely determined idempotent operator with

$$R(A^\pi) = N(A^k), N(A^\pi) = R(A^k). \tag{1.2}$$

Received August 25, 2008.
2000 *Mathematics Subject Classification.* 47A05, 15A09.
Key words and phrases. Drazin inverse, spectral projection, equation of operator.
This work was completed with the support of the National Natural Science Foundation of Anhui (2006KJ256B).

It is well-known that (see [4]) $A^\pi = I - AA^D$.

Recently, the characterization of all matrices B with eigenprojections at zero or all operators B with the spectral projection at zero such that $B^\pi = A^\pi$ have been considered by many authors (see [5, 6, 7]). For given Drazin invertible operator $A \in B(\mathcal{H})$, the main purpose of this note is to determine the representation of $B \in B(\mathcal{H})$ with $B^\pi = A^\pi$.

In this note, using the technique of block operator matrices and solving operator equations, expression of the set $(A^\pi)^{-1} = \{B \in B(\mathcal{H}) : A^\pi = B^\pi\}$ consisting of all such operators B is established and the characterization of all matrices with same eigenprojections at zero obtained by N. Castro etc in [6] are extended through to operators in $B(\mathcal{H})$. It is worthy to point out that the idea and methods used in this note are different from [6].

We start with several preliminary results that will be used later on.

If $A \in B(\mathcal{H})$ has the Drazin inverse A^D and $i(A) = k$, then (see [3]) $R(A^k)$ is a closed invariant subspace of A. Therefore A has the following operator matrix

$$A = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \tag{1.3}$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, where $\oplus$ represents the orthogonal direct sum.

Lemma 1.1 (see [3])**.** *Let $A \in B(\mathcal{H})$ have the operator matrix form* (1) *and $i(A) = k$. If A has the Drazin inverse A^D, then*

$$A^D = \begin{pmatrix} T_{11}^{-1} & \Sigma_{i=0}^{k-1} T_{11}^{i-k-1} T_{12} T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix} \tag{1.4}$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, where $\oplus$ represents the orthogonal direct sum.

The following well-known lemma indicates that the Drazin inverse of an operator is similarly invariant.

Lemma 1.2. *Let $T \in B(\mathcal{H})$ with its Drazin inverse T^D. If $S \in B(\mathcal{H})$ is invertible, then $(STS^{-1})^D = ST^DS^{-1}$.*

2. Representation of the set $(A^\pi)^{-1}$

Now we state the main result of this section. We shall characterize the set

$$(A^\pi)^{-1} = \{B \in B(\mathcal{H}) : A^\pi = B^\pi\},$$

where $A \in B(\mathcal{H})$ is a given operator and $i(A) = k$.

Theorem 2.1. *Let $A \in B(\mathcal{H})$ be a given operator and have the operator matrix form* (3) *with $i(A) = k$. If $B \in B(\mathcal{H})$ such that $A^\pi = B^\pi$, then*

$$B = \begin{pmatrix} B_{11} & B_{11}D - DB_{22} \\ 0 & B_{22} \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, *where* $B_{11} \in B(R(A^k))$ *is invertible,* $B_{22} \in B(R((A^k)^\perp)$ *is nilpotent and*

$$D = \sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i}.$$

Proof. If A is invertible, then $A^D = A^{-1}$. The result holds. So we can assume that A is not invertible below.

If A has the Drazin inverse A^D and $i(A) = k$, by Lemma 1, then A^π has the operator matrix form

$$A^\pi = I - AA^D = \begin{pmatrix} 0 & -D \\ 0 & I \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, where $D = \sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i}$. Put

$$S = \begin{pmatrix} I & D \\ 0 & I \end{pmatrix},$$

thus S is invertible and its inverse

$$S^{-1} = \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix}.$$

By straight calculation, we have

$$SA^\pi S^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Suppose that

$$SBS^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Since $A^\pi = B^\pi$, we have $BA^\pi = A^\pi B$ and so $SBS^{-1}SA^\pi S^{-1} = SA^\pi S^{-1}SBS^{-1}$. Thus we have

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Comparing the two sides of the above equation, we have $B_{12} = 0$ and $B_{21} = 0$. Hence SBS^{-1} has the form $SBS^{-1} = B_{11} \oplus B_{22}$, where $B_{11} \in B(R(A^k))$ and $B_{22} \in B(R(A^k)^\perp)$. Suppose that

$$SB^D S^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Since $SB^D S^{-1} SBB^D S^{-1} = SB^D S^{-1}$ and $A^\pi = B^\pi$, we have

$$SB^D S^{-1} SBB^D S^{-1} = SB^D S^{-1} S(I - A^\pi) S^{-1} = SB^D S^{-1}.$$

Thus

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Comparing the two sides of the above equation, we have $X_{12} = 0$ and $X_{22} = 0$. Since $SB^DS^{-1}SBS^{-1} = SBS^{-1}SB^DS^{-1} = S(I - A^\pi)S^{-1}$, we have

$$\begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix}\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}\begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus we have

$$\begin{cases} X_{11}B_{11} = B_{11}X_{11} = I, \\ X_{21}B_{11} = 0. \end{cases}$$

Observing the above equation implies that B_{11} and X_{11} are invertible and so $X_{21} = 0$. By Lemma 2, we have $(SBS^{-1})^D = SB^DS^{-1}$. Thus $B_{22}^D = 0$ and so B_{22} is nilpotent. Therefore $SBS^{-1} = B_{11} \oplus B_{22}$, where B_{11} is invertible and B_{22} is nilpotent.

By straight calculation, we have

$$B = \begin{pmatrix} B_{11} & B_{11}D - DB_{22} \\ 0 & B_{22} \end{pmatrix}$$

and

$$B^D = \begin{pmatrix} B_{11}^{-1} & B_{11}^{-1}D \\ 0 & 0 \end{pmatrix},$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, where $B_{11} \in B(R(T^k))$ is invertible, $B_{22} \in B(R((T^k)^\perp)$ is nilpotent and

$$D = \sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i}.$$

□

From the proof of the proceeding theorem, we obtain the following useful property immediately.

Corollary 2.2. *Let $A, B \in B(\mathcal{H})$ and $A^\pi = B^\pi$. Thus there exists $S \in B(\mathcal{H})$ is invertible, such that*

$$SAS^{-1} = A_{11} \oplus A_{22}, SBS^{-1} = B_{11} \oplus B_{22},$$

where A_{11} and B_{11} are invertible, A_{22} and B_{22} are nilpotent.

Theorem 2.3. *Let $A \in B(\mathcal{H})$. The following conditions on $B \in B(\mathcal{H})$ are equivalent*:

(1) $A^\pi = B^\pi$;

(2) $A^\pi B = BA^\pi$, *BA^π is nilpotent and $B + A^\pi$ is invertible*;

(3) $I + A^D(B - A)$ *is invertible, $A^\pi B = BA^\pi$ and BA^π is nilpotent.*

Proof. (1) ⇒ (2) By Theorem 3, it is trivial.

(2) ⇒ (3) By Lemma 1, put $S = \left(\begin{smallmatrix} I & -D \\ 0 & I \end{smallmatrix}\right)$, thus

$$A^D = S^{-1}(T_{11}^{-1} \oplus 0)S, \ A^\pi = S^{-1}(0 \oplus I)S.$$

We observe that $A^\pi B = BA^\pi$ implies

$$B = S^{-1}(B_{11} \oplus B_{22})S.$$

Since BA^π is nilpotent and $BA^\pi = S^{-1}(0 \oplus B_{22})S$, we have B_{22} is nilpotent.

Since both

$$B + A^\pi = S^{-1}(B_{11} \oplus (B_{22} + I))S$$

and $B_{22} + I$ are invertible, we have B_{11} is invertible.

By straight calculation,

$$\begin{array}{rcl} I + A^D(B-A) & = & A^D B + A^\pi \\ & = & S^{-1}(T_{11}^{-1} \oplus 0)SS^{-1}(B_{11} \oplus B_{22})S + S^{-1}(0 \oplus I)S \\ & = & S^{-1}(T_{11}^{-1}B_{11} \oplus I)S. \end{array}$$

Hence $I + A^D(B-A)$ is invertible.

(3) $\Rightarrow$ (1) It is verified by a calculation. $\square$

The following corollary indicates the connections between A^D and B^D when $A^\pi = B^\pi$.

Corollary 2.4. *Let $A, B \in B(\mathcal{H})$ and $A^\pi = B^\pi$. Thus*

$$B^D = (I + A^D(B-A))^{-1}A^D$$

and

$$B^D - A^D = A^D(A-B)B^D.$$

Proof. From the proof of the Theorem 2.3, it is trivial by straight calculation. $\square$

Theorem 2.5. *Let $A, B \in B(\mathcal{H})$ and $A^\pi = B^\pi$. Thus*

$$\frac{\|A^D\|}{1 + \|A^D(B-A)\|} \le \|B^D\|.$$

If $\|A^D(B-A)\| < 1$, then

$$\|B^D\| \le \frac{\|A^D\|}{1 - \|A^D(B-A)\|}$$

and

$$\frac{\|B^D - A^D\|}{\|A^D\|} \le \frac{\|A^D(B-A)\|}{1 - \|A^D(B-A)\|}.$$

Proof. By Corollary 2.4, we have $A^D = B^D + A^D(B-A)B^D$. Apply the norm to $A^D = B^D + A^D(B-A)B^D$ to obtain

$$\begin{array}{rcl} \| A^D \| & = & \| B^D + A^D(B-A)B^D \| \\ & \le & \| B^D \| + \| A^D(B-A)B^D \| \\ & \le & \| B^D \| + \| A^D(B-A) \| \| B^D \| \end{array}$$

and so

$$\frac{\| A^D \|}{1 + \| A^D(B-A) \|} \le \| B^D \| .$$

Moreover, suppose $\|A^D(B-A)\| < 1$. Apply the norm to $B^D = A^D + A^D(A-B)B^D$ to obtain $\|B^D\| \le \|A^D\| + \|A^D(A-B)\|\|B^D\|$ and consequently

$$\|B^D\| \le \frac{\|A^D\|}{1-\|A^D(B-A)\|}.$$

Finally apply the norm to $B^D - A^D = A^D(A-B)B^D$ to obtain

$$\begin{aligned} \| B^D - A^D \| &\le \| A^D(B-A) \| (\| A^D \| + \| A^D(A-B) \| B^D \|) \\ &\le \| A^D(B-A) \| (\| A^D \| + \| A^D(A-B) \| \tfrac{\|A^D\|}{1-\|A^D(B-A)\|}) \\ &\le \| A^D \| \| A^D(B-A) \| (1 + \tfrac{\|A^D(A-B)\|}{1-\|A^D(B-A)\|}) \\ &\le \| A^D \| \| \tfrac{\|A^D(B-A)}{1-\|A^D(B-A)\|}) \end{aligned}$$

and then the inequality holds. □

Remark. If $\|A^D\|\|(B-A)\| < 1$ in the preceding theorem, then the inequality becomes

$$\frac{\|B^D - A^D\|}{A^D} \le \frac{\|A^D(B-A)\|}{1-\|A^D(B-A)\|} \le \frac{\kappa_D(A)\triangle}{1-\kappa_D(A)\triangle}$$

where $\kappa_D(A) = \|A^D\|\|A\|$ is the Drazin condition number of A and where $\triangle = \|B-A\|/\|A\|$.

3. Some results on EP operators

As we know, every matrix $A \in C_{n\times n}$ has its Drazin inverse, but it is not true for $A \in B(\mathcal{H})$, in general. The following lemma indicates when an EP operator is Drazin invertible. Recall that an operator $A \in B(\mathcal{H})$ is called an EP operator if $R(T) = R(T^*)$.

Lemma 3.1. *Let $A \in B(\mathcal{H})$ be an EP operator. Thus A is Drazin invertible if and only if $R(A)$ is closed.*

Proof. Sufficiency. Suppose A is Drazin invertible. Since A is an EP operator, $N(A^*) = N(A)$. Therefore A has the matrix form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \overline{R(A)} \oplus N(A) \to \overline{R(A)} \oplus N(A),$$

where $\overline{R(A)}$ denote the closure of $R(A)$. Thus A_1 is injective. Since A is Drazin invertible, A_1 is Drazin invertible and so A_1 is invertible. Hence $R(A)$ is closed.

Necessity. Suppose that $R(A)$ is closed. If A is an EP operator, then $R(A^*) = R(A)$ is closed. Therefore A has the matrix form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : R(A^*) \oplus N(A) \to R(A) \oplus N(A^*),$$

where A_1 is invertible. Therefore A is Drazin invertible with its Drazin inverse $A^D = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. □

Remark. It is well-known that $A \in B(\mathcal{H})$ is Moore-Penrose invertible if and only if $R(A)$ is closed. Recall that If there exists an operator $X \in B(\mathcal{H})$ satisfied the following the relations

$$AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA,$$

then X is called a Moore-Penrose inverse of $A \in B(\mathcal{H})$ and denoted by $A^\dagger$. It is well-known that $A^\dagger$ is unique. One of interesting result is that EP matrices commute with their Moore-inverses [6]. From the proceeding proof, we know that Drazin and Moore-Penrose inverses coincide for all closed range EP operators. Therefore all closed range EP operators commute with their Moore-inverses.

The following result which gives a condition ensuring that a perturbation of a closed range EP operator is again EP, and establishes error bounds for the Drazin inverse of the perturbation.

Theorem 3.2. *Let $A \in B(\mathcal{H})$ be a closed range EP operator and let $B = A+E$, where $E \in B(\mathcal{H})$ satisfies condition $(W) : AA^DEAA^D = E$ and $\| A^DE \| < 1$. Then B is also a closed range EP operator and*

$$\frac{\| B^D - A^D \|}{\| A^D \|} \leq \frac{\| A^DE \|}{1 - \| A^DE \|}.$$

Proof. Since $A \in B(\mathcal{H})$ be a closed range EP operator, A has the matrix form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : R(A) \oplus N(A) \to R(A) \oplus N(A),$$

where A_1 is invertible.

Suppose $E = \left(\begin{smallmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{smallmatrix}\right)$. Since $AA^DEAA^D = E$,

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}.$$

Comparing the two sides of the above equation, we have $E_{12} = E_{21} = E_{22} = 0$.

Since $\|A^DE\| < 1$, we have

$$\|A_1^{-1}E_{11}\| < 1.$$

Thus $I + A_1^{-1}E_{11} = A_1^{-1}(A_1 + E_{11})$ is invertible. Hence $A_1 + E_{11}$ is invertible. By straight calculation, we obtain that B has the matrix form

$$B = A + E = \begin{pmatrix} A_1 + E_{11} & 0 \\ 0 & 0 \end{pmatrix} : R(A) \oplus N(A) \to R(A) \oplus N(A),$$

where $A_1 + E_{11}$ is invertible. Thus B is a closed EP operator. By Theorem 2.3, it is clear that $A^\pi = B^\pi$. Since Theorem 2.5, the inequality is trivial. □

Remark. Recall that $P \in B(\mathcal{H})$ is called an orthogonal projection if $P = P^* = P^2$. If $A \in B(\mathcal{H})$ is Drazin invertible, then $A^\pi = I - AA^D$ is an idempotent, in general, not an orthogonal projection. In this case, by Lemma 1.1, we have $A^\pi = \left(\begin{smallmatrix} I & -D \\ 0 & 0 \end{smallmatrix}\right)$, where $D = \sum_{i=0}^{k-1} T_{11}^{i-k}T_{12}T_{22}^{k-1-i}$. But, from the proceeding proof, it is clear that A^π is an orthogonal projection when A is a closed range EP operator.

Here is a new characterization of closed range EP operators.

Theorem 3.3. *Let $A \in B(\mathcal{H})$ be a closed range operator. Then the following conditions are equivalent*:
(1) *A is EP*;
(2) *$AA^\pi = 0$ and $(A^\pi)^* = A^\pi$*;
(3) *$A^\pi A^* = A^* A^\pi = 0$.*

Proof. (1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) Since $AA^\pi = 0$ and $AA^\pi = A^\pi A$, $A = AA^D A = A^2 A^D$, $R(A) \subseteq R(A^2)$ from Douglas's Range Inclusion Theorem. Since $R(A^2) \subseteq R(A)$, $R(A) = R(A^2)$. Hence $i(A) = des(A) \leq 1$.

Case 1. $i(A) = 0$. Thus A is invertible and $A^D = A^{-1}$. It is trivial.

Case 2. $i(A) = 1$. By straight calculation, then A and A^D have the following operator matrices, respectively

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix},$$

$$A^D = \begin{pmatrix} A_{11}^{-1} & A_{11}^{-2} A_{12} \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = R(A) \oplus R(A)^\perp$.

Since $(A^\pi)^* = A^\pi$, $(A^D A)^* = A^D A$, by straight calculation, we have $R(AA^D) = R(A)$ and so $A_{12} = 0$. Therefore $A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$.

In this case, (3) holds trivially.

(3) $\Rightarrow$ (1) Let A have the Drazin inverse A^D and $i(A) = k$, by Lemma 1.1, then A and A^π have the operator matrix forms, respectively

$$A = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

$$A^\pi = I - AA^D = \begin{pmatrix} 0 & -D \\ 0 & I \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, where $D = \sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i}$.

Since $A^\pi A^* = A^* A^\pi = 0$, $AA^D A^* = A^* AA^D$.

Notice the following: $AA^D A = AAA^D$ and $R(AA^D) = R(A^k)$. It follows that $R(A^k)$ is invariant for both A and A^* and so $T_{12} = 0$. Therefore

$$A = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix},$$

$$A^\pi = I - AA^D = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Since $A^\pi A^* = A^* A^\pi = 0$, we have $T_{22} = 0$ and so $A = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Hence A is EP.

(1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) Since $AA^\pi = 0$ and $AA^\pi = A^\pi A$, $A = AA^D A = A^2 A^D$, $R(A) \subseteq R(A^2)$ from Douglas's Range Inclusion Theorem. Since $R(A^2) \subseteq R(A)$, $R(A) = R(A^2)$. Hence $i(A) = des(A) \leq 1$.

Case 1. $i(A) = 0$. Thus A is invertible and $A^D = A^{-1}$. It is trivial.

Case 2. $i(A) = 1$. By straight calculation, then A and A^D have the following operator matrices, respectively

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix},$$

$$A^D = \begin{pmatrix} A_{11}^{-1} & A_{11}^{-2} A_{12} \\ 0 & 0 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = R(A) \oplus R(A)^\perp$.

Since $(A^\pi)^* = A^\pi$, $(A^D A)^* = A^D A$, by straight calculation, we have $R(AA^D) = R(A)$ and so $A_{12} = 0$. Therefore $A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$.

In this case, (3) holds trivially.

(3) $\Rightarrow$ (1) Let A have the Drazin inverse A^D and $i(A) = k$, by Lemma 1.1, then A and A^π have the operator matrix forms, respectively

$$A = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

$$A^\pi = I - AA^D = \begin{pmatrix} 0 & -D \\ 0 & I \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = R(A^k) \oplus R(A^k)^\perp$, where $D = \sum_{i=0}^{k-1} T_{11}^{i-k} T_{12} T_{22}^{k-1-i}$.

Since $A^\pi A^* = A^* A^\pi = 0$, $AA^D A^* = A^* AA^D$.

Notice the following: $AA^D A = AAA^D$ and $R(AA^D) = R(A^k)$. It follows that $R(A^k)$ is invariant for both A and A^* and so $T_{12} = 0$. Therefore

$$A = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix},$$

$$A^\pi = I - AA^D = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Since $A^\pi A^* = A^* A^\pi = 0$, we have $T_{22} = 0$ and so $A = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Hence A is EP. $\square$

Remark. In [6], they give a characterization of EP matrices. Recall that $A \in C_{n \times n}$ is EP, which can be rewritten in the different equivalent forms (see Theorem 5.2 in [6]). We must point out that the conditions: $I + A^H - A^D A$ is invertible in (iii) and $I + A^D A^H - A^D A$ is invertible in (iv), respectively, can be deleted. In fact, from the proceeding proof, we can easily check that both $I + A^H - A^D A$ and $I + A^D A^H - A^D A$ are invertible if $A^D AA^H = A^H = A^H A^D A$.

Acknowledgment. The authors would like to their sincere thanks to Professor Guoxing Ji for his helpful suggestions. Also, the authors would like to express their sincere thanks to Professor N. Castro González, who located for us reference [4]. At last, the authors wish to express their thanks to the referee for several helpful comments concerning this paper.

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