Commun. Korean Math. Soc. **25** (2010), No. 4, pp. 537–545
DOI 10.4134/CKMS.2010.25.4.537

HYPER MV-DEDUCTIVE SYSTEMS OF HYPER MV-ALGEBRAS

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Abstract. The notions of (weak) hyper MV-deductive systems and (weak) implicative hyper MV-deductive systems are introduced, and several properties are investigated. Relations among hyper MV-deductive systems, weak hyper MV-deductive systems, implicative hyper MV-deductive systems and weak implicative hyper MV-deductive systems are discussed. A characterization of a hyper MV-deductive system is provided. A condition for a weak hyper MV-deductive system to be a weak implicative hyper MV-deductive system is given.

1. Introduction

In this paper, we introduce the notions of (weak) hyper MV-deductive systems and (weak) implicative hyper MV-deductive systems, and we investigates several properties. We discuss relations among hyper MV-deductive systems, weak hyper MV-deductive systems, implicative hyper MV-deductive systems and weak implicative hyper MV-deductive systems. We give a characterization of a hyper MV-deductive system. We give a condition for a weak hyper MV-deductive system to be a weak implicative hyper MV-deductive system.

2. Preliminaries

Definition 2.1 ([1])**.** A *hyper MV-algebra* is a non-empty set M endowed with a hyper operation "$\oplus$", a unary operation "$*$" and a constant "0" satisfying the following axioms:

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(a2) $x \oplus y = y \oplus x$,
(a3) $(x^*)^* = x$,
(a4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(a5) $0^* \in x \oplus 0^*$,
(a6) $0^* \in x \oplus x^*$,

Received December 15, 2009.
2000 *Mathematics Subject Classification.* 06D35, 06F35, 03G25.
Key words and phrases. hyper MV-algebra, (weak) hyper MV-deductive system, (weak) implicative hyper MV-deductive system.

(a7) $x \ll y,\ y \ll x \ \Rightarrow\ x = y$

for all $x, y, z \in M$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every subsets A and B of M, we define

$$A \ll B \ \Leftrightarrow\ (\exists\, a \in A)\,(\exists\, b \in B)\,(a \ll b),$$

$$A \oplus B := \bigcup_{a \in A, b \in B} a \oplus b.$$

We also define $0^* := 1$ and $A^* := \{a^* \mid a \in A\}$.

Proposition 2.2 ([1]). *Every hyper MV-algebra M satisfies the following statem-ents*:

(b1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(b2) $0 \ll x, \quad x \ll 1$,
(b3) $x \ll x$,
(b4) $x \ll y \ \Rightarrow\ y^* \ll x^*$,
(b5) $A \ll B \ \Rightarrow\ B^* \ll A^*$,
(b6) $A \ll A$,
(b7) $A \subseteq B \ \Rightarrow\ A \ll B$,
(b8) $x \ll x \oplus y, \quad A \ll A \oplus B$,
(b9) $(A^*)^* = A$,
(b10) $0 \oplus 0 = \{0\}$,
(b11) $x \in x \oplus 0$,
(b12) $y \in x \oplus 0 \ \Rightarrow\ y \ll x$,
(b13) $y \oplus 0 = x \oplus 0 \ \Rightarrow\ x = y$

for all $x, y, z \in M$ and for all subsets A, B and C of M.

3. Hyper MV-deductive systems

In what follows, let M denote a hyper MV-algebra unless otherwise specified.

Definition 3.1. A nonempty subset D of M is called a *weak hyper MV-deductive system* of M if it satisfies:

(d1) $0 \in D$,
(d2) $(\forall x, y \in M)((x^* \oplus y)^* \subseteq D,\ y \in D \ \Rightarrow\ x \in D)$.

Definition 3.2. A non-empty subset D of M is called a *hyper MV-deductive system* of M if it satisfies (d1) and

(d3) $(\forall x, y \in M)((x^* \oplus y)^* \ll D,\ y \in D \ \Rightarrow\ x \in D)$.

Example 3.3. Let $M := \{0, a, b, 1\}$ be a set with a hyper operation "$\oplus$" and a unary operation "$*$" which are given in the following Cayley tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,b\}$	$\{0,a,b,1\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
b	$\{0,a,b\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$
1	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$	$\{0,a,b,1\}$

x	x^*
0	1
a	b
b	a
1	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easily checked that $D_1 = \{0, a\}$, $D_2 = \{0, b\}$ and $D_3 = \{0, a, b\}$ are weak hyper MV-deductive systems of M.

Example 3.4. Consider the hyper MV-algebra M which is described in Example 3.3. Then $D_1 = \{0, a\}$ is a hyper MV-deductive system of M.

Example 3.5. Let $M := \{0, a, b, 1\}$ be a set with a hyper operation "$\oplus$" and a unary operation "$*$" which are given in the following Cayley tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a\}$	$\{a\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

x	x^*
0	1
a	b
b	a
1	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. It is easily checked that $D_1 = \{0, a\}$ is a hyper MV-deductive systems of M. But $D_2 = \{0, b\}$ is not a hyper MV-deductive system of M, since $(a^* \oplus b)^* = \{0, a, b, 1\} \ll D_2$, $b \in D_2$ but $a \notin D_2$.

Example 3.6. Let $M = \{0, a, b, 1\}$ be a set with a hyper operation "$\oplus$" and a unary operation "$*$" which are given in the following Cayley tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a, b\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a, b\}$	$\{0, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b, 1\}$	$\{b\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

x	x^*
0	1
a	b
b	a
1	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra, but $D := \{0, a, 1\}$ is not a weak hyper MV-deductive system of M, since $(b^* \oplus a)^* = (a \oplus a)^* = \{0, 1\} \subseteq D$ and $b \notin D$. We can easily check that $D_1 := \{0, x\}$, $D_2 := \{0, b, x\}$ and M are weak hyper MV-deductive systems of M for all $x \in M$.

Proposition 3.7. *Let M be a hyper MV-algebra in which $0 \in x \oplus y$ for all $x, y \in M$. Let D be a subset of M such that $0 \in D$ and $1 \notin D$. Then D is a weak hyper MV-deductive system of M.*

Proof. For any $x, y \in M$, assume that $x \notin D$ and $y \in D$. Since $0 \in x^* \oplus y$, we have $1 = 0^* \in (x^* \oplus y)^*$, which shows that $(x^* \oplus y)^*$ is not contained in D. Therefore D is a weak hyper MV-deductive system of M. □

Theorem 3.8. *Let M be a hyper MV-algebra satisfying the following condition:*

$$(\forall x, y \in M)\,(x \in x \oplus y) \tag{3.1}$$

Then every subset of M containing 0 is a weak hyper MV-deductive system of M.

Proof. Let D be a subset of M containing 0, and let $x, y \in M$ be such that $(x^* \oplus y)^* \subseteq D$ and $y \in D$. Assume that $x \notin D$. Then $x^* \in x^* \oplus y$ by (3.1), and so $x = (x^*)^* \in (x^* \oplus y)^*$ by (a3). Hence $(x^* \oplus y)^*$ is not contained in D, which is a contradiction. Therefore $x \in D$, proving that D is a weak hyper MV-deductive system of M. □

A hyper MV-algebra satisfying (3.1) can have a subset containing 0 which is not a hyper MV-deductive system of M. See the following example.

Example 3.9. Consider a hyper MV-algebra $M := \{0, a, b, 1\}$ which is described in Example 3.5. It can be shown that M satisfies (3.1). The set $D := \{0, b\}$ is not a hyper MV-deductive system of M, since $(a^* \oplus b)^* = \{0, a, b, 1\} \ll D$ and $a \notin D$.

Using (b7) and (d3), we have the following theorem.

Theorem 3.10. *Every hyper MV-deductive system is a weak hyper MV-deductive system.*

The following example shows that the converse of Theorem 3.10 is not valid.

Example 3.11. Consider a hyper MV-algebra M which is described in Example 3.3. Then $D_3 := \{0, a, b\}$ is a weak hyper MV-deductive system of M (see Example 3.3). We know that

$$(1^* \oplus a)^* = (0 \oplus a)^* = \{0, a\}^* = \{b, 1\} \ll D$$

and $a \in D$, but $1 \notin D$. This proves that D_3 is not a hyper MV-deductive system of M.

Proposition 3.12. *Every hyper MV-deductive system D of M have the following condition*:

$$(\forall x, y \in M)\,(x \ll y,\ y \in D \ \Rightarrow\ x \in D). \tag{3.2}$$

Proof. Let $x, y \in M$ be such that $x \ll y$ and $y \in D$. Then $0 \in (x^* \oplus y)^*$, and so $(x^* \oplus y)^* \ll D$ by (d1) and (b3). It follows from (d3) that $x \in D$. □

Combining (b2) and Proposition 3.12, we have the following corollary.

Corollary 3.13. *Let D be a hyper MV-deductive system of M. If D contains the element* 1, *then $D = M$.*

By Corollary 3.13, we know that any proper subset of M containing 1 cannot be a hyper MV-deductive system of M.

Proposition 3.14. *Let D be a hyper MV-deductive system of M. For every subsets A and B of M, if $B \subseteq D$ and $(A^* \oplus B)^* \ll D$, then $A \ll D$.*

Proof. We have $(A^* \oplus B)^* = \bigcup_{a \in A,\, b \in B} (a^* \oplus b)^* \ll D$, and so there exists $x \in (a^* \oplus b)^*$ for some $a \in A$ and $b \in B$, and there exists $y \in D$ such that $x \ll y$. It follows that $(a^* \oplus y)^* \ll D$. Since $b \in B \subseteq D$, by (d3) we obtain $a \in D$. Therefore $A \ll D$. □

Remark 3.15. (1) In Proposition 3.14, it is not necessary to mention that $A \subseteq D$. To show this, consider a hyper MV-algebra $M = \{0, a, b, 1\}$ which is given in Example 3.5. Let $A := \{0, a, b\}$ and $B := \{0\}$. Note that $D := \{0, a\}$ is a hyper MV-deductive system of M and $(A^* \oplus B)^* = \{0, a, b, 1\} \ll D$, but A is not contained in D.

(2) In Proposition 3.14, if we use $B \ll D$ instead of $B \subseteq D$, then the result does not hold. Consider a hyper MV-algebra $M = \{0, a, b, 1\}$ which is given in Example 3.5. It can be shown that $D := \{0, a\}$ is a hyper MV-deductive system of M. If we take $A := \{b\}$ and $B := \{0, b\}$, then $B \ll D$ and $(A^* \oplus B)^* = \{0, a, b, 1\} \ll D$, but $A \ll D$ is not valid.

Proposition 3.16. *Every weak hyper MV-deductive system D of M satisfies the following condition*:

$$(\forall A, B \subseteq M)\,((A^* \oplus B)^* \subseteq D,\ B \subseteq D \ \Rightarrow\ A \subseteq D) \tag{3.3}$$

Proof. For all $a \in A$ and $b \in B$, we have $(a^* \oplus b)^* \subseteq (A^* \oplus B)^* \subseteq D$ and $b \in B \subseteq D$. It follows from (d2) that $a \in D$. □

Corollary 3.17. *Every weak hyper MV-deductive system D of M satisfies the condition*:

$$(\forall A, B \subseteq M)\,((A^* \oplus B)^* \subseteq D,\ B \subseteq D \ \Rightarrow\ A \ll D) \tag{3.4}$$

Proof. Straightforward. □

Remark 3.18. In Proposition 3.16, the condition $B \subseteq D$ can not be replaced by $B \ll D$. In fact, in Example 3.6, we see that if $A = \{b\}$, $B = \{a\}$ and $D = \{0, 1\}$, then D is a weak hyper MV-deductive system of M and $B \ll D$. Also, $(A^* \oplus B)^* = (b^* \oplus a)^* = \{0, 1\} \subseteq D$, but $A \nsubseteq D$.

We give a characterization of a hyper MV-deductive system.

Theorem 3.19. *Let D be a non-empty subset of M. Then D is a hyper MV-deductive system of M if and only if it satisfies* (d1) *and*

$$(\forall x, y \in M)((x^* \oplus y)^* \cap D \neq \emptyset,\ y \in D \ \Rightarrow\ x \in D). \tag{e1}$$

Proof. Assume that D is a hyper MV-deductive system of M. Let $x, y \in M$ be such that $y \in D$ and $(x^* \oplus y)^* \cap D \neq \emptyset$. Then there exists $a \in (x^* \oplus y)^* \cap D$. Since $a \ll a$ by (b3), we get $(x^* \oplus y)^* \ll D$. It follows from (d3) that $x \in D$.

Conversely, suppose that D satisfies (d1) and (e1). Let $x, y \in M$ be such that $y \in D$ and $(x^* \oplus y)^* \ll D$. Then there exist $a \in (x^* \oplus y)^*$ and $b \in D$ such that $a \ll b$. Hence $0^* \in a^* \oplus b$, and so $0 \in (a^* \oplus b)^*$. Since $0 \in D$, we have $0 \in (a^* \oplus b)^* \cap D$, i.e., $(a^* \oplus b)^* \cap D \neq \emptyset$. It follows from (e1) that $x \in D$. Therefore D is a hyper MV-deductive system of M. □

Proposition 3.20. *Every hyper MV-deductive system D of M satisfies the following condition*:

$$(\forall A \subseteq M)\,(\forall b \in D)\,((A^* \oplus b)^* \cap D \neq \emptyset \ \Rightarrow\ A \cap D \neq \emptyset) \tag{3.5}$$

Proof. Let $A \subseteq M$ and $b \in D$ be such that $(A^* \oplus b)^* \cap D \neq \emptyset$. Then there exists $x \in D$ such that $x \in (a^* \oplus b)^*$ for some $a \in A$. Thus $(a^* \oplus b)^* \cap D \neq \emptyset$. Since D is a hyper MV-deductive system, it follows from Theorem 3.19 that $a \in D$ so that $A \cap D \neq \emptyset$. □

Theorem 3.21. *For any hyper MV-deductive system D of M, the following are equivalent*:

(i) $(\forall x, y \in M)\ ((x^* \oplus y)^* \ll D)$.
(ii) $(\forall x, y \in M)\ ((x^* \oplus y)^* \cap D \neq \emptyset)$.

Proof. (i) ⇒ (ii). Assume that (i) is valid. Then there exist $a \in (x^* \oplus y)^*$ and $b \in D$ such that $a \ll b$, that is, $0^* \in a^* \oplus b$. Thus $0 \in (a^* \oplus b)^* \subseteq ((x^* \oplus y) \oplus b)^*$, and so $((x^* \oplus y) \oplus b)^* \cap D \neq \emptyset$. It follows from (b9) and Proposition 3.20 that $(x^* \oplus y)^* \cap D \neq \emptyset$.

(ii) ⇒ (i). Straightforward. □

4. Implicative hyper MV-deductive systems

Definition 4.1. A non-empty subset D of M is called a *weak implicative hyper MV-deductive system* of M if it satisfies (d1) and

(d4) $$(\forall x, y, z \in M)\big(((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \subseteq D,\ z \in D \ \Rightarrow\ x \in D\big).$$

Example 4.2. Let $M := \{0, a, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given in the following Cayley tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{b\}$	$\{b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{b, 1\}$	$\{b, 1\}$
b	$\{b\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$

x	x^*
0	1
a	b
b	a
1	0

Then $(M, \oplus, ^*, 0)$ is a hyper MV-algebra. For every $x \in M$, the set $\{0, x\}$ is a weak implicative hyper MV-deductive system of M. Also, $D_1 := \{0, b, 1\}$ is a weak implicative hyper MV-deductive system of M. We know that $D := \{0, a, b\}$ is not a weak implicative hyper MV-deductive system of M, since

$$((1^* \oplus b) \oplus (a^* \oplus 1)^*)^* = \{0, a\} \subseteq D$$

and $b \in D$, but $1 \notin D$.

Definition 4.3. A non-empty subset D of M is called an *implicative hyper MV-deductive system* of M if it satisfies (d1) and

(d5) $$(\forall x, y, z \in M)\big(((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \ll D,\ z \in D \ \Rightarrow\ x \in D\big).$$

Example 4.4. In a hyper MV-algebra $M := \{0, a, b, 1\}$ in Example 4.2, $\{0, a\}$ is an implicative hyper MV-deductive system of M. But $\{0, b\}$ is not an implicative hyper MV-deductive system of M, since

$$((1^* \oplus b) \oplus (a^* \oplus 1)^*)^* = \{0, a\} \ll \{0, b\}.$$

Theorem 4.5. *Every implicative hyper MV-deductive system is a weak implicative hyper MV-deductive system.*

Proof. Let D be an implicative hyper MV-deductive system of M and let $x, y, z \in M$ be such that $z \in D$ and $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \subseteq D$. Using (b7), we have $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \ll D$. It follows from (d5) that $x \in D$. Therefore D is a weak implicative hyper MV-deductive system of M. □

The following example shows that the converse of Theorem 4.5 need not be true in general.

Example 4.6. Consider a hyper MV-algebra $M := \{0, a, b, 1\}$ which is given in Example 3.3. Then it can be easily shown that every subset of M containing 0 is a weak implicative hyper MV-deductive system of M, but $D := \{0, a, 1\}$ is not an implicative hyper MV-deductive system of M, since

$$((b^* \oplus 0) \oplus (0^* \oplus b)^*)^* = M \ll \{0, a, 1\}.$$

Theorem 4.7. *Every implicative hyper MV-deductive system is a hyper MV-deductive system.*

Proof. Let D be an implicative hyper MV-deductive system of M and let $x, y \in M$ be such that $y \in D$ and $(x^* \oplus y)^* \ll D$. Then there exist $a \in (x^* \oplus y)^*$ and $z \in D$ such that $a \ll z$. Using (b2), (b9) and (b11), we obtain

$$a^* \in a^* \oplus 0 \subseteq ((x^* \oplus y)^*)^* \oplus (0^* \oplus x)^* = (x^* \oplus y) \oplus (0^* \oplus x)^*,$$

and so $a \in ((x^* \oplus y) \oplus (0^* \oplus x)^*)^*$. Hence $((x^* \oplus y) \oplus (0^* \oplus x)^*)^* \ll D$ which implies from (d5) that $x \in D$. Therefore D is a hyper MV-deductive system of M. □

The following example shows that the converse of Theorem 4.7 need not be true in general.

Example 4.8. Consider a hyper MV-algebra $M := \{0, a, b, 1\}$ which is given in Example 3.3. Then $D := \{0, a\}$ is a hyper MV-deductive system of M. Since

$$((1^* \oplus 0) \oplus (0^* \oplus 1)^*)^* = M \ll \{0, a\},$$

$D = \{0, a\}$ is not an implicative hyper MV-deductive system of M.

The following example shows that a weak implicative hyper MV-deductive system may not be a weak hyper MV-deductive system.

Example 4.9. Consider a hyper MV-algebra $M := \{0, a, b, 1\}$ which is described in Example 3.6. Then $D := \{0, a, 1\}$ is a weak implicative hyper MV-deductive system of M, but it is not a weak hyper MV-deductive system of M, since $(b^* \oplus a)^* = \{0, 1\} \subseteq D$.

We give a condition for a weak hyper MV-deductive system to be a weak implicative hyper MV-deductive system.

Theorem 4.10. *Let D be a weak hyper MV-deductive system of M satisfying the following condition*:

$$(\forall x, y \in M)\,((x^* \oplus (y^* \oplus x)^*)^* \subseteq D \;\Rightarrow\; x \in D) \tag{4.1}$$

Then D is a weak implicative hyper MV-deductive system of M.

Proof. Let $x, y, z \in M$ be such that $z \in D$ and $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \subseteq D$. Then

$$(((x^* \oplus (y^* \oplus x)^*)^*)^* \oplus z)^* = ((x^* \oplus (y^* \oplus x)^*) \oplus z)^* \subseteq D.$$

Using Proposition 3.16, we obtain $(x^* \oplus (y^* \oplus x)^*)^* \subseteq D$ which implies from (4.1) that $x \in D$. This shows that D is a weak implicative hyper MV-deductive system of M. □

We provide a condition for a weak implicative hyper MV-deductive system to be a weak hyper MV-deductive system, as follows:

Theorem 4.11. *Assume that M satisfies the following condition*:

$$(\forall x \in M)\,\big(|(0^* \oplus x)^*| = 1 = |(x^* \oplus 0)^*|\big). \tag{4.2}$$

Then every weak implicative hyper MV-deductive system of M is a weak hyper MV-deductive system of M.

Proof. If M satisfies (4.2), then $(0^* \oplus x)^* = \{0\}$ and $(x^* \oplus 0)^* = \{x\}$ for all $x \in M$. Hence $(0^* \oplus A)^* = \{0\}$ and $(A^* \oplus 0)^* = A$ for every non-empty subset A of M. Let $x, y \in M$ be such that $y \in D$ and $(x^* \oplus y)^* \subseteq D$. Then

$$((x^* \oplus y) \oplus (0^* \oplus x)^*)^* = ((x^* \oplus y) \oplus 0)^* = (x^* \oplus y)^* \subseteq D,$$

which implies from (d4) that $x \in D$. Therefore D is a weak hyper MV-deductive system of M. □

Theorem 4.12. *Every implicative hyper MV-deductive system D of M satisfies the following condition*:

$$(\forall x, y, z \in M)\big(((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \cap D \neq \emptyset,\; z \in D \;\Rightarrow\; x \in D\big). \tag{e2}$$

Proof. Assume that D is an implicative hyper MV-deductive system of M. Let $x, y, z \in M$ be such that $z \in D$ and $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \cap D \neq \emptyset$. Then there exists $a \in ((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \cap D$. Since $a \ll a$ by (b3), we have $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \ll D$. Using (d5), we obtain $x \in D$, which proves (e2). □

Theorem 4.13. *If a non-empty subset D of M satisfies* (d1) *and* (e2)*, then D is a hyper MV-deductive system of M.*

Proof. Let $x, y \in M$ be such that $y \in D$ and $(x^* \oplus y)^* \cap D \neq \emptyset$. Then there exists $a \in M$ such that $a \in (x^* \oplus y)^*$ and $a \in D$. Using (b2), (b9) and (b11), we have $a^* \in a^* \oplus 0 \subseteq (x^* \oplus y) \oplus (0^* \oplus x)^*$ and so $a \in ((x^* \oplus y) \oplus (0^* \oplus x)^*)^*$. Hence $((x^* \oplus y) \oplus (0^* \oplus x)^*)^* \cap D \neq \emptyset$, which implies from (e2) that $x \in D$. Using Theorem 3.19, we conclude that D is a hyper MV-deductive system of M. □

Theorem 4.14. *Let D be a non-empty subset of M. If D satisfies* (d1) *and* (e2)*, then D is an implicative hyper MV-deductive system of M.*

Proof. Let $x, y, z \in M$ be such that $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \ll D$ and $z \in D$. Then there exist $a, b \in M$ such that $a \in ((x^* \oplus z) \oplus (y^* \oplus x)^*)^*$, $b \in D$ and $a \ll b$, i.e., $0 \in (a^* \oplus b)^*$. It follows that $(a^* \oplus b)^* \cap D \neq \emptyset$. By applying Theorems 3.19 and 4.13 we obtain that $a \in D$. Thus $((x^* \oplus z) \oplus (y^* \oplus x)^*)^* \cap D \neq \emptyset$, which implies from (e2) that $x \in D$. Therefore D is an implicative hyper MV-deductive system of M. □

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