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RELATION BETWEEN ANN-CATEGORIES AND RING CATEGORIES

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Abstract. There are different categorifications of the notion of a *ring* such as *Ann-category* due to N. T. Quang, *ring category* due to M. M. Kapranov and V. A. Voevodsky. The main result of this paper is to prove that every axiom in the definition of a *ring category*, but the axiom $x_0 = y_0$, can be deduced from the axiomatics of an *Ann-category.*

1. Introduction

Categories with monoidal structures $\oplus, \otimes$ (also called *categories with distributivity constraints*) were presented by M. L. Laplaza [3]. M. M. Kapranov and V. A. Voevodsky [2] omitted requirements of the axiomatics due to Laplaza which are related to the commutativity constraints of the operation $\otimes$. These appeared under the name *ring categories.*

In another approach, a monoidal category can be "smoothed" to become a *category with group structure,* when added the invertible objects (see Laplaza [4], Saavedra Rivano [9]). Now, if the ground category is a *groupoid* (i.e., each morphism is an isomorphism), then we have a *group-like monoidal category* (see A. Fröhlich and C. T. C. Wall [1]), or a *Gr-category* (see H. X. Sinh [11]). These categories can be classified by $H^3(\Pi, A)$. Each Gr-category $\mathcal{G}$ is determined by 3 invariants: The group Π of classes of congruence objects, Π-module A of automorphisms of the unit 1, and an element $\overline{h} \in H^3(\Pi, A)$, where h is induced by the associativity constraint of $\mathcal{G}$.

In 1987, in [6], N. T. Quang proposed a notion of an *Ann-category*, as a categorification of the notion of rings, when a symmetric Gr-category (also called Pic-category) is equipped with a monoidal structure $\otimes$. In [8], [7], Ann-categories and *regular* Ann-categories, developed from the ring extension problem, have been classified by, respectively, Mac Lane ring cohomology [5] and Shukla algebraic cohomology [10].

The aim of this paper is to clearly show the relation between these definitions of an *Ann-category* and a *ring category.*

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For convenience, let us recall the definitions. Moreover, let us denote AB or $A.B$ instead of $A \otimes B$.

2. Fundamental definitions

Definition 2.1. The axiomatics of an Ann-category

An *Ann-category* consists of:

i) a groupoid $\mathcal{A}$ together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$.

ii) a fixed object $0 \in \mathcal{A}$ together with naturality constraints a_+, c, g, d such that $(\mathcal{A}, \oplus, a_+, c, (0, g, d))$ is a Pic-category.

iii) a fixed object $1 \in \mathcal{A}$ together with naturality constraints a, l, r such that $(\mathcal{A}, \otimes, a, (1, l, r))$ is a monoidal A-category.

iv) natural isomorphisms $\mathfrak{L}, \mathfrak{R}$:

$$\mathfrak{L}_{A,X,Y} : A \otimes (X \oplus Y) \longrightarrow (A \otimes X) \oplus (A \otimes Y),$$

$$\mathfrak{R}_{X,Y,A} : (X \oplus Y) \otimes A \longrightarrow (X \otimes A) \oplus (Y \otimes A)$$

such that the following conditions are satisfied:

(Ann-1) For each $A \in \mathcal{A}$, the pairs $(L^A, \breve{L}^A), (R^A, \breve{R}^A)$ determined by relations:

$$L^A = A \otimes -, \qquad R^A = - \otimes A,$$

$$\breve{L}^A{}_{X,Y} = \mathfrak{L}_{A,X,Y}, \quad \breve{R}^A{}_{X,Y} = \mathfrak{R}_{X,Y,A}$$

are $\oplus$-functors which are compatible with a_+ and c.

(Ann-2) For all $A, B, X, Y \in \mathcal{A}$, the following diagrams:

(1.1)
$$\begin{array}{ccccc}
(AB)(X \oplus Y) & \xleftarrow{a_{A,B,X\oplus Y}} & A(B(X \oplus Y)) & \xrightarrow{id_A \otimes \breve{L}^B} & A(BX \oplus BY) \\
\downarrow{\scriptstyle \breve{L}^{AB}} & & & & \downarrow{\scriptstyle \breve{L}^{A}} \\
(AB)X \oplus (AB)Y & & \xleftarrow{\quad a_{A,B,X} \oplus a_{A,B,Y} \quad} & & A(BX) \oplus A(BY),
\end{array}$$

(1.1')
$$\begin{array}{ccccc}
(X \oplus Y)(BA) & \xrightarrow{a_{X\oplus Y,B,A}} & ((X \oplus Y)B)A & \xrightarrow{\breve{R}^B \otimes id_A} & (XB \oplus YB)A \\
\downarrow{\scriptstyle \breve{R}^{BA}} & & & & \downarrow{\scriptstyle \breve{R}^{A}} \\
X(BA) \oplus Y(BA) & & \xrightarrow{\quad a_{X,B,A} \oplus a_{Y,B,A} \quad} & & (XB)A \oplus (YB)A,
\end{array}$$

(1.2)
$$\begin{array}{ccccc}
(A(X \oplus Y))B & \xleftarrow{a_{A,X\oplus Y,B}} & A((X \oplus Y)B) & \xrightarrow{id_A \otimes \breve{R}^B} & A(XB \oplus YB) \\
\downarrow{\scriptstyle \breve{L}^{A} \otimes id_B} & & & & \downarrow{\scriptstyle \breve{L}^{A}} \\
(AX \oplus AY)B & \xrightarrow{\breve{R}^B} & (AX)B \oplus (AY)B & \xleftarrow{a \oplus a} & A(XB) \oplus A(YB),
\end{array}$$

(1.3)

$$\begin{array}{ccccc}
(A\oplus B)X\oplus(A\oplus B)Y & \xleftarrow{\breve{L}^{A\oplus B}} & (A\oplus B)(X\oplus Y) & \xrightarrow{\breve{R}^{X\oplus Y}} & A(X\oplus Y)\oplus B(X\oplus Y) \\
\downarrow{\scriptstyle \breve{R}^X\oplus\breve{R}^Y} & & & & \downarrow{\scriptstyle \breve{L}^A\oplus\breve{L}^B} \\
(AX\oplus BX)\oplus(AY\oplus BY) & & \xrightarrow{\quad v \quad} & & (AX\oplus AY)\oplus(BX\oplus BY)
\end{array}$$

commute, where $v = v_{U,V,Z,T} : (U\oplus V)\oplus(Z\oplus T) \longrightarrow (U\oplus Z)\oplus(V\oplus T)$ is the unique functor built from a_+, c, id in the monoidal symmetric category $(\mathcal{A},\oplus)$.
(Ann-3) For the unit object $1\in\mathcal{A}$ of the operation $\oplus$, the following diagrams commute:

(1.4)

$$\begin{array}{ccc}
1(X\oplus Y) & \xrightarrow{\breve{L}^1} & 1X\oplus 1Y \\
 & {\scriptstyle l_{X\oplus Y}}\searrow \quad \swarrow{\scriptstyle l_X\oplus l_Y} & \\
 & X\oplus Y &
\end{array}$$

(1.4')

$$\begin{array}{ccc}
(X\oplus Y)1 & \xrightarrow{\breve{R}^1} & X1\oplus Y1 \\
 & {\scriptstyle r_{X\oplus Y}}\searrow \quad \swarrow{\scriptstyle r_X\oplus r_Y} & \\
 & X\oplus Y &
\end{array}$$

Remark. The commutative diagrams (1.1), (1.1') and (1.2), respectively, mean that:

$$\begin{aligned}
(a_{A,B,-}) &: L^A.L^B &&\longrightarrow L^{AB},\\
(a_{-,A,B}) &: R^{AB} &&\longrightarrow R^A.R^B,\\
(a_{A,-,B}) &: L^A.R^B &&\longrightarrow R^B.L^A
\end{aligned}$$

are $\oplus$-functors. The diagram (1.3) shows that the family $(\breve{L}^Z_{X,Y})_Z = (\mathcal{L}_{-,X,Y})$ is an $\oplus$-functor between the $\oplus$-functors $Z\mapsto Z(X\oplus Y)$ and $Z\mapsto ZX\oplus ZY$, and the family $(\breve{R}^C_{A,B})_C = (\mathcal{R}_{A,B,-})$ is an $\oplus$-functor between the functors $C\mapsto(A\oplus B)C$ and $C\mapsto AC\oplus BC$. The diagram (1.4) (resp. (1.4')) shows that l (resp. r) is an $\oplus$-functor from L^1 (resp. R^1) to the unit functor of the $\oplus$-category $\mathcal{A}$.

Definition 2.2. The axiomatics of a ring category

A *ring category* is a category $\mathcal{R}$ equipped with two monoidal structures $\oplus,\otimes$ (which include corresponding associativity morphisms $a^{\oplus}_{A,B,C}, a^{\otimes}_{A,B,C}$ and unit objects denoted 0, 1) together with natural isomorphisms:

$$u_{A,B} : A\oplus B\to B\oplus A, \qquad v_{A,B,C} : A\otimes(B\oplus C)\to(A\otimes B)\oplus(A\otimes C),$$
$$w_{A,B,C} : (A\oplus B)\otimes C\to(A\otimes C)\oplus(B\otimes C),$$
$$x_A : A\otimes 0\to 0, \qquad y_A : 0\otimes A\to 0.$$

These isomorphisms are required to satisfy the following conditions.

$K1(\bullet\oplus\bullet)$ The isomorphisms $u_{A,B}$ define on $\mathcal{R}$ a structure of a symmetric monoidal category, i.e., they form a braiding and $u_{A,B}u_{B,A}=1$.

$K2(\bullet\otimes(\bullet\oplus\bullet))$ For any objects A,B,C the following diagram commutes:

$$\begin{array}{ccc} A\otimes(B\oplus C) & \xrightarrow{v_{A,B,C}} & (A\otimes B)\oplus(A\otimes C) \\ {\scriptstyle A\otimes u_{B,C}}\downarrow & & \downarrow{\scriptstyle u_{A\otimes B,A\otimes C}} \\ A\otimes(C\oplus B) & \xrightarrow{v_{A,C,B}} & (A\otimes C)\oplus(A\otimes B)\ . \end{array}$$

$K3((\bullet\oplus\bullet)\otimes\bullet)$ For any objects A,B,C the following diagram commutes:

$$\begin{array}{ccc} (A\oplus B)\otimes C & \xrightarrow{w_{A,B,C}} & (A\otimes C)\oplus(B\otimes C) \\ {\scriptstyle u_{A,B}\otimes C}\downarrow & & \downarrow{\scriptstyle u_{A\otimes C,B\otimes C}} \\ (B\oplus A)\otimes C & \xrightarrow{w_{B,A,C}} & (B\otimes C)\oplus(A\otimes C)\ . \end{array}$$

$K4((\bullet\oplus\bullet\oplus\bullet)\otimes\bullet)$ For any objects A,B,C,D the following diagram commutes:

$$\begin{array}{ccccc} (A\oplus(B\oplus C)D) & \xrightarrow{w_{A,B\oplus C,D}} & AD\oplus((B\oplus C)D) & \xrightarrow{AD\oplus w_{B,C,D}} & AD\oplus(BD\oplus CD) \\ {\scriptstyle a^{\oplus}_{A,B,C}\otimes D}\downarrow & & & & \downarrow{\scriptstyle a^{\oplus}_{AD,BD,CD}} \\ ((A\oplus B)\oplus C)D & \xrightarrow{w_{A\oplus B,C,D}} & (A\oplus B)D\oplus CD & \xrightarrow{w_{A,B,D}\oplus CD} & (AD\oplus BD)\oplus CD\ . \end{array}$$

$K5(\bullet\otimes(\bullet\oplus\bullet\oplus\bullet))$ For any objects A,B,C,D the following diagram commutes:

$$\begin{array}{ccccc} A(B\oplus(C\oplus D)) & \xrightarrow{v_{A,B,C\oplus D}} & AB\oplus A(C\oplus D) & \xrightarrow{AB\oplus v_{A,C,D}} & AB\oplus(AC\oplus AD) \\ {\scriptstyle A\otimes a^{\oplus}_{B,C,D}}\downarrow & & & & \downarrow{\scriptstyle a^{\oplus}_{AB,AC,AD}} \\ A((B\oplus C)\oplus D) & \xrightarrow{v_{A,B\oplus C,D}} & A(B\oplus C)\oplus AD & \xrightarrow{v_{A,B,C}\oplus AD} & (AB\oplus AC)\oplus AD\ . \end{array}$$

$K6(\bullet\otimes\bullet\otimes(\bullet\oplus\bullet))$ For any objects A,B,C,D the following diagram commutes:

$$\begin{array}{ccccc} A(B(C\oplus D)) & \xrightarrow{A\otimes v_{B,C,D}} & A(BC\oplus BD) & \xrightarrow{v_{A,BC,BD}} & A(BC)\oplus A(BD) \\ {\scriptstyle a^{\otimes}_{A,B,C\oplus D}}\downarrow & & & & \downarrow{\scriptstyle a^{\otimes}_{A,B,C}\oplus a^{\otimes}_{A,B,D}} \\ (AB)(C\oplus D) & & \xrightarrow{\qquad v_{AB,C,D}\qquad} & & (AB)C\oplus(AB)D\ . \end{array}$$

$K7((\bullet \oplus \bullet) \otimes \bullet \otimes \bullet)$ For any objects A, B, C, D the following diagram commutes:

$$\begin{array}{ccccc}
((A \oplus B)C)D & \xrightarrow{w_{A,B,C\otimes D}} & (AC \oplus BC)D & \xrightarrow{w_{AC,BC,D}} & (AC)D \oplus (BC)D \\
\uparrow {\scriptstyle a^{\otimes}_{A\oplus B,C,D}} & & & & \uparrow {\scriptstyle a^{\otimes}_{A,C,D} \oplus a^{\otimes}_{B,C,D}} \\
(A \oplus B)(CD) & & \xrightarrow{\qquad w_{A,B,CD} \qquad} & & A(CD) \oplus B(CD)\ .
\end{array}$$

$K8(\bullet \otimes (\bullet \oplus \bullet) \otimes \bullet)$ For any objects A, B, C, D the following diagram commutes:

$$\begin{array}{ccccc}
(A(B \oplus C))D & \xrightarrow{v_{A,B,C\otimes D}} & (AB \oplus AC)D & \xrightarrow{w_{AB,AC,D}} & (AB)D \oplus (AC)D \\
\uparrow {\scriptstyle a^{\otimes}_{A,B\oplus C,D}} & & & & \uparrow {\scriptstyle a^{\otimes}_{A,B,D} \oplus a^{\otimes}_{A,C,D}} \\
A((B \oplus C)D) & \xrightarrow{A\otimes w_{B,C,D}} & A(BD \oplus CD) & \xrightarrow{v_{A,BD,CD}} & A(BD) \oplus A(CD)\ .
\end{array}$$

$K9((\bullet \oplus \bullet) \otimes (\bullet \oplus \bullet))$ For any objects A, B, C, D the diagram

$$\begin{array}{ccccc}
(A \oplus B)(C \oplus D) & \longrightarrow & A(C \oplus D) \oplus B(C \oplus D) & \longrightarrow & (AC \oplus AD) \oplus (BC \oplus BD) \\
\downarrow & & & & \downarrow \\
(A \oplus B)C \oplus (A \oplus B)D & & & & ((AC \oplus AD) \oplus BC) \oplus BD \\
\downarrow & & & & \uparrow \\
(AC \oplus BC) \oplus (AD \oplus BD) & & & & (AC \oplus (AD \oplus BC)) \oplus BD \\
\downarrow & & & & \downarrow \\
((AC \oplus BC) \oplus AD) \oplus BD & & \longleftarrow & & (AC \oplus (BC \oplus AD)) \oplus BD
\end{array}$$

is commutative (the notions for arrows have been omitted, they are obvious).

$K10(0 \otimes 0)$ The maps $x_0, y_0 : 0 \otimes 0 \to 0$ coincide.

$K11(0 \otimes (\bullet \oplus \bullet))$ For any objects A, B the following diagram commutes:

$$\begin{array}{ccc}
0 \otimes (A \oplus B) & \xrightarrow{v_{0,A,B}} & (0 \otimes A) \oplus (0 \otimes B) \\
{\scriptstyle y_{A\oplus B}} \downarrow & & \downarrow {\scriptstyle y_A \oplus y_B} \\
0 & \xleftarrow{l^{\oplus}_0 = r^{\oplus}_0} & 0 \oplus 0\ .
\end{array}$$

$K12((\bullet \oplus \bullet) \otimes 0)$ For any objects A, B the following diagram commutes:

$$\begin{array}{ccc}
(A \oplus B) \otimes 0 & \xrightarrow{w_{A,B,0}} & (A \otimes 0) \oplus (B \otimes 0) \\
{\scriptstyle x_{A \oplus B}} \downarrow & & \downarrow {\scriptstyle x_A \oplus x_B} \\
0 & \xleftarrow{l_0^{\oplus} = r_0^{\oplus}} & 0 \oplus 0 \; .
\end{array}$$

$K13(0 \otimes 1)$ The maps $y_1, r_0^{\otimes} : 0 \otimes 1 \to 0$ coincide.

$K14(1 \otimes 0)$ The maps $x_1, l_0^{\otimes} : 1 \otimes 0 \to 0$ coincide.

$K15(0 \otimes \bullet \otimes \bullet)$ For any objects A, B the following diagram commutes:

$$\begin{array}{ccc}
0 \otimes (A \otimes B) & \xrightarrow{a^{\otimes}_{0,A,B}} & (0 \otimes A) \otimes B \\
{\scriptstyle y_{A \otimes B}} \downarrow & & \downarrow {\scriptstyle y_A \otimes B} \\
0 & \xleftarrow{y_B} & 0 \otimes B \; .
\end{array}$$

$K16(\bullet \otimes 0 \otimes \bullet), (\bullet \otimes \bullet \otimes 0)$ For any objects A, B the following diagrams commute:

$$\begin{array}{ccccc}
A \otimes (0 \otimes B) & & \xrightarrow{a^{\otimes}_{A,0,B}} & & (A \otimes 0) \otimes B \\
{\scriptstyle A \otimes y_B} \downarrow & & & & \downarrow {\scriptstyle x_A \otimes B} \\
A \otimes 0 & \xrightarrow{x_A} & 0 & \xleftarrow{y_B} & 0 \otimes B \; ,
\end{array}$$

$$\begin{array}{ccc}
A \otimes (B \otimes 0) & \xrightarrow{a^{\otimes}_{A,B,0}} & (A \otimes B) \otimes 0 \\
{\scriptstyle A \otimes x_B} \downarrow & & \downarrow {\scriptstyle x_{A \otimes B}} \\
A \otimes 0 & \xrightarrow{x_A} & 0 \; .
\end{array}$$

$K17(\bullet(0 \oplus \bullet))$ For any objects A, B the following diagram commutes:

$$\begin{array}{ccc}
A \otimes (0 \oplus B) & \xrightarrow{v_{A,0,B}} & (A \otimes 0) \oplus (A \otimes B) \\
{\scriptstyle A \otimes l^{\oplus}_B} \downarrow & & \downarrow {\scriptstyle x_A \oplus (A \otimes B)} \\
A \otimes B & \xleftarrow{l^{\oplus}_{A \otimes B}} & 0 \oplus (A \otimes B)
\end{array}$$

$K18((0\oplus\bullet)\otimes\bullet), (\bullet\otimes(\bullet\oplus 0)), ((\bullet\oplus 0)\otimes\bullet)$ For any objects A, B the diagrams

$$\begin{array}{ccc}
(0\oplus A)\otimes B & \xrightarrow{w_{0,A,B}} & (0\otimes B)\oplus(A\otimes B) \\
\downarrow{\scriptstyle l^{\oplus}_A\otimes B} & & \downarrow{\scriptstyle y_B\oplus(A\otimes B)} \\
A\otimes B & \xleftarrow{l^{\oplus}_{A\otimes B}} & 0\oplus(A\otimes B)\ ,
\end{array}$$

$$\begin{array}{ccc}
A\otimes(B\oplus 0) & \xrightarrow{v_{A,B,0}} & (A\otimes B)\oplus(A\otimes 0) \\
\downarrow{\scriptstyle A\otimes r^{\oplus}_B} & & \downarrow{\scriptstyle (A\otimes B)\oplus x_A} \\
A\otimes B & \xleftarrow{r^{\oplus}_{A\otimes B}} & (A\otimes B)\oplus 0\ ,
\end{array}$$

$$\begin{array}{ccc}
(A\oplus 0)\otimes B & \xrightarrow{w_{A,0,B}} & (A\otimes B)\oplus(0\otimes B) \\
\downarrow{\scriptstyle r^{\oplus}_A\otimes B} & & \downarrow{\scriptstyle (A\otimes B)\oplus y_B} \\
A\otimes B & \xleftarrow{r^{\oplus}_{A\otimes B}} & (A\otimes B)\oplus 0
\end{array}$$

are commutative.

3. Relation between an Ann-category and a ring category

In this section, we prove that the axiomatics of a ring category, without $K10$, can be deduced from the axiomatics of an Ann-category. First, we can see that, the functor morphisms $a^{\oplus}, a^{\otimes}, u, l^{\oplus}, r^{\oplus}, v, w$, in Definiton 2.2 are, respectively, the functor morphisms $a_+, a, c, g, d, \mathfrak{L}, \mathfrak{R}$ in Definition 2.1. The isomorphisms x_A, y_A coincide with the isomorphisms $\widehat{L}^A, \widehat{R}^A$ referred in Proposition 3.2 below.

We now prove that diagrams which commute in a ring category also hold in an Ann-category.

$K1$ obviously follows from (ii) in the definition of an Ann-category.

The commutative diagrams $K2, K3, K4, K5$ are indeed the compatibility of functor isomorphisms $(L^A, \breve{L}^A), (R^A, \breve{R}^A)$ with the constraints a_+, c (the axiom Ann-1).

The diagrams $K5 - K9$, respectively, are indeed the ones in (Ann-2). Particularly, $K9$ is indeed the decomposition of (1.3) where the morphism v is replaced by its definition diagram:

$$\begin{array}{ccccc}
(P\oplus Q)\oplus(R\oplus S) & \xrightarrow{a_+} & ((P\oplus Q)\oplus R)\oplus S & \xleftarrow{a_+\oplus S} & (P\oplus(Q\oplus R))\oplus S \\
\downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle (P\oplus c)\oplus S} \\
(P\oplus R)\oplus(Q\oplus S) & \xrightarrow{a_+} & ((P\oplus R)\oplus Q)\oplus S & \xleftarrow{a_+\oplus S} & (P\oplus(R\oplus Q))\oplus S.
\end{array}$$

Proofs of $\boldsymbol{K}$17, $\boldsymbol{K}$18

Lemma 3.1. *Let $\mathcal{P}$, $\mathcal{P}^{'}$ be Gr-categories, $(a_+,(0,g,d)),(a^{'}_+,(0^{'},g^{'},d^{'}))$ be respective constraints, and $(F,\breve{F}):\mathcal{P}\to\mathcal{P}^{'}$ be $\oplus$-functor which is compatible with $(a_+,a^{'}_+)$. Then $(F,\breve{F})$ is compatible with the unit constraints $(0,g,d)),(0^{'},g^{'},d^{'}))$.*

First, the isomorphism $\widehat{F}:F0\to 0'$ is determined by the composition

$$u = F0\oplus F0 \xleftarrow{\widetilde{F}} F(0\oplus 0) \xrightarrow{F(g)} F0 \xleftarrow{g'} 0'\oplus F0.$$

Since $F0$ is a regular object, there exists uniquely the isomorphism $\widehat{F}:F0\to 0'$ such that $\widehat{F}\oplus id_{F0}=u$. Then, we may prove that $\widehat{F}$ satisfies the diagrams in the definition of the compatibility of the $\oplus$-functor F with the unit constraints.

Proposition 3.2. *In an Ann-category $\mathcal{A}$, there exist uniquely isomorphisms*

$$\hat{L}^A : A\otimes 0 \longrightarrow 0,\quad \hat{R}^A : 0\otimes A \longrightarrow 0,$$

such that the following diagrams

$$\begin{array}{ccc} AX & \xleftarrow{L^A(g)} & A(0\oplus X)\\ {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \breve{L}^A}\\ 0\oplus AX & \xleftarrow{\hat{L}^A\oplus id} & A0\oplus AX\ , \end{array} \tag{2.1}$$

$$\begin{array}{ccc} AX & \xleftarrow{L^A(d)} & A(X\oplus 0)\\ {\scriptstyle d}\uparrow & & \downarrow{\scriptstyle \breve{L}^A}\\ AX\oplus 0 & \xleftarrow{id\oplus\hat{L}^A} & AX\oplus A0\ , \end{array} \tag{2.1'}$$

$$\begin{array}{ccc} AX & \xleftarrow{R^A(g)} & (0\oplus X)A\\ {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle \breve{R}^A}\\ 0\oplus AX & \xleftarrow{\hat{R}^A\oplus id} & 0A\oplus XA\ , \end{array} \tag{2.2}$$

$$\begin{array}{ccc} AX & \xleftarrow{R^A(d)} & (X\oplus 0)A\\ {\scriptstyle d}\uparrow & & \downarrow{\scriptstyle \breve{R}^A}\\ AX\oplus 0 & \xleftarrow{id\oplus\hat{R}^A} & XA\oplus 0A \end{array} \tag{2.2'}$$

commute, i.e., L^A and R^A are U-functors respect to the operation $\oplus$.

Proof. Since $(L^A, \breve{L}^A)$ are $\oplus$-functors which are compatible with the associativity constraint $a^{\oplus}$ of the Picard category $(\mathcal{A}, \oplus)$, they are also compatible with the unit constraint $(0, g, d)$ thanks to Lemma 3.1. That means there exists uniquely the isomorphism $\hat{L}^A$ satisfying the diagrams (2.1) and (2.1'). The proof for $\hat{R}^A$ is similar. The diagrams commute in Proposition 1 are indeed $K17, K18$. □

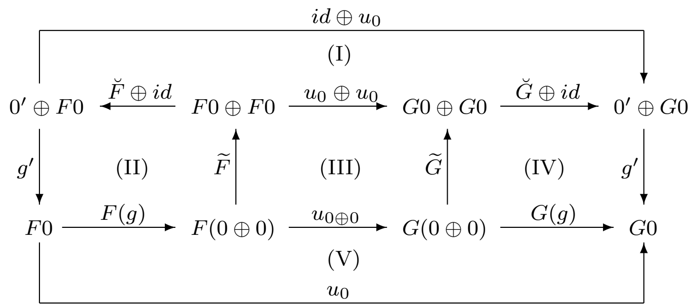
Proofs of $K15$, $K16$

Lemma 3.3. *Let $(F, \breve{F}), (G, \breve{G})$ be $\oplus$-functors between $\oplus$-categories $\mathcal{C}, \mathcal{C}'$ which are compatible with the constraints $(0, g, d), (0', g', d')$ and $\widetilde{F} : F(0) \longrightarrow 0'$, $\widetilde{G} : G(0) \longrightarrow 0'$ are respective isomorphisms. If $\alpha : F \longrightarrow G$ in an $\oplus$-morphism such that α_0 is an isomorphism, then the diagram*

$$\begin{array}{ccc} F0 & \xrightarrow{\alpha_0} & G0 \\ & \hat{F}\searrow \quad \swarrow\hat{G} & \\ & 0' & \end{array}$$

commutes.

Proof. Let us consider the diagram:



In this diagram, the regions (II) and (IV) commute thanks to the compatibility of $\oplus$-functors $(F, \breve{F}), (G, \breve{G})$ with the unit constraint; the region (III) commutes since u is a $\oplus$-morphism; the region (V) commutes thanks to the naturality of g'. Therefore, the region (I) commutes, i.e.,

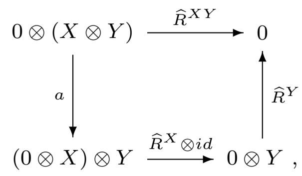
$$\breve{G} \circ u_0 \oplus u_0 = \breve{F} \oplus u_0.$$

Since $F0$ is a regular object, $\breve{G} \circ u_0 = \breve{F}$. □

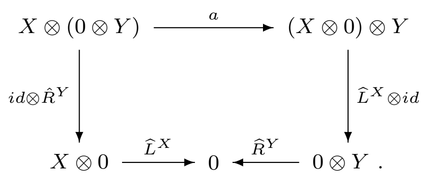
Proposition 3.4. *For any objects $X, Y \in ob\mathcal{A}$ the following diagrams commute*

$$(2.3) \qquad \begin{array}{ccc} X \otimes (Y \otimes 0) & \xrightarrow{id \otimes \widehat{L}^Y} & X \otimes 0 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \widehat{L}^X} \\ (X \otimes Y) \otimes 0 & \xrightarrow{\widehat{L}^{XY}} & 0\ , \end{array}$$

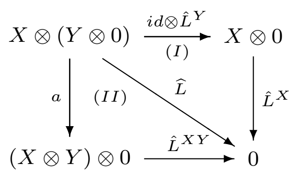
(2.3')



(2.4)

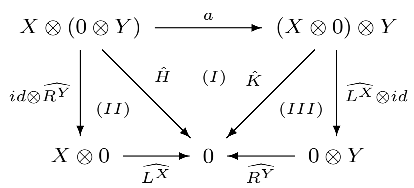


Proof. To prove that the first diagram is commutative, let us consider the diagram:



where $L = L^X \circ L^Y$. According to the axiom (1.1), $(a_{X,Y,Z})_Z$ is an $\oplus$-morphism from the functor $L = L^X \circ L^Y$ to the functor $G = L^{XY}$. Therefore, from Lemma 3.3, the region (II) commutes. The region (I) commutes thanks to the determination of $\hat{L}$ of the composition $L = L^X \circ L^Y$. So the perimeter commutes.

The second diagram is proved similarly, thanks to the axiom (1.1'). To prove that the diagram (2.4) commutes, let us consider the diagram:

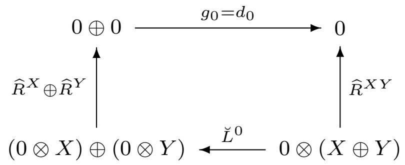


where $H = L^X \circ R^Y$ and $K = R^Y \circ L^X$. Then the regions (II) and (III) commute thanks to the determination of the isomorphisms H and K. From the axiom (1.2), $(a_{X,Y,Z})_Z$ is an $\oplus$-morphism from the functor H to the functor K. So from Lemma 3.3, the region (I) commutes. Therefore, the perimeter commutes. The diagrams in Proposition 3.4 are indeed $K15, K16$. □

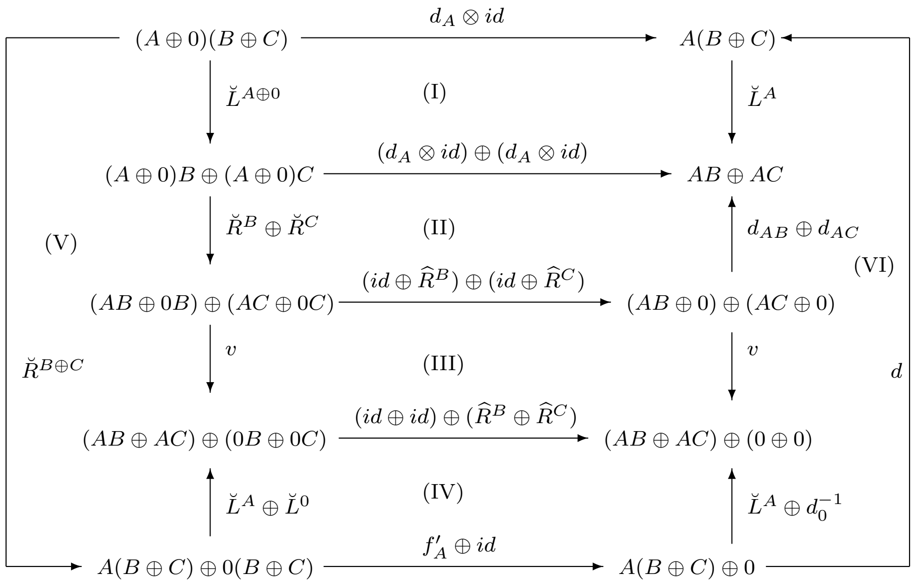
Proof of *K*11

Proposition 3.5. *In an Ann-category, the following diagram commutes*:

(2.5)



Proof. Let us consider the diagram:



In this diagram, the region (V) commutes thanks to the axiom I (1.3), the region (I) commutes thanks to the functorial property of $\mathfrak{L}$; the perimeter and the region (II) commute thanks to the compatibility of the functors $R^{B \oplus C}, R^B, R^C$ with the unit constraint $(0, g, d)$; the region (III) commutes thanks to the functorial property of v; the region (VI) commutes thanks to the coherence for the ACU-functor $(L^A, \breve{L}^A)$. So (IV) commutes. Note that $A(B \oplus C)$ is a regular object respect to the operation $\oplus$, so the diagram (2.5) commutes. We have K11. □

Similarly, we have $K12$.

Proofs of *K*13, *K*14

Proposition 3.6. *In an Ann-category, we have*:

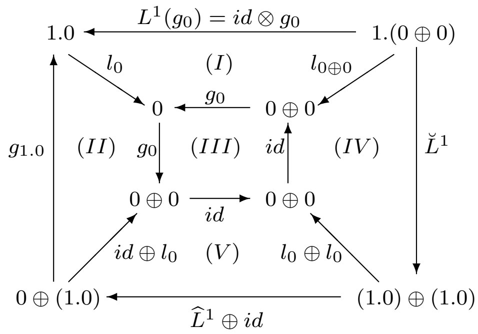
$$\widehat{L}^1 = l_0, \quad \widehat{R}^1 = r_0.$$

Proof. We will prove the first equation, the second one is proved similarly. Let us consider the diagram (2.6). In this diagram, the perimeter commutes thanks to the compatibility of $\oplus$-functor $(L^1, \breve{L}^1)$ with the unit constraint $(0, g, d)$ respect to the operation $\oplus$; the region (I) commutes thanks to the functorial property of the isomorphism l; the region (II) commutes thanks to the functorial property of g; the region (III) obviously commutes; the region (IV) commutes thanks to the axiom I(1.4). So the region (V) commutes, i.e.,

$$\widehat{L}^1 \oplus id_{1.0} = l_0 \oplus id_{1.0}.$$

Since 1.0 is a regular object respect to the operation $\oplus$, $\widehat{L}^1 = l_0$.

(2.6)



We have $K14$.

Similarly, we have $K13$. □

Definition 3.1. An Ann-category $\mathcal{A}$ is *strong* if $\widehat{L}^0 = \widehat{R}^0$.

All the above results can be stated as follows.

Proposition 3.7. *Each strong Ann-category is a ring category.*

Remark. In our opinion, in the axiomatics of a *ring category,* the compatibility of the distributivity constraint with the unit constraint $(1, l, r)$ respect to the operation $\otimes$ is necessary, i.e., the diagrams of (Ann-3) should be added.

Moreover, if the symmetric monoidal structure of the operation $\oplus$ is replaced with the symmetric categorical groupoid structure, then each ring category is an Ann-category.

An open question: May the equation $\widehat{L}^0 = \widehat{R}^0$ be proved to be independent in an Ann-category?

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