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ROUGH FUZZY QUICK IDEALS IN d-ALGEBRAS

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Abstract. Rough sets, rough quick ideals and rough fuzzy quick ideals in d-algebras are established, and some related properties are investigated.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim ([11]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras. In the same paper they also investigated other relations between d-algebras and oriented digraphs. J. Neggers, Y. B. Jun, and H. S. Kim ([12]) discussed ideal theory in d-algebras, and introduced the notions of d-subalgebra, d-ideal, $d^{\#}$-ideal and d^{*}-ideal, and investigated some relations among them. Y. C. Lee and H. S. Kim ([9]) introduced the notion of d-transitive d^{*}-algebra which is another interesting generalization of BCK-algebras. In [2], we introduced the notion of quick ideals and the fuzzification of quick ideals in d-algebras, and investigated some related properties in d-algebras. We also discussed the product of fuzzy quick ideals and projections of fuzzy quick ideals in d-algebras and obtained the fundamental results needed to develop a further theory of these objects.

In this paper, we introduce the notion of a rough set in d-algebras. Using a quick ideal in d-algebras, we obtain some relations between quick ideals and upper (lower) rough quick ideals in d-algebras. Also we consider the notion of rough fuzzy quick ideals in d-algebras and give some properties of such ideals.

2. Preliminaries

A *d-algebra* ([11]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

(I) $x * x = 0$,

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(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

For brevity we also call X a *d-algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

Definition 2.1 ([12])**.** Let X be a d-algebra and let $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$.

I is called a *BCK-ideal* of X if it satisfies:

(D_0) $0 \in I$,

(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

A d-ideal I of X is called a *$d^{\#}$-ideal* of X, if, for arbitrary $x, y, z \in I$,

(D_3) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$.

A $d^{\#}$-ideal I of X is called a *d^*-ideal* of X, if, for arbitrary $x, y, z \in X$,

(D_4) $x * y \in I$ and $y * x \in X$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$.

Definition 2.2 ([2])**.** Let X be a d-algebra and let $0 \in I \subseteq X$. I is called a *quick ideal* of X if for any $x, y \in X$ with $x * y \neq 0$, $x * y \in I$ implies $x, y \in I$.

The notion of a quick ideal is different from the notion of a BCK-ideal in d-algebras.

Example 2.3 ([2])**.** (1) Let $X := \{0, a, b, c\}$ be a d-algebra ([11]) which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	c
b	b	b	0	0
c	c	0	b	0

Then $A := \{0, a, c\}$ is both a quick ideal of X and a d-subalgebra of X, but not a BCK-ideal of X since $b * c = 0, c \in A$, but $b \notin A$.

(2) Let $X := \{0, a, b, c\}$ be a d-algebra ([11]) which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	0	b	0

Then $B := \{0, a, c\}$ is a quick ideal of X, but not a subalgebra of X since $a * c = b \notin B$. Also B is not a BCK-ideal of X since $b * c = 0, c \in B$, but $b \notin B$.

(3) Let $X := \{0, a, b, c\}$ be a d-algebra ([2]) which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	b
c	c	c	c	0

Then $C := \{0, a, c\}$ is a BCK-ideal of X, but not a quick ideal of X since $c * b = c \in C$ and $b \notin C$. Also C is not a subalgebra of X since $a * c = b \notin C$.

Definition 2.4 ([7])**.** Let μ be a fuzzy set in a d-algebra X. Then μ is called a *fuzzy d-subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

μ is called a *fuzzy BCK-ideal of X* if

(F_0) $\mu(0) \geq \mu(x)$,
(F_1) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

μ is called a *fuzzy d-ideal* of X if it satisfies (F_1) and

(F_2) $\mu(x * y) \geq \mu(x)$ for all $x, y \in X$.

Definition 2.5 ([2])**.** Let X be a d-algebra. A map $\mu : X \to [0, 1]$ is called a *fuzzy quick ideal* of X if it satisfies (F_0) and

(F_3) for any $x, y \in X$ with $x * y \neq 0$, $\min\{\mu(x), \mu(y)\} \geq \mu(x * y)$.

Let μ and λ be two fuzzy subsets of X. The inclusion $\lambda \subseteq \mu$ is denoted by $\lambda(x) \subseteq \mu(x)$ for all $x \in X$ and $\mu \cap \lambda$ is defined by

$$(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x) \text{ for all } x \in X.$$

Let μ and ν be fuzzy quick ideals of X. Then $\mu \cap \nu$ is also a fuzzy quick ideal of X.

Theorem 2.6 ([2])**.** *Let μ be fuzzy subset of a d-algebra X. Then μ is a fuzzy quick ideal of X if and only if for any $t \in [0, 1]$ with $\mu_t \neq \emptyset$, μ_t is a quick ideal of X.*

3. Rough sets in d-algebras

In what follows let X denote a d-algebra unless otherwise specified.

Let I be a d^*-ideal of X. Define a relation ρ on X by $(x, y) \in \rho$ if and only if $x * y \in I$ and $y * x \in I$. Then ρ is an equivalence relation on X related to a d^*-ideal I of X. Moreover ρ satisfies $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x * u, y * v) \in \rho$. Hence ρ is a congruence relation on X. We denote by $[a]_\rho$ the ρ-congruence class containing the element $a \in X$. Let X/ρ be the set of all

ρ-equivalence classes on X, i.e., $X/\rho := \{[a]_\rho | a \in X\}$. For any $[x]_\rho, [y]_\rho \in X/\rho$, if we define

$$[x]_\rho * [y]_\rho := [x * y]_\rho = \{z \in X | (z, x * y) \in \rho\},$$

then it is well defined, since ρ is a congruence relation. A congruence relation ρ on a d-algebra X is said to be *regular* if $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$ implies $[x]_\rho = [y]_\rho$ for any $[x]_\rho, [y]_\rho \in X/\rho$.

Theorem 3.1. *Let X be a d-algebra and let ρ be a congruence relation on X. Then ρ is regular if and only if X/ρ is a d-algebra.*

Proof. Straightforward. □

Let X be a d-algebra and ρ be an congruence relation on X and let $\mathcal{P}(X)$ denote the power set of X and $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. For all $x \in X$, let $[x]_\rho$ denote the ρ-congruence class of x. Define the functions $\rho_-,\ \rho^- : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows: for any $\emptyset \neq A \in \mathcal{P}(X)$,

$$\rho_-(A) := \{x \in X | [x]_\rho \subseteq A\}$$

and

$$\rho^-(A) := \{x \in X | [x]_\rho \cap A \neq \emptyset\}.$$

$\rho_-(A)$ is called the *ρ-lower approximation* of A while $\rho^-(A)$ is called the *ρ-upper approximation* of A. For a non-empty subset A of X,

$$\rho(A) = (\rho_-(A), \rho^-(A))$$

is called a *rough set* with respect to ρ of $\mathcal{P}(X) \times \mathcal{P}(X)$ if $\rho_-(A) \neq \rho^-(A)$. A subset A of X is said to be *definable* if $\rho_-(A) = \rho^-(A)$. The pair (X, ρ) is called an *approximation space*.

The following property is useful for our research.

Proposition 3.2 ([1]). *Let ρ and λ be congruence relations on X. Then the following are true*:

(1) $\forall F \in \mathcal{P}^*(X),\ \rho_-(F) \subseteq F \subseteq \rho^-(F)$,
(2) $\forall F, G \in \mathcal{P}^*(X),\ \rho^-(F \cup G) = \rho^-(F) \cup \rho^-(G)$,
(3) $\forall F, G \in \mathcal{P}^*(X),\ \rho_-(F \cap G) = \rho_-(F) \cap \rho_-(G)$,
(4) $\forall F, G \in \mathcal{P}^*(X),\ F \subseteq G \Rightarrow \rho_-(F) \subseteq \rho_-(G)$,
(5) $\forall F, G \in \mathcal{P}^*(X),\ F \subseteq G \Rightarrow \rho^-(F) \subseteq \rho^-(G)$,
(6) $\forall F, G \in \mathcal{P}^*(X),\ \rho_-(F) \cup \rho_-(G) \subseteq \rho_-(F \cup G)$,
(7) $\forall F, G \in \mathcal{P}^*(X),\ \rho^-(F \cap G) \subseteq \rho^-(F) \cap \rho^-(G)$,
(8) $\forall F \in \mathcal{P}^*(X),\ \rho \subseteq \lambda \Rightarrow \lambda_-(F) \subseteq \rho_-(F), \rho^-(F) \subseteq \lambda^-(F)$.

Corollary 3.3. *If ρ and λ are congruence relations on X, then*

(i) $\forall F \in \mathcal{P}^*(X),\ (\rho \cap \lambda)^-(F) \subseteq \rho^-(F) \cap \lambda^-(F)$,
(ii) $\forall F \in \mathcal{P}^*(X),\ \rho_-(F) \cap \lambda_-(F) \subseteq (\rho \cap \lambda)_-(F)$.

Proof. It follows immediately from Proposition 3.2. □

For any $A, B \in \mathcal{P}^*(X)$, we define $A * B := \{a * b | a \in A, b \in B\}$.

Theorem 3.4. *Let ρ be a congruence relation on a d-algebra X and let $\emptyset \neq A, B \subseteq X$. Then $\rho^-(A) * \rho^-(B) \subseteq \rho^-(A * B)$.*

Proof. Let c be any element of $\rho^-(A)*\rho^-(B)$. Then $c = a*b$ for some $a \in \rho^-(A)$ and $b \in \rho^-(B)$. Thus there exist elements $x, y \in X$ such that $x \in [a]_\rho \cap A$ and $y \in [b]_\rho \cap B$. Since ρ is a congruence relation on X, $x * y \in [a]_\rho * [b]_\rho = [a * b]_\rho$. Since $x * y \in A * B$, we have $c = a * b \in \rho^-(A * B)$. □

Theorem 3.5. *Let ρ be a congruence relation on a d-algebra X and let $\emptyset \neq A, B \subseteq X$. If $\rho_-(A * B)$ is non-empty, then $\rho_-(A) * \rho_-(B) \subseteq \rho_-(A * B)$.*

Proof. Let $c \in \rho_-(A) * \rho_-(B)$. Then $c = a * b$ for some $a \in \rho_-(A)$ and $b \in \rho_-(B)$. Thus we have $[a]_\rho \subseteq A$ and $[b]_\rho \subseteq B$. Since ρ is a congruence relation on X, $[a * b]_\rho = [a]_\rho * [b]_\rho \subseteq A * B$ and so $c = a * b \in \rho_-(A * B)$, completing the proof. □

Remark. The condition that $\rho_-(A * B)$ is non-empty in Theorem 3.5 is necessary.

Example 3.6. Let $X := \{0, a, b, c, d\}$ be a d-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $I := \{0, a\}$ is a d^*-ideal of X. Let ρ be a congruence relation on X such that $\{0, a\}, \{b\}, \{c\}$, and $\{d\}$ are all ρ-congruences of X. If we take $A := \{b, c\}$, $B := \{c\}$, then $A * B = \{0\}$ and $\rho_-(A * B) = \emptyset, \rho_-(A) = \{b, c\}, \rho_-(B) = \{c\}$ and hence $\rho_-(A) * \rho_-(B) = \{0\}$.

For any congruence relation ρ on X, we note that
- $\forall F \in \mathcal{P}^*(X)$, $\rho_-(F) \subseteq F$,
- $\forall F, G \in \mathcal{P}^*(X)$, $F \subseteq G \Rightarrow \rho_-(F) \subseteq \rho_-(G)$,
- $\forall F \in \mathcal{P}^*(X)$, $\rho_-(\rho_-(F)) = \rho_-(F)$,

which means that ρ_- is an interior operator on X. This operation induces a topology $\mathcal{T}$ on X such that

$$F \in \mathcal{T} \Leftrightarrow \rho_-(F) = F.$$

Lemma 3.7. *For any congruence relation ρ on X, ρ^- is a closure operator on the topological space $(X, \mathcal{T})$.*

Proof. For any $F \in \mathcal{P}^*(X)$, we have

$$\begin{aligned} x \in \rho^-(F) &\Leftrightarrow [x]_\rho \cap F \neq \emptyset \\ &\Leftrightarrow [x]_\rho \nsubseteq F^c \\ &\Leftrightarrow x \notin \rho_-(F^c) \\ &\Leftrightarrow x \in (\rho_-(F^c))^c, \end{aligned}$$

i.e., $\rho^-(F) = (\rho_-(F^c))^c$, which completes the proof. □

Lemma 3.8. *For any congruence relation ρ on X, we have*
(i) $\forall F \in \mathcal{P}(X)$, $\rho_-(F) = F \Leftrightarrow \rho^-(F^c) = F^c$,
(ii) $\forall F \in \mathcal{P}(X)$, $\rho_-(F) = F \Leftrightarrow \rho^-(F) = F$.

Proof. Straightforward. □

Based on the above two lemmas we have the following result.

Theorem 3.9. *For any $F \subseteq X$ and a congruence relation ρ on X, the following assertions are equivalent*:
(i) *F is definable with respect to ρ.*
(ii) *F is open in the topological space $(X, \mathcal{T})$.*
(iii) *F is closed in the topological space $(X, \mathcal{T})$.*

According to [8], we say that an open set F of X is said to be *free* in approximation space (X, ρ) if $x \notin \rho^-(F \setminus \{x\})$ for all $x \in F$. Since $\rho^-(F \setminus \{x\}) = (\rho_-((F \setminus \{x\})^c))^c$, a non-empty subset F of X is free if and only if $x \in \rho_-(F^c \cup \{x\})$, i.e., if and only if $[x]_\rho \subseteq F^c \cup \{x\}$ for every $x \in F$. Thus for a free subset F and any $(x, y) \in \rho \cap (F \times F)$ we have $y \in F$, which together with $y \in [x]_\rho \subseteq F^c \cup \{x\}$ implies that $y = x$. Therefore $\rho \cap (F \times F) = \{(a, a) | a \in F\}$. Conversely, let

$$\rho \cap (F \times F) = \{(a, a) | a \in F\}$$

and let y be an arbitrary element of $[x]_\rho$. If $y \in F$, then $y = x$, i.e., $y \in \{x\} \subseteq F^c \cup \{x\}$. If $y \notin F$, then $y \in F^c \subseteq F^c \cup \{x\}$. Thus, in each case $[x]_\rho \subseteq F^c \cup \{x\}$, which means that F is free. Consequently, we obtain the following characterization of free subsets.

Theorem 3.10. *$F \subseteq X$ is free if and only if $\rho \cap (F \times F) = \{(a, a) | a \in F\}$.*

Corollary 3.11. *If X is free, then any subset of X is free.*

4. Rough quick ideals

Definition 4.1 ([1])**.** Let ρ be an congruence relation on X related to a d^*-ideal I of X and let $\emptyset \neq A \subseteq X$. Then A is called an *upper* (a *lower*, respectively) *rough subalgebra/ideal* of X if $\rho^-(A)$ ($\rho_-(A)$, respectively) is a d-subalgebra/BCK-ideal of X. If A is both an upper and a lower rough subalgebra/ideal of X, we say that A is a *rough subalgebra/ideal* of X.

Proposition 4.2 ([1]). *Let ρ be a congruence relation on a d-algebra X and let $\emptyset \neq A \subseteq X$. If A is a d-subalgebra of X, then A is an upper rough subalgebra of X.*

Proposition 4.3 ([1]). *Let ρ be a congruence relation on a d-algebra X and let A be a d-subalgebra of X. If $\rho_-(A)$ is non-empty, then it is a d-subalgebra of X, i.e., A is a lower rough subalgebra of X.*

Definition 4.4. Let X be a d-algebra and let $\emptyset \neq A \subseteq X$. Let ρ be a congruence relation on X related to a d^*-ideal of X. Then A is called an *upper* (a *lower*, respectively) *rough quick ideal* if $\rho^-(A)$ ($\rho_-(A)$, respectively) is a quick ideal of X.

Theorem 4.5. *If A is a quick ideal of X, then it is an upper rough quick ideal of X.*

Proof. Since A is a quick ideal of X, $0 \in A$ and so $A \cap [0]_\rho \neq \emptyset$, i.e., $0 \in \rho^-(A)$. Let $x, y \in X$ with $x*y \neq 0$, $x*y \in \rho^-(A)$. Then $([x]_\rho * [y]_\rho) \cap A = [x*y]_\rho \cap A \neq \emptyset$. This means that there exists $\alpha \in A$ such that $\alpha \in [x]_\rho * [y]_\rho$. Thus $\alpha = p * q$ for some $p \in [x]_\rho$ and $q \in [y]_\rho$. Since A is a quick ideal of X, we have $p, q \in A$. Hence $p \in [x]_\rho \cap A, q \in [y]_\rho \cap A$, i.e., $x, y \in \rho^-(A)$, completing the proof. □

Theorem 4.5 shows that the notion of an upper rough quick ideal is an extended notion of a quick ideal in d-algebras. The following example gives that the converse of Theorem 4.5 does not hold in general.

Example 4.6. Let $X := \{0, a, b, c\}$ be a d-algebra ([11]) which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	c
b	b	b	0	0
c	c	0	b	0

Let $I := \{0, a\}$. Then I is a d^*-ideal of X. If we take $A := \{0, c\}$, then it is not a quick ideal of X, since $a * c = c \in A$ and $a \notin A$. On the while, let ρ be a congruence relation on X (related to I) such that $\{0, a\}, \{b\}$ and $\{c\}$ are all ρ-congruences of X. Then $\rho^-(A) = \{0, a, c\}$ is a quick ideal of X.

Theorem 4.7. *Let X be a d-algebra and let A be a quick ideal of X. If $\rho_-(A)$ is non-empty, then A is a lower rough quick ideal of X.*

Proof. By Proposition 4.3, $\rho_-(A)$, if it is non-empty, is a d-subalgebra of X, and hence $0 \in \rho_-(A)$. Let $x, y \in X$ with $x * y \neq 0$, $x * y \in \rho_-(A)$. Then $[x]_\rho * [y]_\rho = [x*y]_\rho \subseteq A$. Let $\alpha \in [x]_\rho$. Then $(\alpha, x) \in \rho$. Since ρ is a congruence relation on X, we have $(\alpha * y, x * y) \in \rho$. Hence $\alpha * y \in [x * y]_\rho \subseteq A$. Since A is a quick ideal of X, we obtain $\alpha, y \in A$. Thus $[x]_\rho \subseteq A$ and so $x \in \rho_-(A)$.

Let $\beta \in [y]_\rho$. Then $(\beta, y) \in \rho$. Since ρ is a congruence relation on X, we have $(x * \beta, x * y) \in \rho$. Hence $x * \beta \in [x * y]_\rho \subseteq A$. Since A is a quick ideal of X, we obtain $x, \beta \in A$. Thus $[y]_\rho \subseteq A$ and so $y \in \rho_-(A)$. □

Let ρ be a regular congruence relation on a d-algebra X. The lower and upper approximations can be presented in an equivalent form as shown below:

$$\begin{aligned} \rho_-(A)/\rho =& \{[x]_\rho \in X/\rho | [x]_\rho \subseteq A\}, \\ \rho^-(A)/\rho =& \{[x]_\rho \in X/\rho | [x]_\rho \cap A \neq \emptyset\}. \end{aligned}$$

Proposition 4.8. *Let ρ be a regular congruence relation on a d-algebra X. If A is a d-subalgebra of X, then $\rho^-(A)/\rho$ is a d-subalgebra of the quotient d-algebra X/ρ.*

Proof. Since A is a d-subalgebra of X, there exists an element $x \in A$ and hence $[x]_\rho \cap A \neq \emptyset$, i.e., $\rho^-(A)/\rho \neq \emptyset$. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho^-(A)/\rho$. Then $[x]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. This means that there exist $a, b \in X$ such that $a \in [x]_\rho \cap A$ and $b \in [y]_\rho \cap A$. Then $a * b \in [x]_\rho * [y]_\rho$. Since A is a d-subalgebra of X, $a*b \in A$. This means that $[x]_\rho*[y]_\rho \in \rho^-(A)/\rho$, completing the proof. □

Proposition 4.9. *Let ρ be a regular congruence relation on a d-algebra X. If A is a d-subalgebra of X, then $\rho_-(A)/\rho$ is, if it is non-empty, a d-subalgebra of the quotient d-algebra X/ρ.*

Proof. Straightforward. □

Theorem 4.10. *Let ρ be a regular congruence relation on a d-algebra X. If A is a quick ideal of X, then $\rho^-(A)/\rho$ is a quick ideal of the quotient d-algebra X/ρ.*

Proof. Since $0 \in A$, $A \cap [0]_\rho \neq \emptyset$ and hence $[0]_\rho \in \rho^-(A)/\rho$. Let $[x]_\rho, [y]_\rho \in X/\rho$ with $[x]_\rho*[y]_\rho \neq [0]_\rho$, $[x]_\rho*[y]_\rho \in \rho^-(A)/\rho$. Then $[x*y]_\rho \neq [0]_\rho$, $[x*y]_\rho \cap A \neq \emptyset$, and hence there exists $\alpha \in [x*y]_\rho \cap A$. Since $[x*y]_\rho \neq [0]_\rho$, we have $\alpha \neq 0$. Thus $\alpha = p*q$ for some $p \in [x]_\rho$ and $q \in [y]_\rho$. Since A is a quick ideal of X, we obtain $p, q \in A$. Hence $p \in [x]_\rho \cap A$ and $q \in [y]_\rho \cap A$, proving $[x]_\rho, [y]_\rho \in \rho^-(A)/\rho$. □

Theorem 4.11. *Let ρ be a regular congruence relation on a d-algebra X. If A is a quick ideal of X, then $\rho_-(A)/\rho$ is, if it is non-empty, a quick ideal of the quotient d-algebra X/ρ.*

Proof. By Proposition 4.9, $\rho_-(A)/\rho$, if it is non-empty, is a d-subalgebra of X/ρ, and hence $[0]_\rho \in \rho_-(A)/\rho$. Let $[x]_\rho, [y]_\rho \in X/\rho$ with $[x]_\rho*[y]_\rho \neq [0]_\rho$, $[x]_\rho*[y]_\rho \in \rho_-(A)/\rho$. Then $[x * y]_\rho \neq [0]_\rho$, $[x]_\rho * [y]_\rho = [x * y]_\rho \subseteq A$.

Let $\alpha \in [x]_\rho$. Then $(\alpha, x) \in \rho$. Since ρ is a congruence relation on X, $(\alpha * y, x * y) \in \rho$. Hence $\alpha * y \in [x * y]_\rho \subseteq A$. Since A is a quick ideal of X, we have $\alpha, y \in A$. Therefore $[x]_\rho \subseteq A$. Thus $[x]_\rho \in \rho_-(A)/\rho$.

Let $\beta \in [y]_\rho$. Then $(\beta, y) \in \rho$. Since ρ is a congruence relation on X, $(x * \beta, x * y) \in \rho$. Hence $x * \beta \in [x * y]_\rho \subseteq A$. Since A is a quick ideal of X, we have $x, \beta \in A$. Therefore $[y]_\rho \subseteq A$. Thus $[y]_\rho \in \rho_-(A)/\rho$. □

Theorem 4.12. *Let ρ be a regular congruence relation on a d-algebra X. If A is an upper rough ideal of X, then $\rho^-(A)/\rho$ is a BCK-ideal of X/ρ.*

Proof. Since $0 \in \rho^-(A)$, we have $[0]_\rho \cap A \neq \emptyset$ and hence $[0]_\rho \in \rho^-(A)/\rho$. Let $[x]_\rho * [y]_\rho = [x*y]_\rho, [y]_\rho \in \rho^-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Then $([x]_\rho * [y]_\rho) \cap A = [x * y]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. Hence $x * y, y \in \rho^-(A)$. Since $\rho^-(A)$ is a BCK-ideal of X, we have $x \in \rho^-(A)$. Thus $x \in [x]_\rho \cap A \neq \emptyset$, proving $[x]_\rho \in \rho^-(A)/\rho$. □

Theorem 4.13. *Let ρ be a regular congruence relation on a d-algebra X. If A is a lower rough ideal of X, then $\rho_-(A)/\rho$ is, if it is non-empty, a BCK-ideal of the quotient d-algebra X/ρ.*

Proof. Since $\rho_-(A)/\rho \neq \emptyset$, $\rho_-(A)/\rho$ is a d-subalgebra of X/ρ and hence $[0]_\rho \in \rho_-(A)/\rho$. Let $[x]_\rho * [y]_\rho, [y]_\rho \in \rho_-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Hence $[x*y]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Therefore $x*y \in \rho_-(A), y \in \rho_-(A)$. Since $\rho_-(A)$ is a BCK-ideal of X, we have $x \in \rho_-(A)$. Therefore $[x]_\rho \subseteq A$. Thus $[x]_\rho \in \rho^-(A)/\rho$. □

5. Approximations of fuzzy sets

Definition 5.1. Let ρ be a congruence relation on a d-algebra X and μ a fuzzy subset of X. We define the fuzzy sets $\rho_-(\mu)$ and $\rho^-(\mu)$ as follows:

$$\rho_-(\mu)(x) := \wedge_{a \in [x]_\rho} \mu(a) \text{ and } \rho^-(\mu)(x) := \vee_{a \in [x]_\rho} \mu(a).$$

The fuzzy sets $\rho_-(\mu)$ and $\rho^-(\mu)$ are called the *ρ-lower* and *ρ-upper approximations* of the fuzzy set μ, respectively. $\rho(\mu) = (\rho_-(\mu), \rho^-(\mu))$ is called a *rough fuzzy set* with respect to ρ if $\rho_-(\mu) \neq \rho^-(\mu)$.

Let μ be a fuzzy subset of a d-algebra X. Then the sets

$$\mu_t := \{x \in X | \mu(x) \geq t\}, \ \mu_t^X := \{x \in X | \mu(x) > t\},$$

where $t \in [0, 1]$ are called *t-level subset* and *t-strong level subset* of μ, respectively. A fuzzy subset μ of a d-algebra X is called an *upper* (a *lower*, respectively) *rough fuzzy ideal* of X if $\rho^-(\mu)$ ($\rho_-(\mu)$, respectively) is a fuzzy BCK-ideal of X. A fuzzy subset μ of a d-algebra X is called an *upper* (a *lower*, respectively) *rough fuzzy quick ideal* of X if $\rho^-(\mu)$ ($\rho_-(\mu)$, respectively) is a fuzzy quick ideal of X.

Theorem 5.2. *Let ρ be a congruence relation on X. If μ is a fuzzy BCK-ideal of X, then $\rho^-(\mu)$ is a fuzzy BCK-ideal of X.*

Proof. Since μ is a fuzzy BCK-ideal of X, $\mu(0) \geq \mu(x)$ for all $x \in X$. Hence we obtain

$$\rho^-(\mu)(0) = \vee_{z \in [0]_\rho} \mu(z) \geq \vee_{x' \in [x]_\rho} \mu(x') = \rho^-(\mu)(x).$$

For any $x, y \in X$, we have

$$\begin{aligned}\rho^-(\mu)(x) = \vee_{x'\in[x]_\rho}\mu(x') &\geq \vee_{x'*y'\in[x]_\rho*[y]_\rho, y'\in[y]_\rho}(\min\{\mu(x'*y'), \mu(y')\})\\ &= \vee_{x'*y'\in[x*y]_\rho, y'\in[y]_\rho}(\min\{\mu(x'*y'), \mu(y')\})\\ &= \min(\vee_{x'*y'\in[x*y]_\rho}\mu(x'*y'), \vee_{y'\in[y]_\rho}\mu(y'))\\ &= \min(\rho^-(\mu)(x*y), \rho^-(\mu)(y)).\end{aligned}$$

Thus $\rho^-(\mu)$ is a fuzzy BCK-ideal of X. □

Theorem 5.3. *Let ρ be a congruence relation on a d-algebra X. If μ is a fuzzy BCK-ideal of X, then $\rho_-(\mu)$ is, if it is non-empty, a fuzzy BCK-ideal of X.*

Proof. Since μ is a fuzzy BCK-ideal of X, $\mu(0) \geq \mu(x)$ for all $x \in X$. Hence for all $x \in X$, we have

$$\rho_-(\mu)(0) = \wedge_{z\in[0]_\rho}\mu(z) \geq \wedge_{z'\in[x]_\rho}\mu(z') = \rho_-(\mu)(x).$$

For any $x, y \in X$, we obtain

$$\begin{aligned}\rho_-(\mu)(x) = \wedge_{x'\in[x]_\rho}\mu(x') &\geq \wedge_{x'*y'\in[x]_\rho*[y]_\rho, y'\in[y]_\rho}(\min\{\mu(x'*y'), \mu(y')\})\\ &= \wedge_{x'*y'\in[x*y]_\rho, y'\in[y]_\rho}(\min\{\mu(x'*y'), \mu(y')\})\\ &= \min(\wedge_{x'*y'\in[x*y]_\rho}\mu(x'*y'), \wedge_{y'\in[y]_\rho}\mu(y'))\\ &= \min(\rho_-(\mu)(x*y), \rho_-(\mu)(y)).\end{aligned}$$

Thus $\rho_-(\mu)$ is a fuzzy BCK-ideal of X. □

Lemma 5.4. *Let ρ be a congruence relation on X. If μ is a fuzzy subset of X and $t \in [0, 1]$, then*

(1) $(\rho_-(\mu))_t = \rho_-(\mu_t)$;
(2) $(\rho^-(\mu))_t^X = \rho^-(\mu_t^X)$.

Proof. (1) We have

$$\begin{aligned}x \in (\rho_-(\mu))_t &\Leftrightarrow \rho_-(\mu)(x) \geq t \Leftrightarrow \wedge_{a\in[x]_\rho}\mu(a) \geq t\\ &\Leftrightarrow \forall a \in [x]_\rho, \mu(a) \geq t \Leftrightarrow [x]_\rho \subseteq \mu_t \Leftrightarrow x \in \rho_-(\mu_t).\end{aligned}$$

(2) Also we have

$$\begin{aligned}x \in (\rho^-(\mu))_t^X &\Leftrightarrow \rho^-(\mu)(x) > t \Leftrightarrow \vee_{a\in[x]_\rho}\mu(a) > t\\ &\Leftrightarrow \exists a \in [x]_\rho, \mu(a) > t \Leftrightarrow [x]_\rho \cap \mu_t^X \neq \emptyset \Leftrightarrow x \in \rho^-(\mu_t^X).\end{aligned}$$

□

Theorem 5.5. *Let μ be a fuzzy quick ideal of a d-algebra X.*

(1) *If ρ is a congruence relation on X, then $\rho^-(\mu)$ is a fuzzy quick ideal of X, i.e., μ is an upper rough fuzzy quick ideal of X.*
(2) *If ρ is a congruence relation on X and $\rho_-(\mu) \neq \emptyset$, then $\rho_-(\mu)$ is a fuzzy quick ideal of X, i.e., μ is a lower rough fuzzy quick ideal of X.*

Proof. (1) Let μ be a fuzzy quick ideal of X. By Theorem 2.6, $\mu_t(t \in [0,1])$ is a quick ideal of X if $\mu_t \neq \emptyset$. Using Theorem 4.5, $\rho^-(\mu_t)$ is a quick ideal of X. It follows from Lemma 5.4(2) that $(\rho^-(\mu))_t$ is a quick ideal of X. By Theorem 2.6, $\rho^-(\mu)$ is a fuzzy quick ideal of X. Thus μ is an upper rough fuzzy quick ideal of X.

(2) It can be seen in a similar way. □

Let μ be a fuzzy subset of X and $(\rho_-(\mu), \rho^-(\mu))$ be a rough fuzzy set. If $\rho_-(\mu)$ and $\rho^-(\mu)$ are fuzzy quick ideals of X, then we call $(\rho_-(\mu), \rho^-(\mu))$ a *rough fuzzy quick ideal* of X. Therefore we have:

Corollary 5.6. *If μ is a fuzzy quick ideal of X, then $(\rho_-(\mu), \rho^-(\mu))$ is a rough fuzzy quick ideal of X. If μ, λ are fuzzy quick ideals of X, then $(\rho_-(\mu \cap \lambda), \rho^-(\mu \cap \lambda))$ is a rough fuzzy quick ideal of X.*

Theorem 5.7. *Let ρ be a congruence relation on a d-algebra X. Then μ is a lower (an upper, respectively) rough fuzzy quick ideal of X if and only if μ_t, μ_t^X are, if they are non-empty, lower (upper, respectively) rough quick ideals of X for every $t \in [0,1]$.*

Proof. By Theorem 2.6 and Lemma 5.4, we can obtain the conclusion easily. □

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