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PRECOVERS AND PREENVELOPES BY MODULES OF FINITE *FGT*-INJECTIVE AND *FGT*-FLAT DIMENSIONS

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Abstract. Let R be a ring and n a fixed non-negative integer. $\mathcal{TI}_n$ (resp. $\mathcal{TF}_n$) denotes the class of all right R-modules of FGT-injective dimensions at most n (resp. all left R-modules of FGT-flat dimensions at most n). We prove that, if R is a right Π-coherent ring, then every right R-module has a $\mathcal{TI}_n$-cover and every left R-module has a $\mathcal{TF}_n$-preenvelope. A right R-module M is called n-TI-injective in case $\mathrm{Ext}^1(N, M) = 0$ for any $N \in \mathcal{TI}_n$. A left R-module F is said to be n-TI-flat if $\mathrm{Tor}_1(N, F) = 0$ for any $N \in \mathcal{TI}_n$. Some properties of n-TI-injective and n-TI-flat modules and their relations with $\mathcal{TI}_n$-(pre)covers and $\mathcal{TF}_n$-preenvelopes are also studied.

1. Notation

In this section, we recall some known notions and facts needed in the sequel.

Throughout this paper, R is an associative ring with identity and all modules are unitary. ${}_R\mathcal{M}$(resp. $\mathcal{M}_R$) stands for the category of all left (resp. right) R-modules. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}^R_n(M, N)$). The character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. The dual module $M^* = \mathrm{Hom}(M, R)$. The cardinality of an R-module M is denoted by $\mathrm{Card}(M)$. We will use the usual notations from [1], [7], [14].

Let $\mathcal{C}$ be the class of R-modules. For an R-module M, a homomorphism $g : C \to M$ is called a $\mathcal{C}$-*cover* (see [6]) of M if $C \in \mathcal{C}$ and the following hold: (1) For any homomorphism $g' : C' \to M$ with $C' \in \mathcal{C}$, there exists a homomorphism $f : C' \to C$ with $g' = gf$. (2) If f is an endomorphism of C with $gf = g$, then f must be an automorphism. If (1) holds but (2) may not, $g : C \to M$ is called a $\mathcal{C}$-*precover*. Dually we have the definition of a $\mathcal{C}$-(pre)envelope. $\mathcal{C}$-covers and $\mathcal{C}$-envelopes may not exist in general, but if they

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exist, they are unique up to isomorphism. If every right R-module has a $\mathcal{C}$-precover, then every right R-module M has a *left $\mathcal{C}$-resolution*, that is, there is a Hom$(\mathcal{C}, -)$ exact complex $\overline{I} = \cdots \to I_1 \to I_0 \to M \to 0$ with each $I_i \in \mathcal{C}$. If $I_0 \to M$, $I_1 \to \mathrm{Ker}(I_0 \to M)$, $I_{i+1} \to \mathrm{Ker}(I_i \to I_{n-1})$ for $i \geq 1$, are $\mathcal{C}$-covers, $\overline{I}$ is called a *minimal left $\mathcal{C}$-resolution* of M. A right R-module M is said to have *left $\mathcal{C}$-dimension* $\leq n$, denoted left $\mathcal{C}$-dim $M \leq n$, if there is a left $\mathcal{C}$-resolution of the form $0 \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to M \to 0$ of M. If there is no such n, we set left $\mathcal{C}$-dim $M = \infty$.

A right R-module T is called *torsionless* if the evaluation map $\sigma : T \to T^{**}$ is injection. A ring R is said to be *right* Π-*coherent* if every finitely generated torsionless right R-module is finitely presented (see [3]). It is well known that right Noetherian rings $\Rightarrow$ right Π-coherent rings $\Rightarrow$ right coherent rings. The *right FGT-injective dimension* of a right R-module M (see [4]), denoted by $FGT - id(M)$, is defined as the least non-negative integer n such that $\mathrm{Ext}^{n+1}(T, M) = 0$ for any finitely generated torsionless right R-module T. The *left FGT-flat dimension* of a left R-module F, denoted by $FGT - fd(F)$, is defined as the least non-negative integer n such that $\mathrm{Tor}_{n+1}(T, F) = 0$ for any finitely generated torsionless right R-module T. A right R-module M is called *FGT-injective* if $\mathrm{Ext}^1(T, M) = 0$ for any finitely generated torsionless right R-module T. A left R-module F is called *FGT-flat* if $\mathrm{Tor}_1(T, F) = 0$ for any finitely generated torsionless right R-module T. We write $\mathcal{TI}_n$ (resp. $\mathcal{TF}_n$) for the class of all right R-modules of FGT-injective dimensions at most n (resp. all left R-modules of FGT-flat dimensions at most n).

The following lemmas due to [4, Corollary 5.5.6] and [4, Proposition 5.6.11], respectively.

Lemma 1.1. *Let R be a right Π-coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules with B FGT-injective. If A is FGT-injective, so is C. If A is not FGT-injective and $FGT - id(A) < \infty$, then $FGT - id(A) = FGT - id(C) + 1$.*

Lemma 1.2. *Let R be a ring. Then*

(1) *$FGT - fd(M) = FGT - id(M^+)$ for any left R-module M.*
(2) *If R is right Π-coherent, then $FGT - id(N) = FGT - fd(N^+)$ for any right R-module N.*

2. Introduction

Precovers and preenvelopes were introduced by Enochs in 1980's [6]. Its turn out to be extremely fruitful for general module theory as well as for representation theory. The idea behind these concepts is to exploit interesting features of a special class of R-modules for the study of the whole module category. In particular, the existence of precovers and preenvelopes is also studied by many authors (see [2], [7], [9], [10], [12], [13], [16]). Let R be a right Π-coherent ring. In Section 3 of this paper, we consider the existence of $\mathcal{TI}_n$-precovers and $\mathcal{TF}_n$-preenvelopes and obtain the relation between $\mathcal{TI}_n$-precovers and $\mathcal{TF}_n$-preenvelopes. Moreover, we show when every right R-module has an epic $\mathcal{TI}_n$-cover and when every left R-module has a monic $\mathcal{TF}_n$-preenvelope.

We introduce the concepts of n-TI-injective and n-TI-flat modules and obtain some interesting properties in Section 4. It is shown that a right R-module M is reduced n-TI-injective if and only if M is the kernel of a $\mathcal{TI}_n$-cover. Furthermore, M is n-TI-injective if and only if it is a direct sum of an injective right R-module and a reduced n-TI-injective right R-module. If R is a commutative ring, we show that a simple R-module S is n-TI-injective if and only if it is n-TI-flat. We get a new characterization of QF-ring in terms of n-TI-injective right R-modules. For a right Π-coherent ring R, if C is the cokernel of a $\mathcal{TF}_n$-preenvelope $f : M \to F$ of a left R-module M with F flat, then C is n-TI-flat, and if L is a finitely presented n-TI-flat right R-module, then L is the cokernel of a $\mathcal{TF}_n$-preenvelope $g : K \to P$ with P flat. We call a ring R *weakly n-Gorenstein* if it is left and right Π-coherent and if $FGT - id(_RR) \leq n$ and $FGT - id(R_R) \leq n$ for integer $n \geq 0$. It is shown that, if R is left and right Π-coherent, then R is weakly 1-Gorenstein if and only if every closed submodule of a finitely generated n-TI-flat (left or right) R-module is n-TI-flat. Finally, we study weakly n-Gorenstein rings with finitely FGT-injective dimensions.

3. $\mathcal{TI}_n$-precovers and $\mathcal{TF}_n$-preenvelopes

The aim of this section is to study the existence of $\mathcal{TI}_n$-(pre)covers and $\mathcal{TF}_n$-preenvelopes. It is easy to verify that $\mathcal{TI}_n$ is closed under extensions, direct products and direct summands, and $\mathcal{TF}_n$ is closed under extensions, direct sums and direct summands. If R is right Π-coherent, then $\mathcal{TI}_n$ is closed under direct sums and $\mathcal{TF}_n$ is closed under direct products. Moreover, we have the following:

Lemma 3.1. *Let R be a right Π-coherent ring. Then $\mathcal{TI}_n$ and $\mathcal{TF}_n$ are closed under pure submodules and pure quotient modules.*

Proof. Let $0 \to A' \to A \to A'' \to 0$ be a pure exact sequence of right R-modules with $FGT - id(A) \leq n$. Then we have a split exact sequence $0 \to (A'')^+ \to A^+ \to (A')^+ \to 0$. By Lemma 1.2(2), $FGT - fd(A^+) \leq n$. Thus $FGT - fd((A')^+) \leq n$ and $FGT - fd((A'')^+) \leq n$. By Lemma 1.2(2), $FGT - id(A') \leq n$ and $FGT - id(A'') \leq n$.

Now let $0 \to A' \to A \to A'' \to 0$ be a pure exact sequence of left R-modules and $FGT - fd(A) \leq n$. Then we have a split exact sequence $0 \to (A'')^+ \to A^+ \to (A')^+ \to 0$. By Lemma 1.2(1), $FGT - id(A^+) \leq n$. Thus $FGT - id((A')^+) \leq n$ and $FGT - id((A'')^+) \leq n$. Therefore, $FGT - fd(A') \leq n$ and $FGT - fd(A'') \leq n$ by Lemma 1.2(1) again. □

The next lemma is a special case of [2, Theorem 5].

Lemma 3.2. *Let R be a ring. Then for each cardinal λ, there is a cardinal κ such that any R-module M and for any $L \leq M$ with* $\mathrm{Card}(M) \geq \kappa$ *and* $\mathrm{Card}(M/L) \leq \lambda$, *the submodule L contains a nonzero submodule that is pure in M.*

Proposition 3.3. *Let R be a right* Π*-coherent ring. There is a cardinal number κ such that any morphism $\varphi : D \to M$ with $D \in \mathcal{TI}_n$ has a factorization $D \to C \to M$ with $C \in \mathcal{TI}_n$ and* $\mathrm{Card}(C) \leq \kappa$.

Proof. Let M be a right R-module with $\mathrm{Card}(M) = \lambda$, and let κ be a cardinal as in Lemma 3.2. Take a morphism $\varphi : D \to M$ with $D \in \mathcal{TI}_n$, $K = \mathrm{Ker}(\varphi)$. If $\mathrm{Card}(D) \leq \kappa$, then consider the factorization of $D \to M$ as $D \to D \to M$, where the first arrow is the identity.

If $\mathrm{Card}(D) > \kappa$. There is K' maximal with the properties that $K' \subseteq K \subseteq D$ and that K' is a pure submodule of D. So φ has the factorization $D \to D/K' \to M$ in terms of [1, Theorem 3.6]. By Lemma 3.1, $D/K' \in \mathcal{TI}_n$. We claim that $\mathrm{Card}(D/K') \leq \kappa$. Otherwise, if $\mathrm{Card}(D/K') > \kappa$, consider $K/K' \subseteq D/K'$. Since D/K is isomorphic to a submodule of M,

$$\mathrm{Card}(\frac{D/K'}{K/K'}) = \mathrm{Card}(D/K) \leq \mathrm{Card}(M) = \lambda.$$

In view of Lemma 3.2, there exists $0 \neq K''/K' \subseteq K/K' \subseteq D/K'$ such that K''/K' is a pure submodule of D/K'. It is clear that $K' \subsetneqq K'' \subseteq K \subseteq D$. By [8, Proposition 7.2], K'' is a pure submodule of D, contradicting the maximality of K'. So let $C = D/K'$, $\mathrm{Card}(C) \leq \kappa$, as desired. □

Theorem 3.4. *Let R be a right* Π*-coherent ring. Then every right R-module has a $\mathcal{TI}_n$-precover.*

Proof. It follows from Proposition 3.3 and [7, Proposition 5.2.2]. □

Remark 3.5. (1) We can prove that $\mathcal{TI}_n$ is closed under direct limits over a right Π-coherent ring. In fact, by [4, Proposition 5.5.3], there is an isomorphism: $\varinjlim \mathrm{Ext}^{n+1}(A, B_i) \cong \mathrm{Ext}^{n+1}(A, \varinjlim B_i)$, where A is a finitely generated torsionless right R-module and $\{B_i | i \in I\}$ is an inductive system of right R-modules. Then, in view of [7, Corollary 5.2.7] and Theorem 3.4, every right R-module has a $\mathcal{TI}_n$-cover.

(2) Let R be a right Π-coherent ring. By [10, Theorem 3.4], every right R-module M has a left $\mathcal{TI}_0$-resolution $\overline{I} = \cdots \to I_n \to \cdots \to I_0 \to M \to 0$. Let $K_0 = M, K_1 = \mathrm{Ker}(I_0 \to M), K_i = \mathrm{Ker}(I_{i-1} \to I_{i-2})$ for $i \geq 2$. We call $K_i (i \geq 0)$ the *nth* $\mathcal{TI}_0$*-syzygy* of M. By [18, Lemma 2.2], $I_n \to K_n$ is a $\mathcal{TI}_n$-precover of K_n.

Theorem 3.6. *If R is a right* Π*-coherent ring, then every left R-module has a $\mathcal{TF}_n$-preenvelope.*

Proof. Let M be a left R-module, and let $\mathrm{Card}(M) = \aleph_\beta$. Then by [7, Lemma 5.3.12], there is an infinite cardinal $\aleph_\alpha$ such that if $FGT - fd(F) \leq n$ and S is a submodule of F with $\mathrm{Card}(S) \leq \aleph_\beta$, there exists a pure submodule G of F such that $S \subset G$ and $\mathrm{Card}(G) \leq \aleph_\alpha$, where cardinal number $\aleph_\alpha$ dependent on $\mathrm{Card}(S)$ and $\mathrm{Card}(R)$. Note that $FGT - fd(G) \leq n$ by Lemma 3.1. In addition, $\mathcal{TF}_n$ is closed under direct products, so M has a $\mathcal{TF}_n$-preenvelope by [7, Corollary 6.2.2]. □

The following proposition elaborates the relationship between $\mathcal{TI}_n$-precovers and $\mathcal{TF}_n$-preenvelopes.

Proposition 3.7. *Let R be a right Π-coherent ring. If $\varphi : M \to F$ is a $\mathcal{TF}_n$-preenvelope of left R-module M, then $\varphi^+ : F^+ \to M^+$ is a $\mathcal{TI}_n$-precover of M^+.*

Proof. By Lemma 1.2(1), $F^+ \in \mathcal{TI}_n$ since $F \in \mathcal{TF}_n$. For any homomorphism $g : D \to M^+$ with $D \in \mathcal{TI}_n$, we have $g^+ : M^{++} \to D^+$, hence $g^+\sigma_M : M \to D^+$, where $\sigma_M : M \to M^{++}$ is an evaluation map. By Lemma 1.2(2), $D^+ \in \mathcal{TF}_n$ since R is right Π-coherent. Thus there exists a morphism $f : F \to D^+$ such that $f\varphi = g^+\sigma_M$. Whence $\sigma_M^+ g^{++} = \varphi^+ f^+$. Since $g^{++}\sigma_D = \sigma_{M^+} g$. Let $f^+\sigma_D : D \to F^+$, note $\sigma_M^+\sigma_{M^+} = 1_{M^+}$, then $\varphi^+ f^+ \sigma_D = \sigma_M^+ g^{++}\sigma_D = \sigma_M^+\sigma_{M^+} g = g$. Therefore $\varphi^+ : F^+ \to M^+$ is a $\mathcal{TI}_n$-precover. □

In general, $\mathcal{TI}_n$-cover need not be an epimorphism and $\mathcal{TF}_n$-preenvelope need not be a monomorphism. In the following theorem, we will consider when every right R-module has an epic $\mathcal{TI}_n$-cover and when every left R-module has a monic $\mathcal{TF}_n$-preenvelope.

Theorem 3.8. *Let R be right Π-coherent. Then the following are equivalent*:

(1) $FGT - id(R_R) \leq n$.
(2) *For any right R-module, there is an epic $\mathcal{TI}_n$-cover.*
(3) *For any left R-module, there is a monic $\mathcal{TF}_n$-preenvelope.*
(4) *Every injective (FP-injective) left R-module belongs to $\mathcal{TF}_n$.*
(5) *Every flat right R-module belongs to $\mathcal{TI}_n$.*

Proof. (1) ⇒ (2). In view of Remark 3.5, every right R-module has a $\mathcal{TI}_n$-cover. By assumption, any projective right R-module belongs to $\mathcal{TI}_n$. Thus any $\mathcal{TI}_n$-cover is epic.

(2) ⇒ (1) is clear since R_R has an epic $\mathcal{TI}_n$-cover.

(1) ⇒ (3). Let M be any left R-module. Then M has a $\mathcal{TF}_n$-preenvelope $f : M \to F$ by Theorem 3.6. Since $(R_R)^+$ is a cogenerator in the category of left R-modules, there is an exact sequence $0 \to M \to \prod (R_R)^+$. By Lemma 1.2 (2), $FGT - fd((R_R)^+) = FGT - id(R_R) \leq n$ since R is right Π-coherent, and so $FGT - fd(\prod (R_R)^+) \leq n$. Thus f is monic, and hence (3) follows.

(3) ⇒ (4). Let N be an FP-injective left R-module. By assumption, there is a pure exact sequence $0 \to N \to F \to L \to 0$ with $F \in \mathcal{TF}_n$. Then N belongs to $\mathcal{TF}_n$ in terms of Lemma 3.1.

(4) $\Rightarrow$ (5). Let M be a flat right R-module. Then M^+ is injective left R-module. By (4), $M^+ \in \mathcal{TF}_n$. Thus $M \in \mathcal{TI}_n$ by Lemma 1.2(2).

(5) $\Rightarrow$ (1) is trivial. □

Let $n = 0$ in Theorem 3.8. Then we have the following result as corollary which have been prove in [10].

Corollary 3.9. *Let R be right Π-coherent. Then the following are equivalent*:

(1) *R_R is FGT-injective.*
(2) *For any right R-module, there is an epic FGT-injective cover.*
(3) *For any left R-module, there is a monic FGT-flat preenvelope.*
(4) *Every injective (FP-injective) left R-module is FGT-flat.*
(5) *Every flat right R-module is FGT-injective.*

A homomorphism $g : M \to C$ with $C \in \mathcal{C}$ is said to be a *$\mathcal{C}$-envelope with the unique mapping property* (see [5]) if for any homomorphism $g' : M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $f : C \to C'$ such that $fg = g'$. Dually, we have the definition of *$\mathcal{C}$-cover with the unique mapping property.*

We conclude this section with the following result which is of independent interest.

Proposition 3.10. *Let R be a ring. If every right R-module has a $\mathcal{TI}_n$-cover with unique mapping property, then $\mathcal{TI}_n$ is closed under direct limits.*

Proof. Let $\{I_i, \varphi_j^i\}$ be a direct system with each $I_i \in \mathcal{TI}_n$. By assumption, $\varinjlim I_i$ has a $\mathcal{TI}_n$-cover $g : I \to \varinjlim I_i$ with the unique mapping property. Suppose that $\alpha : I_i \to \varinjlim I_i$ satisfy $\alpha_i = \alpha_j \varphi_j^i$ whenever $i \leq j$. Then there exists $f_i : I_i \to I$ such that $\alpha_i = gf_i$ for any i, so $gf_i = \alpha_j \varphi_j^i = gf_j \varphi_j^i$. Hence $f_i = f_j \varphi_j^i$ by the unique mapping property of g. Thus there exists $h : \varinjlim I_i \to I$ such that $h\alpha_i = f_i$, hence $(gh)\alpha_i = gf_i = \alpha_i$ for any i. Then $gh = 1_{\varinjlim I_i}$ by the definition of direct limits. So $\varinjlim I_i$ is a direct summand of I, and hence $\varinjlim I_i \in \mathcal{TI}_n$. □

4. n-TI-injective and n-TI-flat modules

Definition 4.1. Let R be a ring, n a fixed non-negative integer. A right R-module M is said to be *n-TI-injective* if $\mathrm{Ext}^1(N, M) = 0$ for any $N \in \mathcal{TI}_n$. A left R-module F is called *n-TI-flat* if $\mathrm{Tor}_1(N, F) = 0$ for any $N \in \mathcal{TI}_n$.

Remark 4.2. (1) By Wakamutsu's Lemma [16, Lemma 2.1.1], any kernel of $\mathcal{TI}_n$-cover is n-TI-injective.

(2) It is clear that 0-TI-injective (resp. 0-TI-flat) R-modules are TI-injective (resp. TI-flat) R-modules in sense of [17]. If $m \geq n$, then m-TI-injective (resp. m-TI-flat) R-modules are n-TI-injective (resp. n-TI-flat) R-modules.

(3) A left R-module F is n-TI-flat if and only if F^+ is n-TI-injective by the standard isomorphism $\mathrm{Ext}^1(N, F^+) \cong \mathrm{Tor}_1(N, F)^+$ for any $N \in \mathcal{TI}_n$.

Proposition 4.3. *The following are equivalent for a right R-module M:*

(1) *M is n-TI-injective.*

(2) *For every exact sequence $0 \to M \to A \to B \to 0$ with $A \in \mathcal{TI}_n$, $A \to B$ is a $\mathcal{TI}_n$-precover of B.*

(3) *M is the kernel of a $\mathcal{TI}_n$-precover $f : A \to B$ with A injective.*

(4) *M is injective with respect to every exact sequence $0 \to K \to A \to C \to 0$, where $C \in \mathcal{TI}_n$.*

Proof. (1)⇒(2) and (1)⇒(4) are trivial.

(2)⇒(3) is obvious since there is an exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$, where $E(M)$ is the injective hull of M.

(3)⇒(1). Let M be a kernel of a $\mathcal{TI}_n$-precover $f : A \to B$ with A injective. Then there is an exact sequence $0 \to M \to A \to A/M \to 0$. For any right R-module $N \in \mathcal{TI}_n$, the sequence $\mathrm{Hom}(N, A) \xrightarrow{\pi} \mathrm{Hom}(N, A/M) \to \mathrm{Ext}^1(N, M) \to 0$ is exact. Note that $A \to A/M$ is also a $\mathcal{TI}_n$-precover, so π is epic. Thus $\mathrm{Ext}^1(N, M) = 0$, and hence M is n-TI-injective.

(4)⇒(1). For any right R-module $C \in \mathcal{TI}_n$, there exists an exact sequence $0 \to K \to A \to C \to 0$ with A projective, which induces an exact sequence $\mathrm{Hom}(A, M) \xrightarrow{\pi} \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(C, M) \to 0$. By assumption, π is epic. So $\mathrm{Ext}^1(C, M) = 0$, and hence M is n-TI-injective. □

It is clear that every injective right R-module (resp. flat left R-module) is n-TI-injective (resp. n-TI-flat) by Definition 4.1. The converse is not true in general. However, if R is a right Π-coherent ring, we have:

Proposition 4.4. *Let R be a right Π-coherent ring. Then the following statements hold.*

(1) *A right R-module M is injective if and only if M is n-TI-injective and $FGT - id(M) \leq n + 1$.*

(2) *A left R-module F is flat if and only if F is n-TI-flat and $FGT - fd(F) \leq n + 1$.*

Proof. (1) (⇒) is clear.

(⇐). Let M be a n-TI-injective right R-module. Then there is an exact sequence $0 \to M \to E \to N \to 0$ with E injective. By Lemma 1.1, $FGT - id(N) \leq n$. Thus $\mathrm{Ext}^1(N, M) = 0$, and hence the exact sequence is split. Then M is injective.

(2) (⇒) is clear.

(⇐). For any n-TI-flat left R-module F. By Remark 4.2(3), F^+ is n-TI-injective right R-module. By Lemma 1.2(1), $FGT - id(F^+) \leq n+1$. Then F^+ is injective by (1). So F is flat. □

A right R-module M is called *reduced* (see [16]) if M has no nonzero injective submodules.

Proposition 4.5. *Let M be a right R-module over a right Π-coherent ring R. Then the following are equivalent:*

(1) *M is reduced n-TI-injective.*
(2) *M is the kernel of a $\mathcal{TI}_n$-cover $f : A \to B$ with A injective.*

Proof. (1)⇒(2). By Proposition 4.3, the nature map $\pi : E(M) \to E(M)/M$ is a $\mathcal{TI}_n$- precover of $E(M)/M$. But $E(M)/M$ has a $\mathcal{TI}_n$-cover by Remark 3.5. $E(M)$ has no nonzero direct summand K contained in M since M is reduced. By [16, Corollary 1.2.8], $\pi : E(M) \to E(M)/M$ is a $\mathcal{TI}_n$-cover of $E(M)/M$.

(2)⇒(1). Let M be the kernel of a $\mathcal{TI}_n$-cover $f : A \to B$ with A injective. So M is n-TI-injective by Proposition 4.3. Now let K be an injective submodule of M. Suppose $A = K \oplus L$. $p : A \to L$ is projection and $i : L \to A$ is inclusion. Note $f(ip) = f$ since $f(K) = 0$. Thus ip is an isomorphism since f is cover. So i is epic, $A = L$. Then $K = 0$, and hence M is reduced. □

Now we get a construction theorem of n-TI-injective R-module.

Theorem 4.6. *Let M be a right R-module over a right Π-coherent ring R. Then the following are equivalent*:

(1) *M is n-TI-injective.*
(2) *M is a direct sum of an injective right R-module and a reduced n-TI-injective right R-module.*

Proof. The proof is modeled on that of [11, Theorem 2.6].

(2)⇒(1) is trivial.

(1)⇒(2). We consider the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. By Proposition 4.3, $E(M) \to E(M)/M$ is a $\mathcal{TI}_n$-precover of $E(M)/M$. Since R is right Π-coherent, by Remark 3.5(1), $E(M)/M$ admits a $\mathcal{TI}_n$-cover $F \to E(M)/M$, and hence we get the following commutative diagram with rows exact:

$$\begin{array}{ccccccc}
0 & \to & K & \xrightarrow{f} & F & \to & E(M)/M & \to 0 \\
 & & \phi\downarrow & & \gamma\downarrow & & \downarrow & \\
0 & \to & M & \to & E(M) & \to & E(M)/M & \to 0 \\
 & & \sigma\downarrow & & \beta\downarrow & & \downarrow & \\
0 & \to & K & \xrightarrow{f} & F & \to & E(M)/M & \to 0.
\end{array}$$

Note that $\beta\gamma$ is an isomorphism, and hence $E(M) \cong \mathrm{Ker}(\beta) \oplus \mathrm{im}(\gamma)$. Thus F and $\mathrm{Ker}(\beta)$ are also injective. Therefore, K is reduced n-TI-injective by Proposition 4.5. On the other hand, by the Five Lemma, we have $\sigma\phi$ is isomorphic. Thus $M \cong \mathrm{Ker}(\sigma) \oplus \mathrm{im}(\phi)$, where $\mathrm{im}(\phi) \cong K$. So we have the commutative diagram:

$$\begin{array}{cccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathrm{Ker}(\sigma) & \to & \mathrm{Ker}(\beta) & \to & 0 & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & M & \xrightarrow{\alpha} & E(M) & \to & E(M)/M & \to 0 \\
 & & \sigma\downarrow & & \beta\downarrow & & \downarrow & \\
0 & \to & K & \xrightarrow{f} & F & \to & E(M)/M & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & . \\
 & & 0 & & 0 & & 0 &
\end{array}$$

Hence $\operatorname{Ker}(\sigma) \cong \operatorname{Ker}(\beta)$ by [14, Exercise 6.16]. This completes the proof. □

Proposition 4.7. *Let S be a simple R-module over a commutative ring R. Then the following are equivalent*:

(1) *S is n-TI-injective.*
(2) *S is n-TI-flat.*

Proof. Suppose that $\{S_i\}_{i\in I}$ is an irredundant set of representatives of the simple R-modules. Let $E = E(\oplus_{i\in I} S_i)$, the injective hull of $\oplus_{i\in I} S_i$. Then E is an injective cogenerator. For any $N \in \mathcal{TI}_n$, there exists an isomorphism $\operatorname{Ext}^1(N, \operatorname{Hom}(S,E)) \cong \operatorname{Hom}(\operatorname{Tor}_1(N,S),E)$. Note that $\operatorname{Hom}(S,E) \cong S$. Thus S is n-TI-injective if and only if $\operatorname{Ext}^1(N,\operatorname{Hom}(S,E)) = 0$ if and only if $\operatorname{Hom}(\operatorname{Tor}_1(N,S),E) = 0$ if and only if $\operatorname{Tor}_1(N,S) = 0$ if and only if S is n-TI-flat. □

Proposition 4.8. *Let R be a commutative Π-coherent ring and F be a flat R-module. Then the following statements hold.*

(1) *M is n-TI-injective if and only if $\operatorname{Hom}(F,M)$ is n-TI-injective.*
(2) *N is n-TI-flat if and only if $F \otimes N$ is n-TI-flat.*

Proof. (1) ($\Leftarrow$) holds by letting $F = R$.

($\Rightarrow$). For any FGT-injective R-module E and flat R-module F, we claim that $E \otimes F$ is FGT-injective. In fact, any finitely generated torsionless R-module T is finitely presented since R is Π-coherent, then there is an exact sequence $0 \to K \to P \to T \to 0$ with P and K finitely generated and P free, so P and K are finitely presented. On the other hand, the sequence $\operatorname{Hom}(P,E)\otimes F \to \operatorname{Hom}(K,E)\otimes F \to 0$ is exact since E is FGT-injective. Furthermore, we have the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}(P,E)\otimes F & \to & \operatorname{Hom}(K,E)\otimes F \to 0 \\
\alpha \downarrow & & \beta \downarrow \\
\operatorname{Hom}(P,E\otimes F) & \to & \operatorname{Hom}(K,E\otimes F)
\end{array}$$

Since P and K are finitely presented, by [7, Theorem 3.2.14], α and β are isomorphisms. Then $\operatorname{Hom}(P,E\otimes F) \to \operatorname{Hom}(K,E\otimes F) \to 0$ is exact. Thus $\operatorname{Ext}^1(T,E\otimes F) = 0$, and hence $E\otimes F$ is FGT-injective.

Then, if $I \in \mathcal{TI}_n$, by the result above and [4, Proposition 5.5.4], $I\otimes F \in \mathcal{TI}_n$.

Now we prove that $\operatorname{Hom}(F,M)$ is n-TI-injective. For any $I \in \mathcal{TI}_n$, there exists an exact sequence $0 \to K_1 \to P_1 \to I \to 0$ with P_1 projective. Then we have an induced exact sequence

$$\operatorname{Hom}(P_1\otimes F, M) \to \operatorname{Hom}(K_1\otimes F, M) \to \operatorname{Ext}^1(I\otimes F, M) = 0.$$

So the sequence

$$\operatorname{Hom}(P_1, \operatorname{Hom}(F,M)) \to \operatorname{Hom}(K_1, \operatorname{Hom}(F,M)) \to 0$$

is exact. Thus $\operatorname{Ext}^1(I, \operatorname{Hom}(F,M)) = 0$. Therefore, $\operatorname{Hom}(F,M)$ is n-TI-injective.

(2) N is n-TI-flat if and only if N^+ is n-TI-injective if and only if Hom(F, N^+) is n-TI-injective by (1) if and only if $(F \otimes N)^+$ is n-TI-injective by the standard isomorphism $(F \otimes N)^+ \cong \mathrm{Hom}(F, N^+)$ if and only if $F \otimes N$ is n-TI-flat. □

In the following proposition, we consider the relationship between n-TI-flat modules and the cokernels of $\mathcal{TF}_n$-preenvelopes.

Proposition 4.9. *Let R be a right Π-coherent ring. Then the following statements hold.*

(1) *If C is the cokernel of a $\mathcal{TF}_n$-preenvelope $f : M \to F$ of a left R-module M with F flat, then C is n-TI-flat.*
(2) *If L is a finitely presented n-TI-flat left R-module, then L is the cokernel of a $\mathcal{TF}_n$-preenvelope $g : K \to P$ with P flat.*

Proof. (1). There is an exact sequence of left R-modules $0 \to \mathrm{im}(f) \to F \to C \to 0$. Using functor $N \otimes -$ with $N \in \mathcal{TI}_n$, we have an exact sequence

$$0 \to \mathrm{Tor}_1(N, C) \to N \otimes \mathrm{im}(f) \to N \otimes F.$$

Note that $\mathrm{im}(f) \to F$ is also a $\mathcal{TF}_n$-preenvelope and $N^+ \in \mathcal{TF}_n$. Then the sequence $\mathrm{Hom}(F, N^+) \to \mathrm{Hom}(\mathrm{im}(f), N^+) \to 0$ is exact. So $(N \otimes F)^+ \to (N \otimes \mathrm{im}(f))^+ \to 0$ is exact. Thus we have exact sequence $0 \to N \otimes \mathrm{im}(f) \to N \otimes F$, so $\mathrm{Tor}_1(N, C) = 0$. Then C is n-TI-flat.

(2). Let L be a finitely presented n-TI-flat left R-module. There is an exact sequence $0 \to K \xrightarrow{i} P \to L \to 0$ with P finitely generated projective and K finitely generated. It is enough to show that $i : K \to P$ is a $\mathcal{TF}_n$-preenvelope. In fact, for any left R-module $F \in \mathcal{TF}_n$, we have $\mathrm{Tor}_1(F^+, L) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{array}{ccccc} 0 & \to & F^+ \otimes K & \overset{1_{F^+} \otimes i}{\to} & F^+ \otimes P \\ & & \alpha \downarrow & & \beta \downarrow \\ & & \mathrm{Hom}(K, F)^+ & \overset{h}{\to} & \mathrm{Hom}(P, F)^+. \end{array}$$

Note that α is an epimorphism and β is an isomorphism by [7, Theorem 3.2.11]. Thus h is a monomorphism, and hence $\mathrm{Hom}(P, F) \to \mathrm{Hom}(K, F)$ is epic, as required. □

Lemma 4.10. *Let R be a right Π-coherent ring. Then*

$$FGT - id(R_R) = \sup\{FGT - fd(_R E) | E \text{ injective left } R\text{-module}\}.$$

Proof. Assume that $FGT - id(R_R) = n < \infty$. Then $\mathrm{Ext}^{n+1}(T, R) = 0$ for every finitely generated torsionless right R-module T. Since R is right Π-coherent, T is finitely presented. Then, for any injective left R-module E,

$$\mathrm{Tor}_{n+1}(T, E) \cong \mathrm{Tor}_{n+1}(T, \mathrm{Hom}(R, E)) \cong \mathrm{Hom}(\mathrm{Ext}^{n+1}(T, R), E) = 0,$$

and so, it follows that $FGT-fd(E)\leq n$. Conversely, let $\sup\{FGT-fd(_RM) \mid M$ injective left R-module$\}= n < \infty$. Since R is right Π-coherent, $FGT-id(R_R) = FGT-fd((R_R)^+) \leq n$ by Lemma 1.2(2). □

Following [4], let $l.FGT-IF.\dim(R) = \sup\{l.FGT-fd(_RE) \mid E$ injective left R-module$\}$. Similarly, we have the definition of $r.FGT-IF.\dim(R)$. By Lemma 4.10, if R is a left and right Π-coherent ring, then $FGT-id(R_R) = l.FGT-IF.\dim(R)$ and $FGT-id(_RR) = r.FGT-IF.\dim(R)$.

Proposition 4.11. *Let R be a left and right Π-coherent ring, $FGT-id(_RR) \leq n$ and $FGT-id(R_R) \leq n$ for integer $n \geq 0$. Then the following are equivalent for any (left or right) R-module M:*

(1) $FGT-id(M) < \infty$.
(2) $FGT-id(M) \leq n$.
(3) $FGT-fd(M) < \infty$.
(4) $FGT-fd(M) \leq n$.

Proof. We only prove the right case. The left case is similar.

(2)⇒(1) and (4)⇒(3) are trivial.

(3)⇒(2). Since $FGT-fd(M_R) < \infty$, in view of [4, Theorem 5.6.16(ii)], $FGT-fd((M_R)^+) \leq l.FGT-IF.\dim(R) = FGT-id(R_R) \leq n$. Thus $FGT-id(M_R) = FGT-fd((M_R)^+) \leq n$ by Lemma 1.2(2).

(1)⇒(4). Assume that $FGT-id(M_R) < \infty$. By [4, Proposition 5.6.16(iii)], $FGT-fd(M_R) \leq r.FGT-IF.\dim(R) = FGT-id(_RR) \leq n$. □

Definition 4.12. A ring R is called *weakly n-Gorenstein* if it is left and right Π-coherent and if $FGT-id(_RR) \leq n$ and $FGT-id(R_R) \leq n$ for integer $n \geq 0$.

Remark 4.13. (1) Obviously, every n-Gorenstein ring [7] (that is, R is a left and right Noetherian ring and $id(R_R) \leq n$ and $id(_RR) \leq n$) is a weakly n-Gorenstein ring. But the converse is not true in general. For example, let F be a field and V be an infinite dimensions vector space over F. Then $R = \mathrm{End}_F V$ is a weakly 0-Gorenstein ring but it is not a 0-Gorenstein ring because it is not Noetherian.

(2) Recall that R is a *QF-ring* [1](i.e., 0-Gorenstein ring) if R is left and right noetherian and R_R and $_RR$ are injective. Here we have a new characterization of QF-ring.

Theorem 4.14. *R is a QF-ring if and only if every (left or right) R-module is n-TI-injective.*

Proof. If R is a QF-ring, then R is weakly 0-Gorenstein ring by Remark 4.13 (1). For any R-module $N \in \mathcal{TI}_n$, in view of Proposition 4.11, N is FGT-injective. By [15, Remark 5], R is also a D-ring, so N is injective in terms of [4, Proposition 5.5.1], and hence N is projective by [1, Theorem 31.9]. Thus every R-module is n-TI-injective. Conversely, note that, for any injective right

R-module M, $FGT - id(M) \leq n$. By assumption, any R-module N is n-TI-injective, so $\mathrm{Ext}^1(M,N) = 0$, hence M is projective. Therefore, R is a QF-ring by [1, Theorem 31.9] again. □

Let K be a submodule of left (or right) R-module M. K is called a *closed submodule* (see [15]) if M/K is torsionless.

Proposition 4.15. *Let R be a left and right Π-coherent ring. Then the following are equivalent.*

(1) *R is weakly 1-Gorenstein.*
(2) *Every closed submodule of a finitely generated n-TI-flat (left or right) R-module is n-TI-flat.*

Proof. (1)⇒(2). Let K be a closed submodule of a finitely generated n-TI-flat left R-module M. For any right R-module $N \in \mathcal{TI}_n$, there is an exact sequence

$$\mathrm{Tor}_2(N, M/K) \to \mathrm{Tor}_1(N, K) \to \mathrm{Tor}_1(N, M) = 0.$$

By Proposition 4.11, $FGT - fd(N_R) \leq 1$. Note that M/K is finitely generated torsionless, so $\mathrm{Tor}_2(N, M/K) = 0$. Thus $\mathrm{Tor}_1(N, K) = 0$, and hence K is n-TI-flat.

(2)⇒(1). For any finitely generated torsionless left R-module M, there is an exact sequence $0 \to K \to F \to M \to 0$, where K is a closed submodule of a finitely generated free R-module F. So there is an induced exact sequence

$$0 = \mathrm{Tor}_2(({}_RR)^+, F) \to \mathrm{Tor}_2(({}_RR)^+, M) \to \mathrm{Tor}_1(({}_RR)^+, K) \to \cdots .$$

By assumption, K is n-TI-flat. Then $\mathrm{Tor}_1(({}_RR)^+, K)$=0, and hence $\mathrm{Tor}_2(({}_RR)^+, M) = 0$. So $FGT - fd(({}_RR)^+) \leq 1$. By Lemma 1.2(2), $FGT - id({}_RR) \leq 1$.

Similarly, we can prove that $FGT - id(R_R) \leq 1$. □

Set $FGT - I.\dim(R) = \sup\{FGT - id(M) \mid M \in \mathcal{M}_R\}$ and call $FGT - I.\dim(R)$ *right FGT-injective dimension of R*. In the end of this article, we give a theorem which character the weakly n-Gorenstein rings with finite FGT-injective dimensions. It needs the following lemmas.

Lemma 4.16. *Let R be a right Π-coherent ring. Then every* $(\mathrm{n}+1)$*th $\mathcal{TI}_0$-syzygy of minimal left $\mathcal{TI}_0$-resolution of any right R-module is n-TI-injective.*

Proof. Let $\overline{I} = \cdots \to I_n \to \cdots \to I_0 \to M \to 0$ be a minimal left $\mathcal{TI}_0$-resolution of M. By Remark 3.5(2), $I_n \to K_n$ is a $\mathcal{TI}_n$-precover, where K_n is the nth $\mathcal{TI}_0$-syzygy of $\overline{I}$. Note that $I_n \to K_n$ is also a $\mathcal{TI}_0$-cover, then $I_n \to K_n$ is a $\mathcal{TI}_n$-cover of K_n. By Remark 4.2(1), the $(n+1)$th $\mathcal{TI}_0$-syzygy K_{n+1} of $\overline{I}$ is n-TI-injective. □

Lemma 4.17. *Let R be a right Π-coherent ring with $FGT - id(R_R) \leq n$ and $n \geq 1$. If M is an $(n-1)$-TI-injective right R-module, then there is an exact sequence $0 \to K \to E \to M \to 0$ such that E is FGT-injective and K is n-TI-injective.*

Proof. The proof is similar to that of [12, Lemma 3.3(1)]. □

Theorem 4.18. *Let R be a weakly n-Gorenstein ring and integer $n \geq 1$. Then the following are equivalent*:

(1) $FGT - I.\dim(R) < \infty$.
(2) $FGT - I.\dim(R) \leq n$.
(3) *Every n-TI-injective right R-module is FGT-injective.*
(4) *Every n-TI-injective right R-module has a monic FGT-injective cover.*
(5) *Every $((n-1)$-TI-injective$)$ right R-module has a monic $\mathcal{TI}_{n-1}$-cover.*

Proof. (1)⇒(2) follows from Proposition 4.11.

(2)⇒(3). For any n-TI-injective right R-module M and any finitely generated torsionless right R-module N, note that $FGT - id(N) \leq n$ by (2), then $\mathrm{Ext}^1(N, M) = 0$. Thus M is FGT-injective.

(3)⇒(4) is clear.

(4)⇒(1). Let M be a right R-module. For any minimal left $\mathcal{TI}_0$-resolution $\overline{I} = \cdots \to I_n \to \cdots \to I_0 \to M \to 0$, the $(n+1)$th $\mathcal{TI}_0$-syzygy K_{n+1} of $\overline{I}$ is n-TI-injective by Lemma 4.16. Thus K_{n+1} has a monic $\mathcal{TI}_0$-cover $f : I \to K_{n+1}$ by (4). But K_{n+1} is a quotient of an FGT-injective right R-module by Lemma 4.17, so f is an isomorphism, and hence K_{n+1} is FGT-injective. Then left $\mathcal{TI}_0$-dim $M \leq n+1$. By [17, Lemma 3.2 and Corollary 3.7], $FGT - I.\dim(R) \leq n+3 < \infty$.

(2)⇒(5). For any right R-module $N \in \mathcal{TI}_{n-1}$ and an exact sequence $0 \to K \to N \to M \to 0$, note that $K \in \mathcal{TI}_n$ by (2), then $M \in \mathcal{TI}_{n-1}$ by [4, Proposition 5.5.5(iii)]. But it is easy to verify that $\mathcal{TI}_{n-1}$ is closed under direct sums. By [9, Proposition 4], every right R-module has a monic $\mathcal{TI}_{n-1}$-cover.

(5)⇒(2). Let M be any right R-module. By Theorem 3.8, M has an epic $\mathcal{TI}_n$-cover $f : I \to M$. Then there is a short exact sequence $0 \to K \to I \to M \to 0$, where $K = \mathrm{Ker}(f)$. Then K is n-TI-injective by Remark 4.2(1). Note that K is also $(n-1)$-TI-injective, so K has a monic $\mathcal{TI}_{n-1}$-cover $g : I' \to K$ by (5). But K is a quotient of an FGT-injective right R-module by Lemma 4.17, then g is an isomorphism, and hence $K \in \mathcal{TI}_{n-1}$. Thus $M \in \mathcal{TI}_n$ by [4, Proposition 5.5.5], as desired. □

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