論文

복합재료 구조 요소의 탄성문제에 대한 해

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Solution to Elasticity Problems of Structural Elements of Composite Materials

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ABSTRACT

The present study describes a method for analytical solution to elastic field in structural elements of general symmetric laminated composite materials. The two dimensional plane stress elasticity problems under mixed boundary conditions are reduced to the solution of a single fourth order partial differential equation, expressed in terms of a single unknown function, called displacement potential function. In addition, all the components of stress and displacement are expressed in terms of the same displacement potential function, which makes the method suitable for any boundary conditions. The method is applied to obtain analytical solutions to two particular problems of structural elements consisting of an angle-ply laminate and a cross-ply laminate, respectively. Some numerical results are presented for both the problems with reference to the glass/epoxy composite. The results are highly accurate and reliable as all the boundary conditions including those in the critical regions of supports and loads are satisfied exactly. This verifies the method as a simple and reliable one as well as capable to obtain exact analytical solution to elastic field in structural elements of composite materials under mixed and any other boundary conditions.

초 록

본 연구는 일반적인 적층 복합재료의 구조요소에서 탄성영역에 대한 해석적 해에 대한 방법을 나타낸 것이다. 혼합된 경계조건 하에서 2차원 평면응력탄성문제는 변위포텐셜함수라 불리는 단일미지함수로 표현된 1/4 부분미분방정식의 해로 축소시켰으며, 응력과 변위의 모든 성분은 어떠한 경계조건에도 적합한 방법을 만드는 동일한 변위포텐셜항으로 표현하였다. 이 방법은 각도를 가진 적층판과 90도 적층판으로 각각 구성된 구조요소의 두 가지 특별문제에 대해서 해석적인 해를 얻는데 적용된다. 본 연구에서 나타낸 몇 가지 수치적인 결과는 두 가지로 적층된 유리섬유복합재료에 관한 것이다. 연구결과는 지지된 하중의 임계영역에서 모든 경계조건이 정확히 만족되어 크게 신뢰할 만한 결과를 나타내었다. 이는 혼합된 어떠한 경계조건하에서도 복합재료의 구조요소에서 탄성영역에 대한 정확한 해석적 해를 얻는 데 적용시킬 수 있을 뿐 아니라 단순한 문제를 해결하는 데도 신뢰할 만한 결과를 얻을 수 있음을 입증한 것이다.

Key Words : 해석적 해(Analytical solution), 탄성문제(Elasticity problems), 복합재료(Composite materials), 변위포텐셜접근법(Displacement potential approach), 혼합경계조건(Mixed boundary conditions), 구조요소(Structural elements)

1. Introduction

Now-a-days, composite materials are the integral part of current technological development as these materials are being increasingly used in almost every branch of engineering because of their outstanding advantages over conventional

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monolithic materials. Among others, structural applications of these materials have received wide attention to ensure better performance in terms of weight, strength, and stiffness. However, the design of a structural element with composite materials requires detail information of its characteristics under actual boundary conditions, e.g., loading and constraint conditions. These boundary conditions may be imposed externally or by an adjacent element of a structure. Very often, the boundary conditions imposed on an element are expected to be mixed type, i.e., a part of the boundary is associated with stress or loading conditions while the other part of the boundary is associated with constraint or displacement conditions. Therefore, a suitable method is inevitable for the analytical solution to elastic field in structural elements of composite materials to know their exact characteristics under mixed boundary conditions.

For the solution of two dimensional elasticity problems, two well known approaches, namely Airy stress function approach [1] and displacement parameter approach [2], are being usually used since long ago. For the problems associated with the stress boundary conditions, the Airy stress function approach can be used to obtain analytical solution. However, this approach appears to be inadequate for the problems associated with displacement boundary conditions. This category of the problems can be treated by displacement parameter approach [2]. However, the displacement parameter approach experiences two major shortcomings. Firstly, formulations of two dimensional elasticity problems in terms of displacement parameters yields two simultaneous second order partial differential equations which are quite difficult to solve analytically. Secondly, the mixed boundary conditions cannot be dealt with by this approach. Thus, it appears that neither of the approaches mentioned above is suitable to address the mixed boundary value problems.

To overcome the above shortcomings and develop a suitable method to address the mixed boundary value problems, more than one decade before, Iddris et al. [3] and Ahmed et al. [4] developed a relatively new approach, called displacement potential approach, for the solution to elasticity problems of isotropic materials. The approach was proved to be applicable for any modes of boundary conditions. Akanda et al. [5] used this approach to solve the stress problems of gear teeth by finite difference method. Later, the approach was extended for an orthotropic lamina and a number of problems under mixed boundary conditions were solved both analytically [6-8] and numerically [9]. However, the extended approach was applicable to orthotropic lamina only, i.e., the angle lamina and laminated composites could not be analyzed by the approach. But, in practice, most of the structural elements consist of laminated composites instead of a single lamina to ensure desired performance. Thus, it is important to modify and extend the approach so as to cover the structural elements of laminated composites for the solution of more practical elasticity problems of composites.

Very recently, the above displacement potential approach has been further extended for laminated composites by Huq [10] and Afsar et al. [11]. It was demonstrated that the method could be applied to obtain the analytical solution of elasticity problems of composite structural elements. However, there was a limitation that the method was not capable to address the angle-ply laminates for any ply angle. Thus, this study is aimed to overcome the limitation associated with the angle-ply laminates. Firstly, the general formulations of the displacement potential approach for a general symmetric laminated composite is discussed by emphasizing its merits over two well-know traditional approaches for the solution to elasticity problems. Secondly, the displacement potential approach is demonstrated to obtain analytical solutions for two specific problems, namely, a deep stiffened cantilever beam of angle-ply laminated composite and a roller guided panel of cross-ply laminated composite. Finally, some numerical results of elastic field (stress and displacement components) are presented to examine the effects of ply-angle and laminate stacking sequence for the cases of cantilever beam and panel, respectively, with reference to the glass/epoxy composite. It should be noted that the panel problem was considered earlier in Ref. [11] where laminate stacking sequence was not taken into account. This study considers the laminate stacking sequence to examine the capability and versatility of the displacement potential approach.

2. Theoretical Formulations

For a general laminate in the Cartesian coordinate system x-y, the stress-strain relation is given by [12, 13]

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} \tag{1}$$

where N denotes the components of in-plane force per unit length, M denotes the components of moment per unit length, ε^0 denotes the components of mid-plane strain, κ denotes the components of mid-plane curvature, and A, B, and D denote the components of extensional stiffness matrix, bending-extension coupling stiffness matrix, and bending stiffness matrix, respectively.

When the laminate is symmetric and acted upon by in-plane load only, the coupling stiffness matrix B and curvature κ in Eq. (1) vanish. Also, the global strains are equal to the mid-plane strains and shear-extension coupling terms of the extensional stiffness matrix are zero for such cases. Thus, Eq. (1) can be written as

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}=\frac{1}{h}\begin{bmatrix}A_{11} & A_{12} & 0\\ A_{12} & A_{22} & 0\\ 0 & 0 & A_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \tag{2}$$

where the average stress components σ_x, σ_y, and σ_{xy} are obtained by dividing the components of in-plane force N (N_x, N_y, N_{xy}) per unit length by the total thickness of the laminate h. The components of the extensional stiffness matrix A are given by

$$A_{11}=\sum_{k=1}^{n}\left[Q_{11}c^4+Q_{22}s^4+2(Q_{12}+2Q_{66})s^2c^2\right]_k(h_k-h_{k-1}) \quad 3(a)$$

$$A_{12}=\sum_{k=1}^{n}\left[(Q_{11}+Q_{22}-4Q_{66})s^2c^2+Q_{12}(c^4+s^4)\right]_k(h_k-h_{k-1}) \quad 3(b)$$

$$A_{22}=\sum_{k=1}^{n}\left[Q_{11}s^4+Q_{22}c^4+2(Q_{12}+2Q_{66})s^2c^2\right]_k(h_k-h_{k-1}) \quad 3(c)$$

$$A_{66}=\sum_{k=1}^{n}\left[(Q_{11}+Q_{22}-2Q_{12}-2Q_{66})s^2c^2+Q_{66}(s^4+c^4)\right]_k(h_k-h_{k-1}) \quad 3(d)$$

Here,

$$Q_{11}=\frac{E_1}{1-\nu_{21}\nu_{12}},\quad Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}},\quad Q_{22}=\frac{E_2}{1-\nu_{21}\nu_{12}},\quad Q_{66}=G_{12},$$

$c=\cos\theta$, $s=\sin\theta$, h_k-h_{k-1} is the thickness of the k-th ply of the laminate, θ is the angle between the x-axis and the fiber direction of a lamina in the laminate, E_1 and E_2 are the Young's modulus in the longitudinal and the transverse directions, respectively, ν_{12} and ν_{21} are the major and minor Poison's ratios, respectively, and G_{12} is the in-plane shear modulus of a lamina in the laminate.

In the absence of body forces, the two dimensional equilibrium equations can be written as [1]

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0 \quad 4(a)$$

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\sigma_{xy}}{\partial x}=0 \quad 4(b)$$

The relationships between the three strain components and the two displacement components are given by [1]

$$\varepsilon_x=\frac{\partial u}{\partial x} \quad 5(a)$$

$$\varepsilon_y=\frac{\partial v}{\partial y} \quad 5(b)$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \quad 5(c)$$

where u and v are the displacement components in the x- and y-directions, respectively.

2.1 Airy stress function approach

By inverse operation, Eq. (2) can be written as

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}I_{11} & I_{12} & 0\\ I_{12} & I_{22} & 0\\ 0 & 0 & I_{66}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}h \tag{6}$$

where the elements of the compliance matrix I and the stiffness matrix A have a relation of $[I]=[A]^{-1}$.

The compatibility condition is [1]

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y} \tag{7}$$

The Airy stress function ϕ is defined as [1]

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2} \quad 8(a)$$

$$\sigma_y=\frac{\partial^2\phi}{\partial x^2} \quad 8(b)$$

$$\sigma_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \quad 8(c)$$

By making use of Eqs. (4) and (6)-(8), one can obtain

$$I_{22}\frac{\partial^4\phi}{\partial x^4}+I_{11}\frac{\partial^4\phi}{\partial y^4}+2\left(I_{12}+I_{66}/2\right)\frac{\partial^4\phi}{\partial x^2\partial y^2}=0 \tag{9}$$

Combination of Eqs. (5), (6), and (8) gives

$$\frac{\partial u}{\partial x}=\left(I_{11}\frac{\partial^2\phi}{\partial y^2}+I_{12}\frac{\partial^2\phi}{\partial x^2}\right)h \quad 10(a)$$

$$\frac{\partial v}{\partial y}=\left(I_{22}\frac{\partial^2\phi}{\partial x^2}+I_{12}\frac{\partial^2\phi}{\partial y^2}\right)h \quad 10(b)$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=-I_{66}\frac{\partial^2\phi}{\partial x\partial y}h \quad 10(c)$$

The solution of elasticity problems following the Airy stress function approach requires the solution of Eq. (9). However, this approach appears to be efficient only for stress boundary conditions as one can readily apply the stress boundary conditions by using Eq. (8). When boundary conditions are prescribed in terms of displacements/constraints, it is quite difficult to directly apply the boundary conditions as one requires integration of Eq. (10) before applying the boundary conditions. Thus, this approach seems to be inconvenient for displacement and mixed boundary conditions.

2.2 Displacement parameter approach

Making use of Eqs. (2), (4), and (5) yields

$$A_{11}\frac{\partial^2 u}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v}{\partial x\partial y}+A_{66}\frac{\partial^2 u}{\partial y^2}=0 \quad 11(a)$$

$$A_{22}\frac{\partial^2 v}{\partial y^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u}{\partial x\partial y}+A_{66}\frac{\partial^2 v}{\partial x^2}=0 \quad 11(b)$$

Equation (11) represents two second order elliptic partial differential equations of equilibrium in terms of two displacement parameters u and v. Although this is suitable for applying displacement boundary conditions, it is quite difficult to obtain analytical solution satisfying these two partial differential equations simultaneously. Further, this approach is not convenient for stress boundary and mixed boundary conditions.

2.3 Displacement potential function approach

As discussed above, both the Airy stress function approach and the displacement parameter approach have some limitations to obtain analytical solution of elasticity problems, particularly the mixed boundary value problems. Thus, the present study is aimed at developinga method for the analytical solution of elasticity problems of structural elements of composite materials under any boundary conditions prescribed in terms of either stress or displacement or any combination of these, i.e. mixed boundary conditions. To realize this, a new function, called displacement potential function $\psi(x, y)$, is introduced. The two displacement parameters u and v are expressed in terms of this displacement potential function $\psi(x, y)$ as

$$u=\frac{\partial^2\psi}{\partial x\partial y} \quad 12(a)$$

$$v=-\frac{A_{11}}{A_{12}+A_{66}}\frac{\partial^2\psi}{\partial x^2}-\frac{A_{66}}{A_{12}+A_{66}}\frac{\partial^2\psi}{\partial y^2} \quad 12(b)$$

Using of this definition of $\psi(x, y)$ in to Eq. (11) satisfies the first equation of Eq. (11) automatically and the second equation is reduced to the following fourth partial differential equation.

$$\frac{\partial^4\psi}{\partial x^4}+\left[\frac{A_{22}}{A_{66}}-\frac{A_{12}^2}{A_{11}A_{66}}-\frac{2A_{12}}{A_{11}}\right]\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{A_{22}}{A_{11}}\frac{\partial^4\psi}{\partial y^4}=0 \tag{13}$$

Now there is only one equation (Eq. (13)) to be dealt with as the first equation (Eq. 11(a)) is satisfied automatically. Thus, the major advantage of the present approach is that the mixed boundary problems of elasticity has been reduced to the solution of a single equation of the single parameter ψ. This reduces the computational time and cost for both the analytical and numerical solutions of mixed boundary value problems. Using Eqs. (2), (5) and (12), the components of stress can be expressed in terms of the same displacement potential function $\psi(x, y)$ as

$$\sigma_x=\frac{A_{66}}{h\left(A_{12}+A_{66}\right)}\left[A_{11}\frac{\partial^3\psi}{\partial x^2\partial y}-A_{12}\frac{\partial^3\psi}{\partial y^3}\right] \quad 14(a)$$

$$\sigma_y=\frac{1}{h\left(A_{12}+A_{66}\right)}\left[\left(A_{12}^2+A_{12}A_{66}-A_{11}A_{22}\right)\frac{\partial^3\psi}{\partial x^2\partial y}-A_{22}A_{66}\frac{\partial^3\psi}{\partial y^3}\right] \quad 14(b)$$

$$\sigma_{xy}=\frac{A_{66}}{h\left(A_{12}+A_{66}\right)}\left[A_{12}\frac{\partial^3\psi}{\partial x\partial y^2}-A_{11}\frac{\partial^3\psi}{\partial x^3}\right] \quad 14(c)$$

It is obvious that this approach is based on the solution of the displacement potential function $\psi(x, y)$ from Eq. (13). Once the displacement potential function $\psi(x, y)$ is found, the components of displacement and stress can be readily found from Eqs. (12) and (14), respectively. It is interesting to note that all the components of stress and displacement have been expressed in terms of the displacement potential function $\psi(x, y)$. Thus, this approach is suitable for any boundary conditions whether they are prescribed in terms of either stress or displacement or any combination of them. In the present study, this approach is applied to two particular problems of composite elements under mixed boundary conditions and analytical solution of different components of stress and displacement is presented.

By considering only a single ply in Eq. (3) and assuming the fibers in the x-direction, one can readily obtain the corresponding set of formulations of displacement potential approach for a single lamina as [9]

$$E_1 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 (E_1 - 2\mu_{12} G_{12}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + E_2 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{15}$$

$$u = \frac{\partial^2 \psi}{\partial x \partial y} \quad 16(a)$$

$$v = -\frac{1}{Z}\left[E_1^2 \frac{\partial^2 \psi}{\partial x^2} + G_{12}\left(E_1 - \mu_{12}^2 E_2\right)\frac{\partial^2 \psi}{\partial y^2} \right] \quad 16(b)$$

$$\sigma_x = \frac{E_1 G_{12}}{Z}\left[E_1 \frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial y^3} \right] \quad 17(a)$$

$$\sigma_y = \frac{E_1 E_2}{Z}\left[(\mu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - G_{12} \frac{\partial^3 \psi}{\partial y^3} \right] \quad 17(b)$$

$$\sigma_{xy} = -\frac{E_1 G_{12}}{Z}\left[E_1 \frac{\partial^3 \psi}{\partial x^3} - \mu_{12} E_2 \frac{\partial^3 \psi}{\partial x \partial y^2} \right] \quad 17(c)$$

where

$$Z = \mu_{12} E_1 E_2 + G_{12}(E_1 - \mu_{12}^2 E_2) \tag{18}$$

3. Application of Displacement Potential Approach

In this section, the displacement potential function approach is applied to obtain analytical solution to elastic field in two specific structural elements of laminated composite. The first problem is associated with a deep stiffened cantilever beam of symmetric angle ply laminated composite while the second one is a roller guided rectangular panel of symmetric cross-ply laminated composite.

3.1 A deep stiffened cantilever beam

A deep stiffened cantilever beam of symmetric angle-ply laminated composite is shown in Fig. 1. The depth and length of the beam are denoted by a and b, respectively. The left lateral edge is rigidly fixed to a support and the two longitudinal edges parallel to the x-axis are stiffened. It is worthy to mention that stiffeners are usually used in structural elements in order to increase the stiffness, which, in turn, reduces the degree of deformation of the structural elements. The right lateral edge is subjected to a parabolic shear load $\sigma_{xy}^0 = 4P(y^2 - ay)/a^2$, where P is the maximum value of the shear stress at $y = a/2$. It is usual practice that a concentrated load at the tip of a cantilever beam is simulated by the parabolic shear load to ease the analysis [1]. The fiber angle is defined by θ. The boundary conditions of the present beam problem are

(i) $u_x(x, 0) = u_x(x, a) = 0;\ 0 \le x \le b$
(ii) $\sigma_y(x, 0) = \sigma_y(x, a) = 0;\ 0 \le x \le b$
(iii) $u_x(0, y) = u_y(0, y) = 0;\ 0 \le y \le a$
(iv) $\sigma_x(b, y) = 0, \sigma_{xy}(b, y) = \sigma_{xy}^0 = 4P(y^2 - ay)/a^2; 0 \le y \le a$

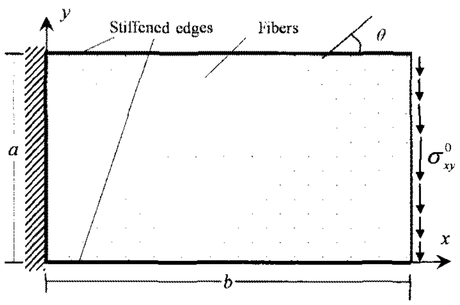

Fig. 1 A deep stiffened cantilever beam of symmetric angle-ply laminated composite.

The stiffeners at the top ($y = a$) and bottom ($y = 0$) boundaries of the beam are mathematically defined by the fact that there is no displacement along their length (boundary condition (i)). To obtain the solution of the present beam problem, Eq. (13) should be solved for $\psi(x, y)$ so that the components of displacement and stress determined from $\psi(x, y)$ satisfy the above boundary conditions.

For the present problem, the solution of Eq. (13) is assumed in the form of Fourier series as

$$\psi=\sum_{m=1}^{\infty} X_m \cos\alpha y + Mx^3 \tag{19}$$

where X_m is a function of x only, M is an arbitrary constant, and $\alpha = m\pi/a$. It should be noted that any other forms of $\psi(x, y)$ may be assumed. However, it is important to verify that all the boundary conditions above are identically satisfied. It is interesting to note that the assumption for $\psi(x, y)$ in Eq. (19) automatically satisfies the first four boundary conditions in items (i) and (ii). Thus, only the boundary conditions (iii) and (iv) are remaining to be satisfied.

Substitution of Eq. (19) into Eq. (13) yields

$$X_m'''' - BX_m''\alpha^2 + CX_m\alpha^4 = 0 \tag{20}$$

where $B=\frac{A_{22}}{A_{66}}-\frac{A_{12}^2}{A_{11}A_{66}}-\frac{2A_{12}}{A_{11}}$ and $C=\frac{A_{22}}{A_{11}}$. The general solution of Eq. (20) is

$$X_m = A_m e^{n_1 x} + B_m e^{n_2 x} + C_m e^{n_3 x} + D_m e^{n_4 x} \tag{21}$$

where

$$n_1, n_2 = \alpha\sqrt{(B\pm\sqrt{B^2-4C})/2} \qquad 22(a)$$

$$n_3, n_4 = -\alpha\sqrt{(B\pm\sqrt{B^2-4C})/2} \qquad 22(b)$$

and A_m, B_m, C_m, and D_m are arbitrary constants that should be determined from the remaining four boundary conditions. Now making use of Eqs. (13) and (21) in Eqs. (12) and (14), the components of displacement and stress can be expressed in terms of the four constants A_m, B_m, C_m, and D_m as

$$u=\sum_{m=1}^{\infty}\left[-\left(n_1A_me^{n_1x}+n_2B_me^{n_2x}+n_3C_me^{n_3x} + n_4D_me^{n_4x}\right)\alpha\sin\alpha y\right] \qquad 23(a)$$

$$v=\frac{1}{A_{12}+A_{66}}\sum_{m=1}^{\infty}\left[\{A_{66}(A_me^{n_1x}+B_me^{n_2x}+C_me^{n_3x} + D_me^{n_4x})\alpha^2 - A_{11}\left(n_1^2A_me^{n_1x}+n_2^2B_me^{n_2x} + n_3^2C_me^{n_3x}+n_4^2D_me^{n_4x}\right)\}\cos\alpha y - 6MxA_{11}\right] \qquad 23(b)$$

$$\sigma_x = -\frac{A_{66}}{h(A_{12}+A_{66})}\sum_{m=1}^{\infty}\left[A_{12}(A_me^{n_1x}+B_me^{n_2x}+C_me^{n_3x} + D_me^{n_4x})\alpha^3 + A_{11}(n_1^2A_me^{n_1x}+n_2^2B_me^{n_2x} + n_3^2C_me^{n_3x}+n_4^2D_me^{n_4x})\alpha\right]\sin\alpha y \qquad 24(a)$$

$$\sigma_y = -\frac{1}{h(A_{12}+A_{66})}\sum_{m=1}^{\infty}\left[A_{22}A_{66}(A_me^{n_1x}+B_me^{n_2x}+C_me^{n_3x} + D_me^{n_4x})\alpha^3 + (A_{12}^2+A_{12}A_{66}-A_{11}A_{22})(n_1^2A_me^{n_1x} + n_2^2B_me^{n_2x}+n_3^2C_me^{n_3x}+n_4^2D_me^{n_4x})\alpha\right]\sin\alpha y \qquad 24(b)$$

$$\sigma_{xy} = -\frac{A_{66}}{h(A_{12}+A_{66})}\sum_{m=1}^{\infty}\left[A_{12}(n_1A_me^{n_1x}+n_2B_me^{n_2x}+n_3C_me^{n_3x} + n_4D_me^{n_4x})\alpha^2 + A_{11}(n_1^3A_me^{n_1x}+n_2^3B_me^{n_2x}+n_3^3C_me^{n_3x} + n_4^3D_me^{n_4x})\right]\cos\alpha y - \frac{6MA_{11}A_{66}}{h(A_{12}+A_{66})} \qquad 24(c)$$

The parabolic applied load at the right boundary ($x = b$) can be expressed in terms of Fourier series as

$$\sigma_{xy}^0 = 4P\left(y^2-ay\right)/a^2 = E_0 + \sum_{m=1}^{\infty} E_m\cos\alpha y \tag{25}$$

where $E_0 = -2P/3$ and $E_m = 8P[1+\cos(m\pi)]/m^2\pi^2$. Now, applying the boundary conditions (iii) and (iv) into Eqs. (23), 24(a), and 24(c), one can readily obtain the constant M as

$$M = hP\left(A_{12}+A_{66}\right)/9A_{11}A_{66} \tag{26}$$

and the following four simultaneous equations for the determination of the four constants A_m, B_m, C_m, and D_m.

$$\begin{bmatrix} n_1 & n_2 & n_3 & n_4 \\ P_1 & P_2 & P_3 & P_4 \\ Q_1 & Q_2 & Q_3 & Q_4 \\ R_1 & R_2 & R_3 & R_4 \end{bmatrix}\begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{E}_m \end{bmatrix} \tag{27}$$

where $\bar{E}_m = -E_m h\left(A_{12}+A_{66}\right)/A_{66}$ and

$$\left.\begin{array}{l} P_i = A_{11}n_i^2 - A_{66}\alpha^2 \\ Q_i = \left(A_{11}n_i^2\alpha + A_{12}\alpha^3\right)e^{n_i b} \\ R_i = \left(A_{11}n_i^3 + A_{12}n_i\alpha^2\right)e^{n_i b} \end{array}\right] ; \; i = 1,\ 2,\ 3,\ 4.$$

Once Eq. (27) is solved for the constants A_m, B_m, C_m, and D_m, the explicit expressions for the components of displacement and stress can be found from Eqs. (23) and (24) which are valid throughout the entire region of the beam.

3.2 A roller guided panel

Figure 2 shows a rectangular panel of symmetric cross-ply laminated composite. The left lateral edge is fixed and the two longitudinal edges parallel to the x-axis are roller guided. The right lateral edge is subjected to a linearly varying tensile load $\sigma_x^0 = P(1-2y/a)$, where P is the maximum value of the load. The roller guided edges can freely move in the x-direction while restricted in the y-direction. Thus, the boundary conditions are

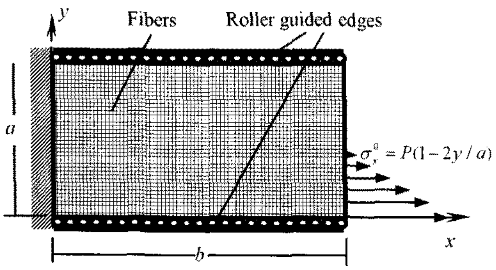

Fig. 2 A rectangular panel of symmetric cross-ply laminated composite.

(i) $u_y\ (x,\ 0) = u_y\ (x,\ 0) = 0;\ 0 \le x \le b$

(ii) $\sigma_{xy}\ (x,\ 0) = \sigma_{xy}\ (x,\ 0) = 0;\ 0 \le x \le b$

(iii) $u_x\ (0,\ y) = u_y\ (0,\ y) = 0;\ 0 \le y \le a$

(iv) $\begin{array}{l} \sigma_x(b,y) = \sigma_x^0(b,y) = P(1-2y/a); 0 \le y \le a/2 \\ \sigma_{xy}(b,y) = 0; 0 \le y \le a \end{array}$

The solution to the displacement potential function $\psi(x,\ y)$ for the panel problem is assumed in the form of

$$\psi(x,y) = \sum_{m=1}^{\infty} X_m \sin\alpha y + Mx^2y + Ny^3 \qquad (28)$$

As beore, X_m is a function of x only, M and N are arbitrary constants, and $\alpha = m\pi/a$. The rest of the solution procedure is the same as that of the cantilever beam problem. Thus, following the same procedure, one can finally obtain [11]

$$\begin{bmatrix} n_1 & n_2 & n_3 & n_4 \\ P_1 & P_2 & P_3 & P_4 \\ Q_1 & Q_2 & Q_3 & Q_4 \\ R_1 & R_2 & R_3 & R_4 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{E}_m \\ 0 \end{bmatrix} \qquad (29)$$

where

$$\left.\begin{array}{l} P_i = A_{11}n_i^2 - A_{66}\alpha^2 \\ Q_i = \left(A_{11}n_i^2\alpha + A_{12}\alpha^3\right)e^{n_i b} \\ R_i = \left(A_{11}n_i^3 + A_{12}n_i\alpha^2\right)e^{n_i b} \end{array}\right] ; \; i = 1,\ 2,\ 3,\ 4 \qquad 30(a)$$

$$\bar{E}_m = \frac{E_m h\left(A_{12} + A_{66}\right)}{A_{66}} \qquad 30(b)$$

$$E_m = \frac{4P}{m^2\pi^2}\left(1 - \cos\frac{m\pi}{2}\right) \qquad 30(c)$$

The components of displacement and stress are obtained as

$$u = \sum_{m=1}^{\infty}(n_1A_me^{n_1x} + n_2B_me^{n_2x} + n_3C_me^{n_3x} + n_4D_me^{n_4x})\alpha\cos\alpha y + 2Mx \qquad 31(a)$$

$$v = \frac{1}{A_{12}+A_{66}}\left[\sum_{m=1}^{\infty}\left\{A_{66}(A_me^{n_1x} + B_me^{n_2x} + C_me^{n_3x} + D_me^{n_4x})\alpha^2 - A_{11}(n_1^2A_me^{n_1x} + n_2^2B_me^{n_2x} + n_3^2C_me^{n_3x} + n_4^2D_me^{n_4x})\right\}\sin\alpha y - 2y(A_{11}M + 3A_{66}N)\right] \qquad 31(b)$$

$$\sigma_x = \frac{A_{66}}{h(A_{12}+A_{66})}\left[\sum_{m=1}^{\infty}\left\{A_{12}(A_me^{n_1x} + B_me^{n_2x} + C_me^{n_3x} + D_me^{n_4x})\alpha^3 + A_{11}(n_1^2A_me^{n_1x} + n_2^2B_me^{n_2x} + n_3^2C_me^{n_3x} + n_4^2D_me^{n_4x})\alpha\right\}\cos\alpha y + 2(A_{11}M - 3A_{12}N)\right] \qquad 32(a)$$

$$\sigma_y = \frac{1}{h\left(A_{12}+A_{66}\right)}\left[\sum_{m=1}^{\infty}\left\{A_{22}A_{66}(A_me^{n_1x} + B_me^{n_2x} + C_me^{n_3x} + D_me^{n_4x})\alpha^3 + (A_{12}^2 + A_{12}A_{66} - A_{11}A_{22})(n_1^2A_me^{n_1x} + n_2^2B_me^{n_2x} + n_3^2C_me^{n_3x} + n_4^2D_me^{n_4x})\alpha\right\}\cos\alpha y + 2\left(A_{12}^2 + A_{12}A_{66} - A_{11}A_{22}\right)M - 6A_{22}A_{66}N\right] \qquad 32(b)$$

$$\sigma_{xy} = -\frac{A_{66}}{h\left(A_{12}+A_{66}\right)}\sum_{m=1}^{\infty}\left[A_{12}(n_1A_me^{n_1x} + n_2B_me^{n_2x} + n_3C_me^{n_3x} + n_4D_me^{n_4x})\alpha^2 + A_{11}(n_1^3A_me^{n_1x} + n_2^3B_me^{n_2x} + n_3^3C_me^{n_3x} + n_4^3D_me^{n_4x})\right]\sin\alpha y \qquad 32(c)$$

where $M = hP/8A_{11}$ and $N = -hP/24A_{66}$.

4. Results and Discussion

The formulations developed in the preceding section are demonstrated for a cantilever beam and a roller guided panel of glass/epoxy composite in order to present some numerical results of the analytical solution to elastic field. Here, it is noted that the formulations can be applied to any composites. However, the glass/epoxy composite has been selected here merely as an example. The mechanical properties of the glass/epoxy composite are: E_1 = 38.6 GPa, E_2 = 8.3 GPa, G_{12} = 4.1 GPa, and ν_{12} = 0.26. The maximum value of the applied load Pis considered to be 1000 MPa. It has been verified that the results of the series solution converge very well only for the number of terms m = 10 in the series. This verifies the advantage of the present approach that accurate results can be obtained with only a few terms in the series which, in turn, reduces the computational time and cost. However, for further accuracy of the results, the number of terms in the series is taken as 20to calculate all the numerical results in this paper.

4.1 Results of stiffened cantilever beam

The cantilever beam is considered to be consisted of angle-ply laminates of 4 plies. For such a cantilever beam, the effects of ply-angle on the elastic field (stress and displacement) have been investigated and some representative results corresponding to y/a = 0.25, 0.50, and 0.75 on the lateral section x/b = 0.75 are presented.

Figure 3 shows the effect of ply-angle on the longitudinal stress component. Near the top (y/a = 0.75) and bottom (y/a = 0.25) surfaces of the beam, the longitudinal stress varies significantly as the ply-angle θ varies from 0 to 45 degrees. After that the longitudinal stress remains almost uninfluenced by the variation of the ply-angle. However, the stress at the mid-way of the vertical section (y/a = 0.5) is totally uninfluenced by the ply-angle. Further, it is noted that the stress at all the points is zero for θ = about 45 and 60 degrees. It is noted that the zero degree is the critical ply-angle for the longitudinal stress components.

The effect of ply-angle on the lateral stress is illustrated in Fig. 4. The lateral stress at the mid-way of the vertical section (y/a = 0.50) is also uninfluenced by the ply-angle. However, near the top and bottom surfaces of the beam, the magnitude of the stress increases as the ply-angle varies from 0 to 35 degrees. Then it falls as the ply-angle further increases. For this case, θ = 37 degree is the critical ply-angle for which the value of the lateral stress near the top and bottom surfaces is the maximum.

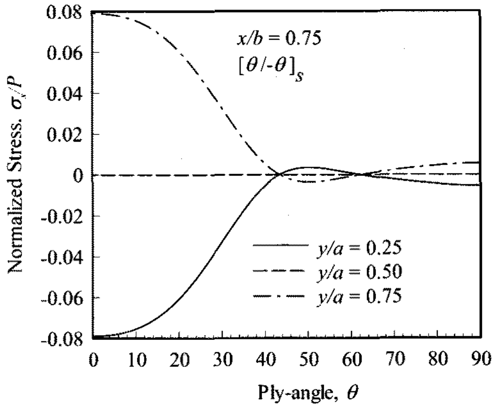

Fig. 3 Effect of ply-angle on the longitudinal stress component.

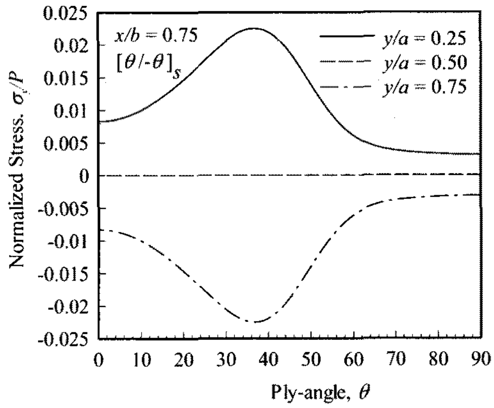

Fig. 4 Effect of ply-angle on the lateral stress component.

Shown in Fig. 5 is the effect of ply-angle on the shear stress component. The shear stress exhibits the reverse trend of that of the longitudinal and lateral stress components. Here, the stress at the mid-way of the vertical section has a significant variation with the variation of the ply-angle. On the contrary, the stress near the top and bottom surfaces of the beam has a little variation with the ply-angle. In addition, the shear stress at the points of equal distance from the mid-point of the vertical section is the same and θ = 34 degree is the critical value of the ply-angle.

The effect of ply-angle on the longitudinal displacement is similar to that on the longitudinal stress component as seen from Fig. 6. However, the lateral displacement has different characteristics with the variation of ply-angle as shown in Fig. 7. The magnitude of the lateral displacement is the minimum at θ = 45 degrees and its value increases as the ply-angle deviates in either direction from 45 degrees.

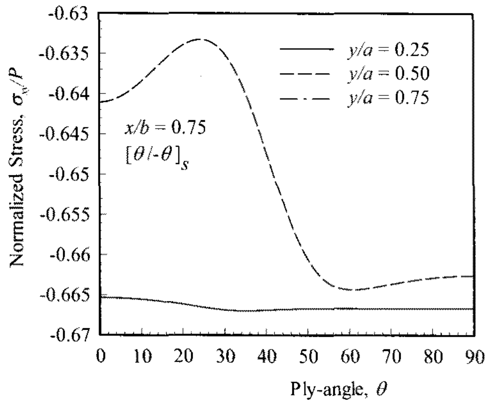

Fig. 5 Effect of ply-angle on the shear stress component.

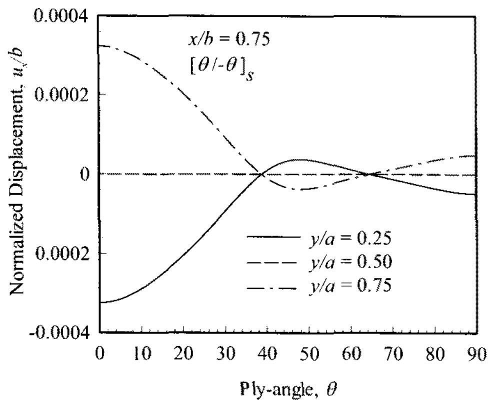

Fig. 6 Effect of ply-angle on the longitudinal displacement component.

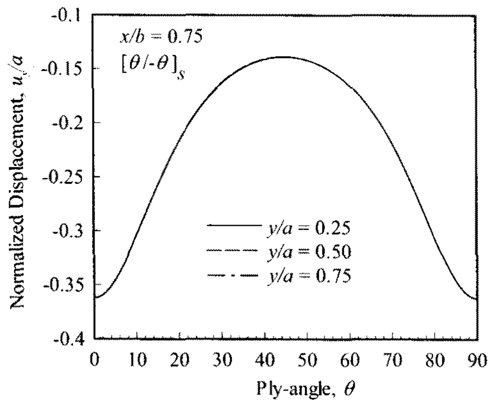

Fig. 7 Effect of ply-angle on the lateral displacement component.

4.2 Results of roller guided panel

To present some numerical results of elastic field in the roller guided panel of laminated composite, a five-ply symmetric cross-ply laminate is considered. The six different possible staking sequences, $[0_5]$, $[0_2/\overline{90}]_s$, $[90/0_3/90]$, $[0/90_3/0]$, $[90_2/\bar{0}]_s$, and $[90_5]$ are taken into account to investigate their effects on the elastic field.

Figure 8 shows the effect of stacking sequence on the longitudinal displacement component. It is seen that the magnitude of this displacement is the minimum when all the plies are arranged with 0° orientation. Its value increases as the number of plies with 90° fiber orientation increases. The effect of stacking sequence on the lateral displacement component is illustrated in Fig. 9 which shows the reverse characteristics of the longitudinal displacement. The magnitude of this displacement is the maximum for 0° fiber orientation in all the plies. Its magnitude decreases with the increase of the plies with 90° fiber orientation. Further, it is noted that the lateral displacement is zero at the top and bottom surfaces of the panel that agrees with the boundary conditions of the problem.

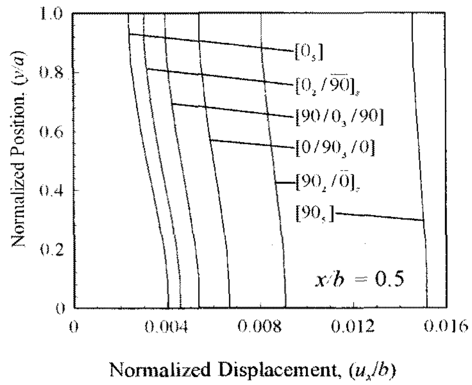

Fig. 8 Effect of stacking sequence on longitudinal displacement component.

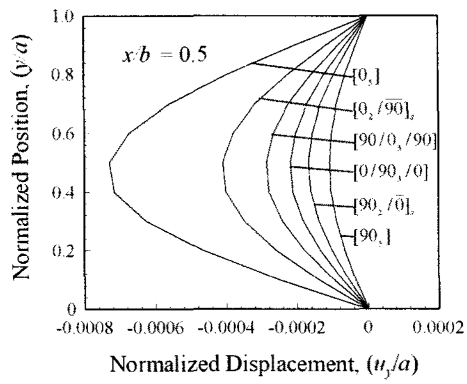

Fig. 9 Effect of stacking sequence on lateral displacement component.

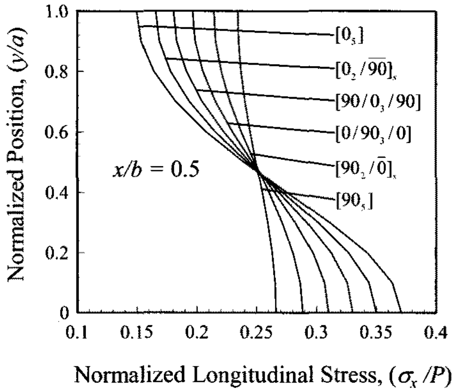

Fig. 10 Effect of stacking sequence on longitudinal stress component.

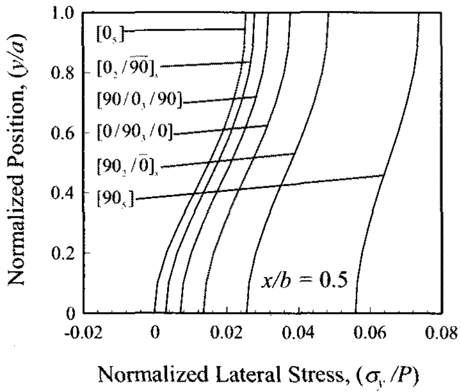

Fig. 11 Effect of stacking sequence on lateral stress component.

Figure 10 exhibits the longitudinal stress component for different stacking sequence. For any stacking sequence, the stress varies from its maximum value at the bottom surface to the minimum value at the top surface of the panel. The range of variation reduces as the number of plies with 90° fiber orientation increases. The lateral stress as a function of stacking sequence is displayed in Fig. 11. The maximum stress is developed when all the plies have 90° fiber orientation. On the other hand, the stress is the minimum for 0° orientation of fibers in all the plies. The magnitude of shear stress is the maximum for the fibers with 0° orientation in all the plies as shown in Fig. 12. Its magnitude decreases as the number of plies with 90° fiber orientation increases. For any stacking sequence, the shear stress is zero at the top and bottom surfaces of the panel that conforms to the boundary conditions of the problem.

Figure 13 presents a comparison of the two displacement components corresponding to different stacking sequences. The

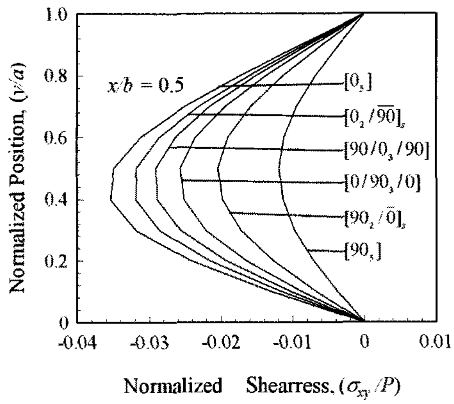

Fig. 12 Effect of stacking sequence on shear stress component.

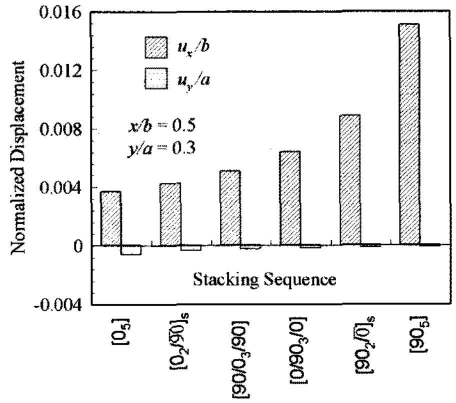

Fig. 13 Normalized displacement for various stacking sequence.

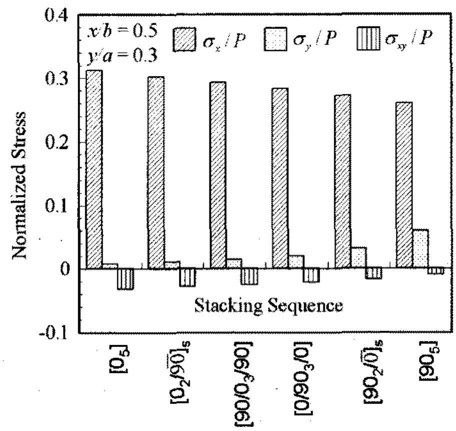

Fig. 14 Normalized stresses for various stacking sequence.

longitudinal displacement is considerably larger than the lateral displacement. The comparison of three stress components is shown in Fig. 14 corresponding to various stacking sequences. The longitudinal stress is many times higher than the lateral and shear stress components.

4.3 Validity of Solution

To verify the validity of the present approach to the solution to mixed boundary value elasticity problems, a roller-guided panel of three ply ([0/90/0]) laminated composite is solved by finite element method using the commercial software ANSYS. The in-plane engineering constants [12] of the laminate are used as the material properties for the present orthotropic panel ([0/90/0]). The whole domain of the panel is meshed using the quadratic 8-node-82 type of element. The size of the panel considered is 30×10mm and the total number of elements used in the calculations is 300 (30×10). The size and distribution of elements are uniform throughout the domain of the panel. The convergence and accuracy of the solution are confirmed by varying the number of elements.

Figures 15 to 19 compare the analytical and finite element solutions [11] of different components of and stress displacement at different sections of the panel. It is seen that both the analytical and finite element solutions agree very well with each other. The maximum difference is found to be 2.7% for the shear stress component (Fig. 17) at y/a = 0.3 of the section x/b = 0.9. This small discrepancy in the results is attributed to the error in numerical calculations by finite element method. This verifies the validity of the present analytical solution based on the displacement potential approach.

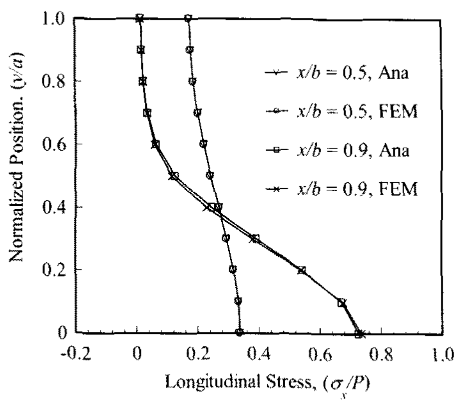

Fig. 15 Comparison of analytical and finite element solutions of longitudinal stress.

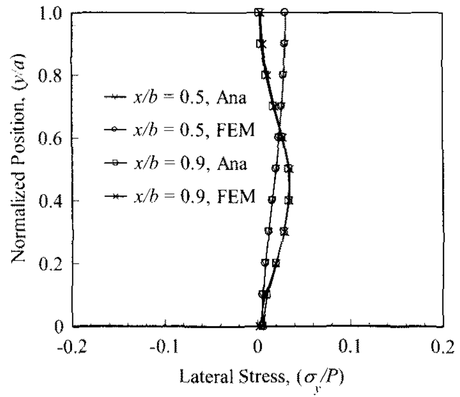

Fig. 16 Comparison of analytical and finite element solutions of lateral stress.

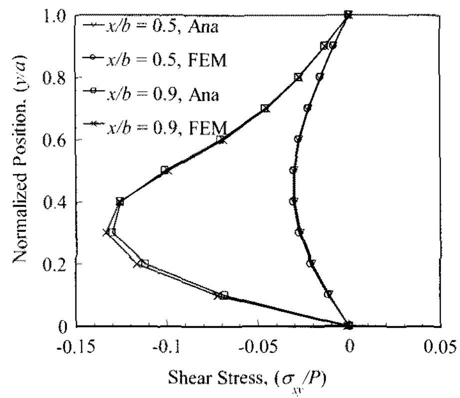

Fig. 17 Comparison of analytical and finite element solutions of shear stress.

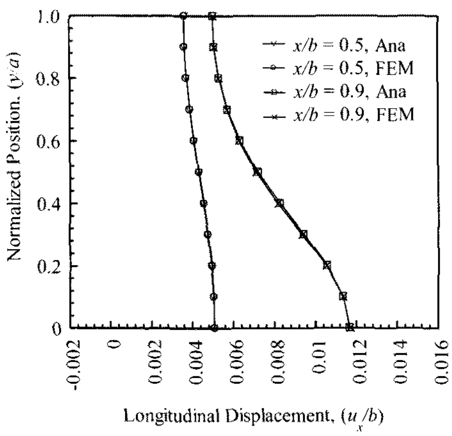

Fig. 18 Comparison of analytical and finite element solutions of longitudinal displacement.

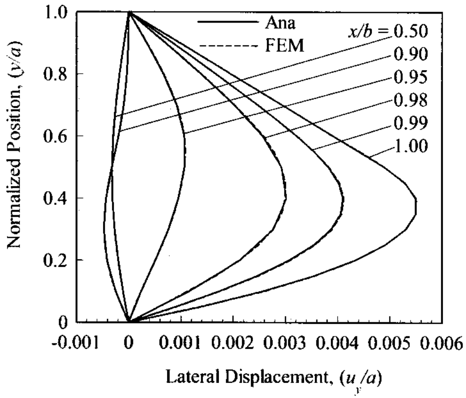

Fig. 19 Comparison of analytical and finite element solutions of lateral displacement.

5. Conclusions

A method has been discussed for the analytical solution to elastic field in structural elements of composite materials. The outstanding advantage of the method is that the solution of the two dimensional elasticity problems requires finding a single parameter, called displacement potential function, from a single differential equation as one of the equilibrium equations is satisfied automatically. Also, the method is applicable to any modes of boundary conditions, whether they are prescribed in term of stress or constraints or any combination of these. The numerical results obtained for two different structural elements of composite materials establish the fact that the method is simple and, at the same time, capable to produce exact analytical solution to elastic field in structural elements of composite materials under any modes of boundary conditions. The comparison of analytical and finite element solution justifies the validity of the analytical solution.

Acknowledgement

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