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PRECISE ASYMPTOTICS IN STRONG LIMIT THEOREMS FOR NEGATIVELY ASSOCIATED RANDOM FIELDS

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ABSTRACT. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a field of identically distributed and negatively associated random variables with mean zero and set $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{Z}_+^d$, $d \geq 2$. We investigate precise asymptotics for $\sum_{\mathbf{n}}$ $|\mathbf{n}|^{r/p-2}P(|S_{\mathbf{n}}| \geq \epsilon|\mathbf{n}|^{1/p})$ and $\sum_{\mathbf{n}} \frac{(\log|\mathbf{n}|)^\delta}{|\mathbf{n}|} P(|S_{\mathbf{n}}| \geq \epsilon\sqrt{|\mathbf{n}|\log|\mathbf{n}|})$, $(0 \leq \delta \leq 1)$ as $\epsilon \searrow 0$.

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1. Introduction

Let $d \geq 2$ be a positive integer and $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The field $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ is called negatively associated(NA) if, for every pair of disjoint subsets S, T of $\mathbb{Z}_+^d$ and any pair of coordinatewise increasing functions $f(X_{\mathbf{i}}, \mathbf{i} \in S), g(X_{\mathbf{j}}, \mathbf{j} \in T)$ with $E\Big[f^2(X_{\mathbf{i}}, \mathbf{i} \in S)\Big] < \infty$ and $E\Big[f^2(X_{\mathbf{j}}, \mathbf{j} \in T)\Big] < \infty$ $Cov(f(X_{\mathbf{i}}), g(X_{\mathbf{j}})) \leq 0$ holds. The concept of NA was introduced by Joag-Dev and Proschan(1983). As pointed out and proved by Joag-Dev and Proschan(1983), a number of well-known multivariate distributions possess the NA property, such as multinomial distribution, negatively correlated normal distribution, multivariate hypergeometric distribution, etc. Negative association has found application in reliability theory, statistical mechanics and multivariate statistical analysis. The interested reader is referred to Roussas(1999). Roussas(1994) proved the central limit theorem and Zhang and Wen(2001) investigated the weak convergence for negatively associated random fields. Recently, Li(2009) obtained convergence rate in the law of the iterated logarithm and Ko(2009)showed the complete convergence for

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negatively associated random fields. Now let $\mathbb{Z}_+^d (d \geq 2)$ denote positive integer d-dimensional lattice with coordinatewise partial ordering $\leq$. The notation $\mathbf{m} \leq \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \cdots, m_d)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_d)$, means that $m_k \leq n_k$ for $k = 1, 2, \cdots, d$. We also use $|\mathbf{n}|$ for $\prod_{k=1}^{d} n_k$, and $\mathbf{n} \to \infty$ is to be interpreted as $n_k \to \infty$, for $k = 1, 2, \cdots, d$ and set $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$.

Gut and *Spătaru*(2003) proved the following precise asymptotics in strong limit theorems for a field of i.i.d. random variables:

Theorem A. *Let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ *be a field of i.i.d. random variables with* $EX_{\mathbf{1}} = 0$ *and let* N *denote a standard normal random variable. Suppose that* $E\Big[|X_{\mathbf{1}}|^r (\log^+ |X_{\mathbf{1}}|)^{d-1}\Big] < \infty$, $r \geq 2$, *set* $\sigma^2 = EX_{\mathbf{1}}^2$. *Then, for* $1 \leq p < 2$,

$$\lim_{\epsilon \searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log \epsilon)^{d-1}} \sum_{|\mathbf{n}|} |\mathbf{n}|^{r/p-2} P(|S_{\mathbf{n}}| \geq \epsilon |\mathbf{n}|^{\frac{1}{p}})$$

$$= \frac{1}{(d-1)!} \left(\frac{2p}{2-p}\right)^{d-1} \frac{p}{r-p} \sigma^{\frac{2p}{2-p}(\frac{r}{p}-1)} E|N|^{\frac{2p}{2-p}(\frac{r}{p}-1)}$$

(here and in the sequel $\log^+ x = \log(e \vee x), x \geq 0$*).*

Theorem B. *Let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ *be a field of i.i.d. random variables with* $EX_{\mathbf{1}} = 0$ *and let* N *denote a standard normal random variable. Suppose that* $E\Big[|X_{\mathbf{1}}|^2 (\log^+ |X_{\mathbf{1}}|)^{d-1}\Big] < \infty$ *and set* $\sigma^2 = EX_{\mathbf{1}}^2$. *Then, for* $0 \leq \delta \leq 1$,

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2d} \sum_{\mathbf{n}} \frac{(\log |\mathbf{n}|)^\delta}{|\mathbf{n}|} P(|S_{\mathbf{n}}| \geq \epsilon \sqrt{|\mathbf{n}| \log |\mathbf{n}|})$$

$$= \frac{1}{(d-1)!} \frac{\sigma^{2\delta+2d}}{\delta+d} E|N|^{2\delta+2d}.$$

In this paper we will extend similar results as above to the NA random fields.

2. Preliminaries

Let $d(j) = Card\{\mathbf{k} : |\mathbf{k}| = j\}$ *and* $M(j) = Card\{\mathbf{k} : |\mathbf{k}| \leq j\}$. Then following asymptotics holds:

$$\frac{M(j)}{j(\log j)^{d-1}} \to \frac{1}{(d-1)!} \text{ as } j \to \infty. \tag{2.1}$$

An important fact is that,

$$\sum_{\mathbf{n}} \cdots = \sum_{j \geq 1} \sum_{|\mathbf{n}|=j} \cdots . \tag{2.2}$$

Whenever the function involving $\mathbf{n}$ only depends on the value of $|\mathbf{n}|$, the summation can be simplified as follows. For example, for sum in Theorem A we have

$$\sum_{\mathbf{n}} |\mathbf{n}|^{r/p-2} P(|S_{\mathbf{n}}| \geq \epsilon |\mathbf{n}|^{1/p}) = \sum_{j\geq 1} \sum_{|\mathbf{n}|=j} |\mathbf{n}|^{r/p-2} P(|S_{\mathbf{n}}| \geq \epsilon |\mathbf{n}|^{1/p}) \tag{2.3}$$
$$= \sum_{j\geq 1} d(j) j^{\frac{r}{p}-2} P(|S_{\pi(j)}| \geq \epsilon j^{1/p})$$

where $\pi(j) = (j, 1, 1, \cdots, 1)$, $j \geq 1$. (See Gut and *Spătaru*(2003) for more detail)

The following lemma will be used to prove the main results.

Lemma 2.1 (Gut and *Spătaru*(2003)). *Let X be a random variable and $r \geq 2$, assume that $E[|X|^r(\log^+ |X|)^{d-1}] < \infty$, and set $\rho(\epsilon) = \epsilon^{-2p/(2-p)}$, where $1 \leq p < 2$. Then, for any constant $a > 0$,*

$$\sum_{j>\rho(\epsilon)} d(j) j^{r/p-1} P(|X| \geq a\epsilon j^{\frac{1}{p}}) \leq C\epsilon^{-r} a^{-r} E[|X|^r (\log^+ |X|)^{d-1}] < \infty. \tag{2.4}$$

Lemma 2.2 (Gut and *Spătaru*(2003)). *Let X be a random variable. Assume that $E[|X|^2(\log^+ |X|)^{d-1}] < \infty$ and set $c(\epsilon) = \epsilon^{M/\epsilon^2}$, where $M > 1$. Let $0 \leq \delta \leq 1$. For any positive constant a,*

$$\sum_{j>c(\epsilon)} d(j)(\log j)^{\delta} P(|X| \geq a\epsilon\sqrt{j \log j})$$
$$\leq C\epsilon^{-2\delta} M^{\delta-1} a^{-2} \left(E\left[|X|^2(\log^+ |X|)^{d-1}\right] + (-\log \epsilon)^{d-1} EX^2 \right) < \infty.$$

Lemma 2.3 (Gut and *Spătaru*(2003)). *For $\delta \geq -d+1$,*

$$\sum_{j=2}^{k} \frac{d(j)(\log j)^{\delta}}{j} \sim \frac{1}{(d-1)!} \sum_{j=2}^{k} \frac{(\log j)^{\delta+d-1}}{j} \sim \frac{(\log k)^{\delta+d}}{(d-1)!(\delta+d)} \text{ as } k \to \infty.$$

Lemma 2.4 (Gut and *Spătaru*(2003)). *For $\gamma > -1$,*

$$\sum_{j=1}^{k} d(j) j^{\gamma} \sim \frac{1}{(d-1)!} \sum_{j=1}^{k} j^{\gamma} (\log j)^{d-1} \sim \frac{1}{(d-1)!} \frac{k^{\gamma+1}(\log k)^{d-1}}{\gamma+1} \text{ as } k \to \infty.$$

Finally we introduce some results on NA sequences.

Lemma 2.5 (Shao(2000)). *Let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of identically distributed NA random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Then for $x > 0$, $y > 0$,*

$$P(|S_{\pi(j)}| \geq x) \leq 2jP(|X_1| \geq y) + 4\exp\left\{ -\frac{x^2}{8jEX_1^2} \right\} + 4\left(\frac{jEX_1^2}{4(xy + jEX_1^2)} \right)^{\frac{x}{12y}}.$$

Finally, we introduce the central limit theorem for NA random variables obtained by Newman(1984).

Theorem 2.6 (Newman(1984)). *Assume that* $\{X_n, n \geq 1\}$ *is a strictly stationary sequence of NA random variables with* $EX_1 = 0$ *and* $EX_1^2 < \infty$. *If* $0 < \sigma^2 = EX_1^2 + 2\sum_{j=2}^{\infty} Cov(X_1, X_j) < \infty$, *then*

$$\frac{S_n}{\sigma\sqrt{n}} \to^{\mathcal{D}} N(0,1) \text{ as } n \to \infty.$$

3. Main results

Throughout this section we assume that $\left\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\right\}$ are identically distributed NA random variables with $\sigma^2 = EX_1^2 + 2\sum_{\mathbf{k}\in Z_+^d} EX_{\mathbf{1}}X_{\mathbf{k}} = 1$.

Theorem 3.1. *Let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ *be a field of identically distributed NA random variables with* $EX_1 = 0$ *and* $EX_1^2 < \infty$ *and let* N *denote a standard normal random variable. Assume that* $E\Big[|X_1|^r(\log^+|X_1|)^{d-1}\Big] < \infty$ $r \geq 2$. *Then, for* $r \geq 2$ *and* $1 \leq p < 2$

$$\lim_{\epsilon\searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log\epsilon)^{d-1}} \sum_{|\mathbf{n}|} |\mathbf{n}|^{r/p-2} P(|S_{\mathbf{n}}| \geq \epsilon|\mathbf{n}|^{\frac{1}{p}}) \tag{3.1}$$
$$= \frac{1}{(d-1)!}(\frac{2p}{2-p})^{d-1}\frac{p}{r-p}E|N|^{\frac{2p}{2-p}(\frac{r}{p}-1)}$$

To prove Theorem 3.1 we need the following Propositions:

Proposition 3.2 (Gut, *Spătaru*(2003)). *Let* N *be a standard normal random variable. For* $r \geq 2$ *and* $1 \leq p < 2$, *we have*

$$\lim_{\epsilon\searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log\epsilon)^{d-1}} \sum_{j\geq 1} d(j) j^{\frac{r}{p}-2} P(|N| \geq \epsilon j^{\frac{1}{p}}) \tag{3.2}$$
$$= \frac{1}{(d-1)!}\left(\frac{2p}{2-p}\right)^{d-1}\frac{p}{r-p}E|N|^{\frac{2p}{2-p}(\frac{r}{p}-1)}.$$

Proposition 3.3 *Let* $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ *be a field of identically distributed NA random variables with* $EX_1 = 0$ *and let* $\rho(\epsilon) = \epsilon^{-2p/(2-p)}$. *Then, for* $r \geq 2$ *and*

$1 \le p < 2$,

$$\lim_{\epsilon \searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log \epsilon)^{d-1}} \sum_{j \le \rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} |P(|S_{\pi(j)}| \ge \epsilon j^{\frac{1}{p}}) - |P(|N| \ge \epsilon j^{\frac{1}{p}-\frac{1}{2}})| = 0. \tag{3.3}$$

Proof. Let $M > 1$ be a positive number and set $\Delta_{\pi(j)} = \sup_x |P(|S_{\pi(j)}| \ge \sqrt{j}x) - P(|N| \ge x)|$. Then we can easily get that $\Delta_{\pi(j)} \to 0$ as $j \to \infty$ by Theorem 2.6 and we first conclude that by Lemma 2.4 with $\gamma = \frac{r}{p} - 2$ and Toepliz lemma(Stout(1995), p 120)

$$\lim_{m\to\infty} \frac{1}{m^{\frac{r}{p}-1}(\log m)^{d-1}} \sum_{j \le m} d(j) j^{\frac{r}{p}-2} \Delta_{\pi(j)} = 0. \tag{3.4}$$

Letting $\epsilon \searrow 0$, we then obtain

$$\lim_{\epsilon \searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log \epsilon)^{d-1}} \sum_{j \le \rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} \Delta_{\pi(j)} \tag{3.5}$$

$$\begin{aligned}
&= \lim_{\epsilon \searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log \epsilon)^{d-1}} \lim_{M\to\infty} [\rho(\epsilon)M]^{\frac{r}{p}-1} (\log[\rho(\epsilon)M])^{d-1} \\
&\quad \frac{1}{[\rho(\epsilon)M]^{\frac{r}{p}-1}(\log[\rho(\epsilon)M])^{d-1}} \sum_{j \le [\rho(\epsilon)M]} d(j) j^{\frac{r}{p}-2} \Delta_{\pi(j)} \\
&\le \lim_{\epsilon \searrow 0} \lim_{M\to\infty} C M^{\frac{r}{p}-1} \left(\frac{2p}{2-p} - \frac{\log M}{\log \epsilon} \right)^{d-1} \times \frac{1}{[\rho(\epsilon)M]^{\frac{r}{p}-1}(\log[\rho(\epsilon)M])^{d-1}} \\
&\quad \times \sum_{j \le [\rho(\epsilon)M]} d(j) j^{\frac{r}{p}-2} \Delta_{\pi(j)} = 0 \textit{ by } (3.4).
\end{aligned}$$

Proposition 3.4 (Gut and *Spătaru*(2003)). *Let N be a standard normal random variable. Then, we have, for $r \ge 2$ and $1 \le p < 2$*

$$\lim_{M\to\infty} \limsup_{\epsilon \searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log \epsilon)^{d-1}} \sum_{j > \rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} P(|N| \ge \epsilon j^{\frac{1}{p}-\frac{1}{2}}) = 0. \tag{3.6}$$

Proposition 3.5. *Let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of identically distributed NA random variables with $EX_{\mathbf{1}} = 0$. Assume that $E\Big[|X_{\mathbf{1}}|^r (\log^+ |X_{\mathbf{1}}|)^{d-1}\Big] < \infty$. Then, for $r \ge 2$ and $1 \le p < 2$*

$$\lim_{M\to\infty} \limsup_{\epsilon \searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log \epsilon)^{d-1}} \sum_{j > \rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} P(|S_{\pi(j)}| \ge \epsilon j^{\frac{1}{p}}) = 0. \tag{3.7}$$

Proof. Let $M > 1$. Lemma 2.5 with $x = \epsilon j^{\frac{1}{p}}$ and $y = \epsilon j^{\frac{1}{p}}/12\gamma$ with $\gamma = r/(2-p)$ yields

$$\begin{aligned}
&\sum_{j>\rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} P(|S_{\pi(j)}| \geq \epsilon j^{1/p}) \qquad (3.8)\\
&< 2 \sum_{j>\rho(\epsilon)M} d(j) j^{\frac{r}{p}-1} P(|X_1| \geq \epsilon j^{1/p}/12\gamma)\\
&+4 \sum_{j>\rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} \exp\left(-\frac{\epsilon^2 j^{2/p}}{8jEX_1^2}\right)\\
&+4 \sum_{j>\rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} \left(\frac{12\gamma j(EX_1^2)}{4\epsilon^2 j^{2/p}}\right)^{\gamma}\\
= \ & I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, it follows from Lemma 2.1 that

$$\begin{aligned}
I_1 &= 2 \sum_{j>\rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} P(|X_1| \geq \epsilon j^{\frac{1}{p}}/12\gamma)\\
&\leq 2 \sum_{j>\rho(\epsilon)} d(j) j^{\frac{r}{p}-1} P(|X_1| \geq \epsilon j^{\frac{1}{p}}/12\gamma)\\
&\leq C\epsilon^{-r}\gamma^r E\Big[|X_1|^r (\log^+ |X_1|)^{d-1}\Big].
\end{aligned}$$

Hence, we obtain

$$\lim_{M\to\infty} \limsup_{\epsilon\searrow 0} \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log\epsilon)^{d-1}} I_1 = 0 \tag{3.9}$$

by the fact that $\frac{2p}{2-p}(\frac{r}{p}-1) \geq r$. It follow from (2.1) that

$$\sum_{j\geq k} \frac{d(j)}{j^{\eta+1}} \leq C \sum_{j\geq k} \frac{M(j)}{j^{\eta+2}} \leq C \sum_{j\geq k} \frac{(\log j)^{d-1}}{j^{\eta+1}} \leq C \frac{(\log k)^{d-1}}{k^{\eta}},$$

which yields

$$\begin{aligned}
\frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log\epsilon)^{d-1}} I_2 &= \frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log\epsilon)^{d-1}} \sum_{j>\rho(\epsilon)M} d(j) j^{\frac{r}{p}-2} \exp\left(-\frac{\epsilon^2 j^{2/p}}{8jEX_1^2}\right) \qquad (3.10)\\
&\leq C\left(\frac{2p}{2-p} - \frac{\log M}{\log\epsilon}\right)^{d-1} M^{\frac{r}{p}-1} \exp(-bM^{\frac{2}{p}-1}),\ b > 0\\
&\quad \to 0 \ \textit{as}\ M \to \infty \ \textit{where}\ b^{-1} = 8EX_1^2,
\end{aligned}$$

and

$$\frac{\epsilon^{\frac{2p}{2-p}(\frac{r}{p}-1)}}{(-\log\epsilon)^{d-1}} I_3 \leq CM^{-1}\left(\frac{2p}{2-p} - \frac{\log M}{\log\epsilon}\right)^{d-1} \to 0 \ \textit{as}\ M \to \infty \tag{3.11}$$

by Lemma 2.3.

From (3.8), (3.9), (3.10) and (3.11) the result (3.7) follows. □

Proof of Theorem 3.1: The triangle inequality and combining Propositions 3.2 - 3.5 yield (3.1).

Theorem 3.6. *Let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of identically distributed NA random variables with $EX_{\mathbf{1}} = 0$ and $EX_{\mathbf{1}}^2 < \infty$ and let N note a standard normal random variable. Assume*

$$E\Big[|X_{\mathbf{1}}|^2(\log^+ |X_{\mathbf{1}}|)^{d-1}\Big] < \infty, \tag{3.12}$$

$$\sigma^2 = EX_{\mathbf{1}}^2 + 2\sum_{\mathbf{k}\in Z_+^d} EX_{\mathbf{1}}X_{\mathbf{k}} = 1. \tag{3.13}$$

Then, for $0 \le \delta \le 1$,

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2d} \sum_{\mathbf{n}} \frac{(\log|\mathbf{n}|)^\delta}{\mathbf{n}} P\Big(|S_{\mathbf{n}}| \ge \epsilon\sqrt{|\mathbf{n}|\log|\mathbf{n}|}\Big) = \frac{1}{(d-1)!(\delta+d)} E|N|^{2\delta+2d}. \tag{3.14}$$

Proposition 3.7. *Let N be a standard normal random variable. For $0 \le \delta \le 1$*

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2d} \sum_{j\ge 1} \frac{d(j)(\log j)^\delta}{j} P(|N| \ge \epsilon\sqrt{\log j}) = \frac{1}{(d-1)!(\delta+d)} E|N|^{2\delta+2d}. \tag{3.15}$$

Proposition 3.8. *Let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of identically distributed NA random variables with $EX_{\mathbf{1}} = 0$ and let $c(\epsilon) = e^{M/\epsilon^2}$, where $M > 1$. Then*

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2d} \sum_{j\le c(\epsilon)} d(j)\frac{(\log j)^\delta}{j} |P(|S_{\pi(j)}| \ge \epsilon\sqrt{j\log j}) - P(|N| \ge \epsilon\sqrt{\log j})| = 0.$$

Proof. Let $\Delta_{\pi(j)} = \sup_x |P(|S_{\pi(j)}| \ge \sqrt{j}x) - P(|N| \ge x)|$. Then by Theorem 2.6 $\Delta_{\pi(j)} \to 0$ *as* $j \to \infty$. It follows from Lemma 2.4 that

$$\lim_{\epsilon\searrow 0} \frac{\epsilon^{2\delta+2d}}{M^{\delta+d}} \sum_{j\le c(\epsilon)} d(j)\frac{(\log j)^\delta}{j}\Delta_{\pi(j)} = 0.$$

□

Proposition 3.9 (Gut and *Spătaru*(2003)). *Let N be a standard normal random variable and let $c(\epsilon) = \exp(M/\epsilon^2)$ where $M > 1$. Then*

$$\lim_{M\to\infty} \limsup_{\epsilon\searrow 0} \epsilon^{2\delta+2d} \sum_{j>c(\epsilon)} d(j)\frac{(\log j)^\delta}{j} P(|N| > \epsilon\sqrt{\log j}) = 0.$$

Proposition 3.10. *Let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of identically distributed NA random variables with $EX_1 = 0$, $EX_1^2 < \infty$ and satisfying (3.12) and (3.13) and let $c(\epsilon) = \exp(M/\epsilon^2)$ where $M > 1$. Then, for $0 \le \delta \le 1$*

$$\lim_{M\to\infty} \limsup_{\epsilon \searrow 0} \epsilon^{2\delta+2d} \sum_{j>c(\epsilon)} d(j) \frac{(\log j)^\delta}{j} |P(|S_{\pi(j)}|$$

$$\ge \epsilon\sqrt{j\log j}) - P(|N| \ge \epsilon\sqrt{\log j})| = 0.$$

Proof.

$$\begin{aligned}
&\sum_{j>c(\epsilon)} d(j)\frac{(\log j)^\delta}{j}|P(|S_{\pi(j)}| \ge \epsilon\sqrt{j\log j}) \\
&\le 2\sum_{j>c(\epsilon)} d(j)(\log j)^\delta |P(|X_1| \ge \frac{\epsilon}{12(d+2)}\sqrt{j\log j}) \\
&+4\sum_{j>c(\epsilon)} d(j)\frac{(\log j)^\delta}{j}\exp(-\frac{\epsilon^2\log j}{8EX_1^2}) \\
&+4\sum_{j>c(\epsilon)} d(j)\frac{(\log j)^\delta}{j}\left(\frac{j12(d+2)(EX_j^2)}{4\epsilon^2 j\log j}\right)^{d+2} \\
&= I_4 + I_5 + I_6.
\end{aligned}$$

Since $j > c(\epsilon)$ implies $(\log j)^\delta \le (\epsilon^2/M)^{1-\delta} j \log j$, by Lemma 2.2 we estimate

$$\begin{aligned}
I_4 &\le C\sum_{j>c(\epsilon)} (\log j)^{d-1}(\log j)^\delta P(\epsilon^2 j\log j \le a^{-2}X_1^2 < \epsilon^2(j+1)\log(j+1)) \\
&\le C\epsilon^{-2\delta}M^{\delta-1}\sum_{j>c(\epsilon)} \epsilon^2 j\log j(\log j)^{d-1} \\
&\quad \times P(\epsilon^2 j\log j \le a^{-2}X_1^2 < \epsilon^2(j+1)\log(j+1))) \\
&\le C\epsilon^{-2\delta}M^{\delta-1}\left(\sum_{j>c(\epsilon)} \epsilon^2 j\log j(\log(\epsilon^2 j\log j))^{d-1} + (-2\log\epsilon)^{d-1}\right) \\
&\quad \times P\Big(\epsilon^2 j\log j \le a^{-2}X_1^2 < \epsilon^2(j+1)\log(j+1)\Big) \\
&\le C\epsilon^{-2\delta}M^{\delta-1}\Big(E[X_1^2\log(1+|X_1|)^{d-1}] + (-\log\epsilon)^{d-1}EX_1^2\Big)
\end{aligned}$$

where $a = \dfrac{1}{12(d+2)}$. Hence,

$$
\begin{aligned}
&\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2\delta+2d}I_4 \\
\le\ & C\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2d}M^{\delta-1}(EX_1^2(\log(1+|X_1|))^{d-1}+(-\log\epsilon)^{d-1}EX_1^2) \\
&=0.
\end{aligned}
$$

For I_5, we also estimate

$$
\begin{aligned}
I_5 &\le C\sum_{j>c(\epsilon)}\frac{(\log j)^{d-1+\delta}}{j}\exp\left(-\frac{\epsilon^2}{8EX_1^2}\log j\right) \\
&= C\int_{c(\epsilon)}^{\infty}\frac{(\log x)^{d-1+\delta}}{x}\exp(-\frac{\epsilon^2}{8EX_1^2}\log x)dx \\
&\le C\int_{c(\epsilon)}^{\infty}\frac{(\log x)^{d-1+\delta}}{x}\exp(-\frac{M}{8EX_1^2})dx \\
&\qquad \textit{since } \frac{(\log x)^{d-1+\delta}}{x} \textit{ is a decreasing function} \\
&\le C(M/\epsilon^2)^{d+\delta}\exp\left(-\frac{M}{8EX_1^2}\right),
\end{aligned}
$$

from which it follows that

$$\lim_{M\to\infty}\lim_{\epsilon\searrow 0}\epsilon^{2\delta+2d}I_5=0.$$

$$
\begin{aligned}
I_6 &= C\sum_{j>c(\epsilon)}(\epsilon^{-2})^{d+2}(\log j)^{\delta-3}/j \\
&\le C(\epsilon^{-2})^{d+2}(M/\epsilon^2)^{\delta-2},
\end{aligned}
$$

which yields

$$\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}\epsilon^{2d+2\delta}I_6\le C\lim_{M\to\infty}\limsup_{\epsilon\searrow 0}M^{\delta-2}=0.$$

Proof of Theorem 3.6: From triangle inequality and Propositions 3.7-3.10, (3.14) follows.

Note that Fu and Zhang(2007) have already established a similar result for $d=1$ case as follows:

Theorem 3.11. $\{X_n, n\ge 1\}$ *be a sequence of strictly stationary NA random variables with* $EX_1=0$ *and* $0<EX_1^2<\infty$ *and set* $0<\sigma^2=EX_1^2+2\sum_{j=2}^{\infty}Cov(X_1,X_j)=1<\infty$*. For* $b>-1$ *if* $EX_1^2(\log|X_1|)^{b-1}<\infty$*, then*

$$\lim_{\epsilon\searrow 0}\epsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{(\log n)^b}{n}P(|S_n|\ge\epsilon\sqrt{n\log n})=\frac{E|N|^{2(b+1)}}{b+1}.$$

References

1. Fu, K.A. and Zhang, L.X.(2007) *Precise rates in the law of the iterated logarithm for negatively associated random variables*, Computer. Math. Appl. **54** 687-698
2. Gut, A.(1978) *Marcinkiewicz laws and convergence rates on the law of large numbers for random variables with nultidimensional indices*, Ann. Probab. **6** 469-482
3. Gut, A.(1980) *Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indeces*, Ann. Probab. **8** 298-313
4. Gut, A. and *Spătaru*(2003) *Precise asymptotics in some strong limit theorems for multidimensinally indexed random variables*, J. Multi. Anal. **86** 398-442
5. Joag-Dev, K. and Proschan, F.(1983) *Negative association of random variables with applications*, Ann. Statist. **11** 286-295
6. Ko, M.H.(2009) *On the complete convergence for negatively associated random fields*, Taiwanese J. Math. accepted.
7. Li, Y.X. (2009) *Convergence rates in the law of the iterated logarithm for negatibily associated random variables with multidimensional indices*, Statist. Probab. Lett. **79** 1038-1043
8. Newman, C.M.(1984) *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, in: Y.L.Tong, ed, IMS Vol. 5 pp. 127-140
9. Roussas, G.G.(1994) *Asymptotic normality of random fields of positively or negatively associated processes*, J.Multi. Anal. **50** 152-173
10. Roussas, G.G.(1999) *Positive and negative dependence with some statistical application*, In: Ghosh, S.(Ed.) Asymptotics, Nonparametrics and Time series. Marcel Dekker, New York, pp. 757-788
11. Stout, W.F.(1995) *Almost sure convergence*, Academic, New-York.
12. Shao, Q.M.(2000) *A coparison on maximum inequalities between negatively associated and independent random variables*, J. Theort. Probab. **13** 343-356
13. Zhang, L.X. and Wen, J.(2000) *A weak convergence for negatively associated random fields*,Statist. Probab. Lett. **53** 259-267

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